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A concept of characteristic for semigroups and semirings

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§ 1. Introduction

The characteristic $\gamma(R)$ of a Ring $R=(R, +, \cdot)$ corresponds to a congruence on the ring $\mathbf{Z}$ of integers via the ideal $\mathfrak{n}(R)=(\gamma(R))$, the annihilator $\mathfrak{n}(R)$ of $(R, +)$ regarded as a $\mathbf{Z}$-module in the natural way. Likewise, the *characteristic* $\gamma(a)$ *of an element* $a\in R$ is defined by the annihilator $\mathfrak{n}(a)=(\gamma(a))$, and it determines the structure of the submodule

$$\langle a\rangle = \mathbf{Z}a \cong \mathbf{Z}/(\gamma(a)).$$

Moreover, the characteristic $\gamma(R)$ of R is the least common multiple of all $\gamma(a)$, corresponding to the intersection $(\gamma(R))=\bigcap\{(\gamma(a))|a\in R\}$ of ideals or congruences. Clearly, $\gamma(a)=o(a)$ if the (additive) order $o(a)=|\langle a\rangle|$ of $a\in R$ is finite, and $\gamma(a)=0$ if $o(a)=\infty$. In particular, these considerations do not depend on the multiplication of R, and may be used to define the *characteristic* $\gamma(R)$ *of any (not necessarily commutative) group* $(R, +)$.

In a similar way we shall introduce the characteristic of a *semiring* $(S, +, \cdot)$, defined to be an algebra such that $(S, +)$ and $(S, \cdot)$ are arbitrary semigroups connected by ring-like distributivity, dealing basically with the characteristic of a semigroup $(S, +)$. For the latter, the additive notation does not mean any restriction, and may be changed if one is interested in semigroups only.

Let $(S, +)$ be a semigroup. For each $a\in S$, the cyclic subsemigroup

$$\langle a\rangle = \mathbf{N}a \cong \mathbf{N}/\varkappa(a)$$

is determined by a congruence $\varkappa(a)$ on the semigroup $(\mathbf{N}, +)$ of positive integers. Let $K=K(\mathbf{N})$ be the complete lattice of all congruences on $(\mathbf{N}, +)$. Then, analogously to the above procedure concerning rings or groups, the intersection $\bigcap\{\varkappa(a)|a\in S\}\in K(\mathbf{N})$ will be a first candidate for the characteristic of

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$(S, +)$ we want to define. However, the characteristics $\gamma(a)$ and $\gamma(R)$ above are integers, corresponding to congruences on $\mathbf{Z}$ by a lattice isomorphism of $(K(\mathbf{Z}), \supseteqq)$ onto $(\mathbf{N}_0, |)$ with the divisiblity relation $|$, and this arithmetical aspect is important. As a substitute for the latter, we define a lattice monomorphism χ of $(K(\mathbf{N}), \supseteqq)$ into a complete lattice $(L, \supseteqq)$, the dual of the direct product $(\mathbf{N}_0^\infty, \leqq)\times(\mathbf{N}_0, |)$, determining each congruence $\varkappa$ on $(\mathbf{N}, +)$ by a pair $\chi(\varkappa)=(v, g)\in L$. In particular, the pair $\chi(\varkappa(a))=(v(a), g(a))$ corresponding to the congruence $\varkappa(a)$, will be called the characteristic of the element $a\in S$.

After these preparations in § 2, we define in § 3 the characteristic of a semigroup $(S, +)$ as the intersection $\chi(S)=(v(S), g(S))$ of all $\chi(\varkappa(a))=(v(a), g(a))$, the characteristics of the elements $a\in S$. In fact there are two ways to do this, depending on whether one takes this intersection in L or in $\chi(K)\subset L$. Both resulting concepts, clearly not very different, will prove fruitful and well-behaved e.g. with respect to subsemigroups and epimorphic images. In particular, if S is a ring or a group, the second component $g(S)$ of $\chi(S)$ will coincide in both cases with the usual characteristic $\gamma(S)$ discussed above.

In fact we deal in this paper (§ 3) with the more general concept of characteristic, taking the intersection in L, since it contains the first one by simplification, and provides more information in some cases. More details as well as some remarks concerning another concept of characteristic introduced in [5] and [6], are given in the text. Of our results, some of them being independent on any concept of "characteristic" used to prove them, we mention here the following ones on semirings: All elements of a semiring S which are multiplicatively (left, right or weakly) cancellable in S, have the same characteristic $\chi(s)$, coinciding with the characteristic $\chi(S)$ of S. Let S be a semiring which consists only of those elements (and, possibly, of a zero); then $\chi(S)$ is either $(0, 1)$, or $(0, p)$ for some prime p, or $(\infty, 0)$. If such a semiring S contains at most one idempotent and no element of infinite order, both with respect to $(S, +)$, then it is a ring (cf. Prop. 7, Thm. 8).

In § 4, we deal with semirings S embeddable into one with right identity or even with identity. Let T_r be a semiring containing S and a right identity e_r. Then $\chi(S)\supseteqq\chi(T_r)=\chi(e_r)=\lambda\supseteqq(\infty, 0)$ holds, and for each $\lambda\in\chi(K)$ contained in this interval there exists a semiring T_r' with a right identity e_r' such that $\chi(T_r')=\lambda$. The corresponding statements hold for a semiring S embeddable into one with identity (cf. Thm. 9). Further, using concepts due to [2] and given in the text, the universal identity extension U of such a semiring S has characteristic $\chi(U)=(\infty, 0)$, and, if $\chi(S)\in\chi(K)$, at least one strict identity extension T_0 of S has characteristic $\chi(T_0)=\chi(S)$. Moreover, for each $\lambda\in\chi(K)$ such that $\chi(S)\supseteqq\lambda\supseteqq(\infty, 0)$, there exists an identity extension U_λ of S with characteristic $\chi(U_\lambda)=\lambda$ which is universal with respect to all identity extensions T of S with characteristic $\chi(T)\supseteqq\lambda$: each such T is an epimorphic image of U_λ (cf. Thm. 11).

§ 2. Basic concepts

The semiring (N, +, ·) of positive integers operates in a natural way on each semigroup $(S, +)$ or on each semiring $(S, +, \cdot)$ by

$$(1) \qquad na = an = \sum_{i=1}^{n} a \quad \text{for all} \quad n \in \mathbf{N}, a \in S,$$

satisfying $(n+m)a = na+ma$, $(nm)a = n(ma)$ and, in case of semirings,

$$(1') \qquad n(a \cdot b) = (na) \cdot b = a \cdot (nb) \quad \text{for all} \quad n \in \mathbf{N}, \quad a, b \in S.$$

Sometimes we shall assume, for all $a, b \in S$ or some particular ones, that

$$(1'') \qquad n(a+b) = na+nb \quad \text{for all} \quad n \in \mathbf{N}$$

holds, clearly not true in general and weaker than $a+b=b+a$. (For example, a semiring S embeddable into one with identity satisfies (1″) for all $a, b \in S$, and there are such semirings with non commutative addition, cf. [2].) Moreover, for each $a \in S$,

$$(2) \qquad \varphi_a\colon \mathbf{N} \to \langle a \rangle = \mathbf{N}a \quad \text{defined by} \quad n \to n^{\varphi_a} = na$$

is an epimorphism of $(\mathbf{N}, +)$ onto the cyclic subsemigroup $\langle a \rangle$ of $(S, +)$, and the corresponding congruence on $(\mathbf{N}, +)$ will be denoted by $\varkappa(a)$.

According to the introduction, *we want to define the characteristic of* $a \in S$ *by* $\chi(a) = \chi(\varkappa(a))$ *with a suitable monomorphism* χ *of the lattice* K *of all congruences on* $(\mathbf{N}, +)$ *into a lattice* L, which extends the arithmetical structure of $(\mathbf{N}_0, |)$. We do this step by step in the following way.

Each congruence $\varkappa \in K$ on $(\mathbf{N}, +)$ is either the equality $\iota_{\mathbf{N}}$ or uniquely determined by two integers $v \in \mathbf{N}_0$ and $g \in \mathbf{N}$ (the minimal ones such that $v+1 \equiv v+1+g(\varkappa)$ holds, cf. [5], § 20) according to

$$(3) \qquad (n, m) \in \varkappa \Leftrightarrow n \equiv m(\varkappa) \Leftrightarrow \begin{cases} n = m & \text{or} \\ n \equiv m & (g) \quad \text{for} \quad n, m > v, \end{cases}$$

where $n \equiv m(g)$ means the usual congruence of integers modulo g. Conversely, each pair $(v, g) \in \mathbf{N}_0 \times \mathbf{N}$ defines by (3) a congruence $\varkappa \neq \iota_{\mathbf{N}}$ on $(\mathbf{N}, +)$. Thus we can define the bijection

$$(4) \qquad \chi\colon K \setminus \{\iota_{\mathbf{N}}\} \to L' = \mathbf{N}_0 \times \mathbf{N} \quad \text{by} \quad \varkappa \to \chi(\varkappa) = (v, g).$$

Applying this via (2) to an element $a \in (S, +)$, we obtain: If $\varkappa(a) = \iota_{\mathbf{N}}$, all elements $a, 2a, \ldots$ of $\langle a \rangle$ are mutually distinct, and $o(a) = |\langle a \rangle| = \infty$. If $\varkappa(a) \neq \iota_{\mathbf{N}}$, we use (4) to define the *characteristic of such an element* $a \in S$ by

$$\chi(a) = \chi(\varkappa(a)) = (v(a), g(a)) = (v, g).$$

It determines the mutually distinct elements of $\langle a \rangle$,

$$(5) \qquad a, \ldots, va; \quad (v+1)a, \ldots, (v+g)a$$

such that $(v+g+1)a = (v+1)a$ is the first repetition, $v = v(a)$ the length of the *aperiodic part* $V(a)=\{a, \ldots, va\}$ of $\langle a\rangle$, and $g=g(a)$ the length or order of the *periodic part* $G(a)=\{(v+1)a, \ldots, (v+g)a\}$ of $\langle a\rangle$, known to be a group (which follows immediately by Lemma 3). Clearly we have $v(a)+g(a)=o(a)$.

Lemma 1. *Let* $(K, \subseteqq)$ *be the lattice of all congruences on* $(\mathbf{N}, +)$, *regarded as partially ordered set with respect to the inclusion relation* $\varkappa_2 \subseteqq \varkappa_1$. *Define, further, a relation on* $L'=\mathbf{N}_0\times\mathbf{N}$ *by*

$$(v_1, g_1) \supseteqq (v_2, g_2) \Leftrightarrow v_1 \leqq v_2 \text{ and } g_1|g_2. \tag{6}$$

Then $(L', \subseteqq)$ *is dual-isomorphic to the direct product of* $(\mathbf{N}_0, \leqq)$ *and* $(\mathbf{N}, |)$, *hence is likewise a lattice, and* (4) *becomes an isomorphism of the lattice* $(K\setminus\{\iota_\mathbf{N}\}, \subseteqq)$ *onto* $(L', \subseteqq)$ *according to*

$$\varkappa_1 \supseteqq \varkappa_2 \Leftrightarrow \chi(\varkappa_1) \supseteqq \chi(\varkappa_2) \Leftrightarrow v_1 \leqq v_2 \text{ and } g_1|g_2. \tag{7}$$

Proof. It is easily seen by (3) that (7) holds for all $\varkappa_1, \varkappa_2\in K\setminus\{\iota_\mathbf{N}\}$. Hence (4) becomes an isomorphism of the partially ordered sets $(K\setminus\{\iota_\mathbf{N}\}, \subseteqq)$ and $(L', \subseteqq)$, due to (6); since $(L', \subseteqq)$ is a lattice, so is $(K\setminus\{\iota_\mathbf{N}\}, \subseteqq)$.

In order to include $\iota_\mathbf{N}\in K$ in this context, we also want to define $\chi(\iota_\mathbf{N})$ as a pair (v, g) such that (3) remains meaningful. This could be done by choosing $g=0$ (for each v) or $v=\infty$ (for each g), adjoining a greatest element ∞ to $(\mathbf{N}_0, \leqq)$. With respect to the structure of L', we do both and define

$$\chi(\iota_\mathbf{N}) = (\infty, 0)\in L = \mathbf{N}_0^\infty\times\mathbf{N}_0, \tag{4'}$$

hence $\chi(a)=\chi(\varkappa(a))=(\infty, 0)$ for the *characteristic of an element* $a\in(S, +)$ *of infinite order*.

Lemma 2. *We use* (6) *to define a relation on* $L=\mathbf{N}_0^\infty\times\mathbf{N}_0$. *Then* $(L, \subseteqq)$ *is dual-isomorphic to the direct product of* $(\mathbf{N}_0^\infty, \leqq)$ *and* $(\mathbf{N}_0, |)$, *hence is likewise a complete lattice. Moreover, the bijection* $\chi\colon K\to\chi(K)=L'\cup\{(\infty, 0)\}$ *defined by* (4) *and* (4′) *is a lattice monomorphism of* $(K, \subseteqq)$ *into* $(L, \subseteqq)$. *Hence* $(\chi(K), \subseteqq)$ *is a sublattice of* $(L, \subseteqq)$ *as well as a complete lattice, but* $\chi(K)$ *is not closed with respect to infinite intersections in* $(L, \subseteqq)$.

The proof is immediate using Lemma 1 and assertions like $\bigcap\{(v, 1)|v\in\mathbf{N}_0\}= =(\infty, 1)$ or $\bigcap\{(0, g)|g\in\mathbf{N}\}=(0, 0)$. We conclude these preliminary considerations with the following statements, denoting by [ ] the greatest integer, by (,)* the greatest common divisor, and by [,]* the least common multiple.

Lemma 3. *Let* $\chi(a)=(v(a), g(a))$ *be the characteristic of an element* $a\in(S, +)$, $o(a)<\infty$, *and consider an element* $ha\in\langle a\rangle$, $1\leqq h\leqq o(a)$. *Then*

$$v(ha) = \left[\frac{v(a)}{h}\right], \quad g(ha) = \frac{g(a)}{(g(a), h)^*} \tag{8}$$

holds, implying $\chi(ha)=(v(ha), g(ha))\supseteqq\chi(a)$.

Proof. Suppose $\chi(ha)=(v', g')$. Then, by (3) or (5), $v'\in\mathbf{N}_0$ has to be maximal such that $v'ha$ is contained in the aperiodic part of $\langle a\rangle$, hence v' is given by the first formula (8). Similarly, $g'\in\mathbf{N}$ has to be minimal such that $(v'+1+g')ha=(v'+1)ha$ holds, which is equivalent to $(v'+1+g')h\equiv(v'+1)\cdot h$ modulo $g(a)$; the smallest solution $g'\in\mathbf{N}$ of this congruence is given by the second formula of (8).

As a consequence of (8), the periodic part $G(a)=\{ha|v(a)<h\leqq o(a)\}$ of $\langle a\rangle$ contains a unique idempotent h_0a, i.e. an element with characteristic (0, 1), determined by $(g(a), h_0)^*=g(a)$, or $g(a)|h_0$. This yields $\chi((h_0+1)a)=(0, g(a))$, hence $(h_0+1)a\in G(a)$ generates — like each element with a characteristic of this type — a cyclic group of order $g(a)$, proving that $G(a)$ is such a group with h_0a as its neutral element.

For formulas being equivalent to (8) cf. [6], § 2. Note that (8) as well as $v(a)+$ $+g(a)=\infty$ also hold in case $o(a)=\infty$, dealing with ∞ in a suitable way. (One can look at $(\mathbf{N}_0^\infty +, \cdot, \leqq)$ as an ordered semiring.) The proof of the next lemma, similar to that above, will be omitted.

Lemma 4. a) *Let $(S, +)$ be a semigroup and a, b elements of S satisfying* (1″). *Then we have*

$$v(a+b) \leqq \max\{v(a), v(b)\}, \quad g(a+b)|[g(a), g(b)]^*, \tag{9}$$

i.e. $\chi(a+b)\supseteq\chi(a)\cap\chi(b)$ *and likewise* $\chi(b+a)\supseteq\chi(a)\cap\chi(b)$.

b) *Let $(S, +, \cdot)$ be a semiring and $a, b\in S$. Then we have*

$$v(ab) \leqq \min\{v(a), v(b)\}, \quad g(ab)|(g(a), g(b))^*, \tag{9'}$$

i.e. $\chi(ab)\supseteq\chi(a)\cup\chi(b)$ *and likewise* $\chi(ba)\supseteq\chi(a)\cup\chi(b)$.

§ 3. The characteristic of semigroups and semirings

Definition. The *characteristic* $\chi(S)$ *of a semigroup* $(S, +)$ is defined by

$$\chi(S) = (v(S), g(S)) = (\sup\{v(a)|a\in S\}, \operatorname{lcm}\{g(a)|a\in S\}), \tag{10}$$

i.e. by the intersection of all characteristics $\chi(a)=\chi(\varkappa(a))=(v(a), g(a))$, $a\in S$, taken in the lattice $(L, \subseteq)$ introduced in Lemma 2. The *characteristic of a semiring* $(S, +, \cdot)$ is defined to be that of $(S, +)$.

If $(S, +)$ contains an element of infinite order, then $\chi(S)=(\infty, 0)$ by (4′) [1]. Hence, suppose $o(a)<\infty$, i.e. $\chi(a)=(v(a), g(a))\in L'$ for all $a\in S$. Then we have $v(S)<\infty$ or $v(S)=\infty$ depending on whether $\{v(a)|a\in S\}\subseteq\mathbf{N}_0$ has a maximum,

[1]) A possibility to distinguish this case from the following one with $\chi(S)=(\infty, 0)$ is to replace L by $\mathbf{N}_0^\infty\times\mathbf{N}_0^\infty$ and to define $\chi(\iota_\mathbf{N})=(\infty, \infty)$ instead of (4′).

and likewise $g(S) \neq 0$ or $g(S)=0$ with $\{g(a)|a \in S\} \subseteq \mathbf{N}$, and clearly there are semigroups and semirings which correspond to each of the resulting four cases.

In particular, if $v(S)<\infty$ and $g(S) \neq 0$, i.e. $\chi(S) \in L'$ (obviously satisfied if $|S|<\infty$), then each $\langle a \rangle \subseteq (S, +)$ is an epimorphic image of a single finite cyclic semigroup $(C, +)$, determined by $\chi(C)=\chi(S)$. Moreover, since a congruence on $(\mathbf{N}, +)$ is also one on $(\mathbf{N}, +, \cdot)$, the semiring $(\mathbf{N}, +, \cdot)$ operating on $(S, +)$ by (1) can be replaced by the semiring $(\mathbf{N}/\varkappa, +, \cdot)$ with $\varkappa=\chi^{-1}(\chi(S))$ if $\chi(S) \in \chi(K)$, and $\varkappa$ is the maximal congruence on $(\mathbf{N}, +)$ of this kind.

Note further that only in the mixed cases $v(S)<\infty$, $g(S)=0$ and $v(S)=\infty$, $g(S) \neq 0$, the characteristic $\chi(S)$ defined above is not contained in $\chi(K)=L' \cup \{(\infty, 0)\}$. Leaving certain information out of consideration, one could decide to define a characteristic $\bar{\chi}(S)$ such that $\bar{\chi}(S)=(\infty, 0)$ also holds in these two cases, or equivalently, to define $\bar{\chi}(S)$ by the intersection $\cap \{\chi(a)|a \in S\}$ taken in the sublattice $\chi(K)$ of L. But this would not simplify things considerably, and so we stay here [2]) with the more general concept defined above. Clearly, corresponding results for the other one may be obtained by identification of all $(v, g) \in L \setminus \chi(K)$ with $(\infty, 0)$.

Proposition 5. *Let E be a set of generators of the semigroup $S=(S, +)$ or of the semiring $S=(S, +, \cdot)$, of course using both operations in the latter case, and assume that* (1″) *holds for all $a, b \in S$. Then one can replace* (10) *by*

$$(10') \qquad \chi(S) = (\sup\{v(a)|a \in E\}, \operatorname{lcm}\{g(a)|a \in E\}) = \cap\{\chi(a)|a \in E\}.$$

In particular, $|E|<\infty$ implies $\chi(S) \in \chi(K)$ if (1″) *holds for all $a, b \in S$.*

The proof is immediate by Lemma 4. The next statements on semigroups clearly apply to semirings, too.

Proposition 6. a) *Let $(H, +)$ be a subsemigroup or an epimorphic image of a semigroup $(S, +)$. Then we have $\chi(H) \supseteq \chi(S)$.*

b) *If $(S, +)$ is cancellative, we have $v(a) \in \{0, \infty\}$ for all $a \in S$, hence $v(S) \in \{0, \infty\}$. If $(S, +)$ is a group, $g(S)$ is the usual characteristic of S as defined in* § 1.

The proof is straightforward. Note that even a commutative semigroup $(S, +)$ with $\varkappa(S)=(0, p)$, p a prime, need not be cancellative. Any direct sum in the semigroup-theoretical sence (cf. [1], II, § 9.4) of two or more cyclic groups of order p provides a counter example.

An element s of a semigroup $(S, \cdot)$ is called *weakly cancellable* in $(S, \cdot)$ if $sx=sy$ and $xs=ys$ for $x, y \in S$ imply $x=y$ (cf. [3], I. 2).

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[2]) Some announcements given in [2], concerning the characteristic of semirings S, refer directly to the intersection $\varkappa(S)=\cap\{\varkappa(a)|a \in S\} \in K$, hence to a concept which clearly corresponds to $\bar{\chi}(S)$ above.

Proposition 7. *Let* $(S, +, \cdot)$ *be a semiring.*

a) *If* $s \in S$ *is weakly cancellable in* $(S, \cdot)$, *then* $\chi(s) \subseteqq \chi(a)$ *holds for all* $a \in S$. *Hence all elements* $s, s' \in S$ *which are weakly cancellable in* $(S, \cdot)$ *have the same characteristic, and* $\chi(s) = \chi(S) \in \chi(K)$ *holds if such an element* s *exists.*

b) *If* $(S, +)$ *has a neutral element, called the zero* o *of* $(S, +, \cdot)$, *and if* o *is weakly cancellable in* $(S, \cdot)$, *then* $\chi(S) = (0, 1)$ *or, equivalently,* $(S, +)$ *is idempotent.*

Proof. By assumption on s, $n(as) = m(as)$ and $n(sa) = m(sa)$ together imply $na = ma$, from which $\chi(as) \cap \chi(sa) \subseteqq \chi(a)$ follows. Using (9′), we obtain $\chi(s) \subseteqq \subseteqq \chi(as) \cap \chi(sa) \subseteqq \chi(a)$ for all $a \in S$, i.e. $\chi(s) \subseteqq \chi(a)$. This implies the other statements in a) and also b), since (0, 1) is the characteristic of idempotent elements in $(S, +)$, and the greatest element of L.

An example of a semiring S such that S has a zero o which is even cancellable in $(S, \cdot)$, is given in [8]. Moreover, a semiring S is called *multiplicatively cancellative*, briefly *mult. can.*, if each $a \neq o$ of S (meaning each $a \in S$ if there is no zero o of S) is cancellable in $(S, \cdot)$. Note that if S has a zero o and $|S| \geqq 2$, then either o is also cancellable in $(S, \cdot)$, or o is annihilating (i.e. $ao = oa = o$ for all $a \in S$) and $(S \setminus \{o\}, \cdot)$ is a cancellative subsemigroup of $(S, \cdot)$. A mult. can. semiring S does not have (proper) zero divisors, whereas the converse does not hold in general (cf. [8], [10]).

We introduce a wider class of semirings: A semiring S, containing a zero o or not, is called *weakly mult. can.*, if each $a \neq o$ of S is weakly cancellable in $(S, \cdot)$.

Theorem 8. a) *Let* S *be a weakly mult. can. semiring. Then all elements* $s \neq o$ *of* S *have the same characteristic* $\chi(s)$, *coinciding with* $\chi(S)$, *which is either* (0, 1), *or* $(0, p)$ *for some prime* p, *or* $(\infty, 0)$.

b) *Let* S *be a weakly mult. can. semiring with zero* o. *Then either* $(S, +)$ *is idempotent, or* o *is the only idempotent of* $(S, +)$ *and annihilating. Clearly, the first case corresponds to* $\chi(S) = (0, 1)$, *the second one to* $(0, p)$ *or* $(\infty, 0)$.

c) *Let* S *be a weakly mult. can. semiring,* $|S| \geqq 2$, *which contains at most one idempotent and no element of infinite order of* $(S, +)$. *Then* S *is a ring, whose additive group* $(S, +)$ *is the direct sum of cyclic groups of order* p.

Proof. a) By Prop. 7, a) we obtain $\chi(S) = \chi(s)$ for all $s \neq o$ of S. If $\chi(S) \neq \neq (\infty, 0)$, we have $v(s) = v(S) < \infty$ and $g(s) = g(S) \neq 0$ for each $s \neq o$; hence for each $h \in \mathbf{N}$, $1 \leqq h \leqq v(s) + g(s)$, either $hs = o$ holds or (8) implies

$$v(hs) = \left[\frac{v(s)}{h}\right] = v(s) \quad \text{and} \quad g(hs) = \frac{g(s)}{(g(s), h)^*} = g(s).$$

This proves $v(s) = 0$ and either $g(s) = 1$ or $g(s)$ prime.

b) If $(S, +)$ is not idempotent, we have $(0, 1)=\chi(o)\neq\chi(s)=\chi(S)$ for all $s\neq o$ of S by a). Hence o is the only idempotent of $(S, +)$, and $\chi(ao)\supseteqq\chi(o)$, $\chi(oa)\supseteqq\chi(o)$ by (9′) implies $ao=oa=o$ for all $a\in S$.

c) Using a) and the assumptions, we have $\chi(S)=(0,p)$. Hence each $s\neq o$ generates a group $\langle s\rangle$ of order p, whose zero o_s has characteristic $(0, 1)$. Again by a), there is at most one element in S whose characteristic differs from $(0,p)$, hence all o_s collapse to the zero o of $(S, +)$, and $(S, +)$ is a group with $\varkappa(S)=(0,p)$. By well known facts on groups or modules, $(S, +)$ is the direct sum of cyclic groups of order p, if it is commutative. Suppose the contrary, then $(S, +, \cdot)$ would be a ring with non commutative addition, which always contains a two-sided annihilator ideal different from $\{o\}$ (cf. [9]), contradicting that S is weakly mult. can. .

An example of a weakly mult. can. semiring S such that each element of $(S, \cdot)$ is neither left nor right cancellable is given by the tables

$+$	a_1	a_2	b_1	b_2
a_1	a_1	a_2	b_1	b_2
a_2	a_2	a_2	b_2	b_2
b_1	b_1	b_2	b_1	b_2
b_2	b_2	b_2	b_2	b_2

$\cdot$	a_1	a_2	b_1	b_2
a_1	a_1	a_2	a_1	a_2
a_2	a_1	a_2	a_1	a_2
b_1	b_1	b_2	b_1	b_2
b_2	b_1	b_2	b_1	b_2.

Note that S is the direct composition (cf. the definition given in the proof of Prop.10) of two semirings $\{a, b\}$ and $\{1, 2\}$ with operations obvious from these tables, and that the zero a_1 of S is weakly cancellable in $(S, \cdot)$, too. On the other hand, one easily proves that a ring S is weakly mult. can. iff it is mult. can.. Further, using the other parts of Thm. 8, part c) may be reformulated as follows: A weakly mult. can. semiring S such that $\chi(S)=(0,p)$ is a ring. The corresponding statements in the other cases, $\chi(S)=(0, 1)$ or $\chi(S)=(\infty, 0)$, are far away from being true, even for semirings S which are mult. can.. In fact, such a semiring need not have a zero (e.g. $S=(\mathbf{N}, +, \cdot)$ for $\chi(S)=(\infty, 0)$, for $\chi(S)=(0, 1)$ see [8]).

Concluding this section, we mention a concept of characteristic introduced in [5], § 23 for semigroups, and similarly in [6], § 3 for semirings. Working only with the order of elements of $(S, +)$, the characteristic of S is defined to be

0 iff $o(a)=\infty$ for all $a\neq o$ of S,

n iff there is a (minimal) $n\in\mathbf{N}$ such that $o(a)|n$ for all $a\in S$, and

∞ for all other cases.

Applied to a weakly mult. can. semiring S, by Thm. 8a) S has either characteristic $n=1$, or $n=p$, or 0 in this sense, a result stated in [6], Satz 1 for mult. left (or right) can. semirings.

§ 4. Semirings embeddable into semirings with (one sided) identity

A semiring S need neither be embeddable into a semiring with right identity nor into one with left identity. There are also semirings S for which only one kind of these extensions exists, and semirings S which have both, extensions with right and no left identity as well as extensions with left and no right identity. The latter case is equivalent to S being embeddable into a semiring with identity. For corresponding examples as well as for necessary and sufficient conditions we refer to [2]. The following statements deal with the characteristic of a semiring S and of its extensions with a one-sided (say, right) or two-sided identity; assertions or concepts needed from [2] will be given.

Theorem 9. a) *Let S be a semiring embeddable into one with right identity. Then for each semiring T_r with a right identity e_r, containing S as a subsemiring, the characteristic $\lambda=\chi(T_r)$ satisfies $\chi(S)\supseteqq\chi(T_r)$ and $\lambda=\chi(T_r)=\chi(e_r)\in\chi(K)$, hence*

$$\chi(S) \supseteqq \lambda \supseteqq (\infty, 0), \quad \lambda\in\chi(K). \tag{11}$$

In particular, if $\chi(S)\notin L'=\mathbf{N}_0\times\mathbf{N}$, the characteristic of T_r is uniquely determined by $\lambda=\chi(T_r)=(\infty, 0)$.

Conversely, let S be a semiring as above and let $\lambda\in L$ satisfy (11). *Then there exists at least one such extension T_r of S satisfying $\chi(T_r)=\lambda$.*

b) *Let S be a semiring embeddable into one with identity. Then the same statements hold for the characteristic $\lambda=\chi(T)$ of each semiring T with identity which contains S, and for each $\lambda\in L$ satisfying* (11).

Remark. By the converse statements, *for a semiring S embeddable into one with (right) identity, there exists such an extension T_r or T of the same characteristic $\chi(T_r)=\chi(S)$ or $\chi(T)=\chi(S)$ if and only if $\chi(S)\subseteqq\chi(K)$ holds. This is always the case if S is finitely generated* (by Prop. 5, since (1″) holds for all $a, b\in S$ if S is embeddable as assumed above), *and also if $(S, \cdot)$ contains a weakly cancellable element* (by Prop. 7. a). But there are semirings S, embeddable as above, such that $\chi(S)\notin\chi(K)$, hence $\chi(S)\supset\chi(T_r)$ or $\chi(S)\supset\chi(T)$ for all extensions under discussion. For an example, let S be the zero ring on the Prüfer group $(S, +)$ (cf. [5], § 23); then S is even embeddable into a ring with identity, but $\chi(S)=(0, 0)\notin\chi(K)$.

Proof of Thm. 9. The first part of a) follows directly by Prop. 6a) and by Prop. 7a), and likewise the corresponding part of b). Moreover, both converse statements of a) and b) become trivial if $\chi(S)\notin L'$, since then only $\lambda=(\infty, 0)$ satisfies (11). Thus it remains to prove these statements with the assumption $\chi(S)\in L'$.

If S is embeddable into a semiring with right identity, in the proof of Thm. 1, [2] we have constructed a semiring T_r with the following properties: S is a subsemiring, T_r contains the identity mapping $\iota=\iota_S$ of S, $\iota+\iota$ is defined by $a(\iota+\iota)=a+a$ for all $a\in S$, and $e_r=\iota_S$ is the right identity of T_r. Hence for each $(n, m)\in\mathbf{N}\times\mathbf{N}$ we have

(12) $$na = ma \text{ for all } a\in S \Leftrightarrow ne_r = me_r.$$

Since $\chi(S)=\bigcap\chi(a)\in L'$ by assumption, (12) yields $\chi(S)=\chi(e_r)=\chi(T_r)$. In the two-sided case, the semiring T constructed in the proof of Thm. 2, [2] in a similar way, satisfies (12) with respect to the identity e of T, hence $\chi(S)=\chi(e)=\chi(T)$. Thus in both cases there are extensions T_r resp. T of S whose characteristic equals $\chi(S)$, and the proof will be complete applying the following

Proposition 10. a) *Let T_r be a semiring with a right identity e_r. Then for each $\lambda\in\chi(K)$ such that $\chi(T_r)\supseteqq\lambda\supseteqq(\infty, 0)$, there exists an extension T_r' of T_r with a right identity e_r' of characteristic $\chi(T_r')=\lambda$.*

b) *The corresponding statement holds in the case of two-sided identities.*

Proof. It will be enough to deal with a). Since $\lambda=(v, g)$ corresponds to $\varkappa=\chi^{-1}(\lambda)\in K$, which is also a congruence on the semiring $(\mathbf{N}, +, \cdot)$, the semiring $\mathbf{N}'=\mathbf{N}/\varkappa=\{1', 2', \dots\}$ satisfies $\chi(\mathbf{N}')=\chi(1')=\lambda$ and $v=v(1')$, $g=g(1')$. If $\lambda=(0, g)$, $\mathbf{N}'\cong\mathbf{Z}/(g)$ has $g'=o'$ as annihilating zero, and we write $\mathbf{N}'=\mathbf{N}_0'$. In each other case (including $\mathbf{N}'=\mathbf{N}$ for $\lambda=(\infty, 0)$) we adjoin an annihilating zero o' to $\mathbf{N}'$ and obtain a semiring $\mathbf{N}_0'=\{o', 1', 2', \dots\}$, sharing with $\mathbf{N}'$ all properties stated above.

Now we use the semiring T_r, and define operations on the set $T_r'=T_r\times\mathbf{N}_0'$ by

(13) $$(t_1, n_1')+(t_2, n_2') = (t_1+t_2, n_1'+n_2').$$

This construction, called the *direct composition of T_r and $\mathbf{N}_0'$*, may clearly be applied to any two (or more) semirings, yielding a semiring again. In our case, T_r' contains an isomorphic copy of T_r by $t\to(t, o')$; hence we can consider T_r' as an extension of T_r. Moreover, $e_r'=(e_r, 1')$ is a right identity of T_r'. Since $\chi(T_r)=\chi(e_r)\supseteqq\lambda=$ $=\chi(1')$, we obtain $\chi(e_r')=\chi(1')$, again by (13), i.e. $\chi(T_r')=\lambda$ as we were to prove.

Now let S be a subsemiring of a semiring T with identity e_T. We call T an *identity extension of S*, and write $T=[S, e_T]$, if T is generated by S and e_T. In this case, each element $t\in T$ equals a sum

$$t = \sum_{i=1}^{n} t_i \quad \text{with} \quad t_i\in S\cup\langle e_T\rangle, \quad n\in\mathbf{N}.$$

Note that the addition in T is not assumed to be commutative. Clearly, each extension T' of S with an identity e' contains the identity extension $[S, e']\subseteqq T'$. Moreover, by Thm. 4 of [2], there exists a *universal identity extension* $U=[S, e_U]$ *of* S, defined by the property that for each $T=[S, e_T]$ there is an epimorphism

(14) $$\psi\colon U\to T \quad \text{such that} \quad a\to a \quad \text{for all} \quad a\in S, \quad \text{and} \quad e_U\to e_T.$$

By (14), U is unique up to isomorphisms (relative w.r.t. S). Conversely, each congruence τ on $(U, +, \cdot)$ such that $\tau|S\times S$ is the equality on S, corresponds to an identity extension $T\cong U/\tau$ of S. Applying Zorn's Lemma to the set Θ of all these congruences (in fact a complete lattice), there is at least one maximal $\tau_0\in\Theta$, hence an identity extension $T_0\cong U/\tau_0$ of S with the property: Each epimorphism (relative w.r.t. S) of T_0 onto an identity extension is an isomorphism. We call such a T_0 (being "minimal with respect to epimorphisms") a *strict identity extension* of S. Note that T_0 also has no proper subsemirings containing S and any identity, and that there are semirings S with non isomorphic strict identity extensions (cf. [2]).

Theorem 11. *Let S be a semiring embeddable into one with identity.*

a) *The universal identity extension $U=[S, e_U]$ has characteristic $\chi(U)=(\infty, 0)$, regardless of the characteristic $\chi(S)$ of S.*

b) *For each $\lambda\in\chi(K)$ such that $\chi(S)\supseteqq\lambda\supseteqq(\infty, 0)$ (i.e. (11)), there exists an identity extension U_λ of S of characteristic $\chi(U_\lambda)=\lambda$, which is universal for all identity extensions T' of S of characteristic $\chi(T')$ satisfying $\chi(S)\supseteqq\chi(T')\supseteqq\lambda$. Clearly, U_λ is uniquely determined by S and λ, up to isomorphisms relative w.r.t. S.*

c) *If $\chi(S)\in\chi(K)$, then there exists at least one strict identity extension T_0 of S of characteristic $\chi(T_0)=\chi(S)$.*

Proof. a) By Thm. 9, there is a semiring T containing S and an identity e_T such that $\chi(T)=(\infty, 0)$. The identity extension $[S, e_T]\subseteqq T$ is an epimorphic image of U, ψ: $U\to[S, e_T]$, hence $(\infty, 0)=\chi([S, e_T])\supseteqq\chi(U)$ by Prop. 6a), proving $\chi(U)=(\infty, 0)$.

b) As just stated, the subsemiring $\mathbf{N}e_U=\langle e_U\rangle$ of $U=[S, e_U]$ is isomorphic to $(\mathbf{N}, +, \cdot)$, and may be identified with the latter. As a consequence of Thm. 9, for each $\lambda\in\chi(K)$ satisfying (11), there is an identity extension T of S with $\chi(T)=\lambda$. Hence for the congruence $\tau\in\Theta$ on $(U, +, \cdot)$ such that $T\cong U/\tau$, the restriction $\tau|\mathbf{N}\times\mathbf{N}$ coincides with $\chi^{-1}(\lambda)=\varkappa\in K$. Considering $\varkappa$ as a relation on U, let $\Theta_\varkappa$ be the set of all $\tau'\in\Theta$ such that $\tau'\supseteqq\varkappa$. Since $\tau\in\Theta_\varkappa$, the intersection $\tau_\varkappa=\bigcap\{\tau'\in\Theta_\varkappa\}\in\Theta_\varkappa$ satisfies $\tau_\varkappa|\mathbf{N}\times\mathbf{N}=\varkappa$, like τ. Hence $\tau_\varkappa$ provides an identity extension $U_\lambda\cong U/\tau_\varkappa$ of S which has characteristic $\chi(U_\lambda)=\chi(\varkappa)=\lambda$. Moreover, for each identity extension T' of S with $\chi(T')=\lambda'\supseteqq\lambda$, the corresponding congruence $\tau'\in\Theta$ such that $T'\cong U/\tau'$ satisfies $\tau'\in\Theta_\varkappa$, since $\tau'|\mathbf{N}\times\mathbf{N}=\chi^{-1}(\lambda')\supseteqq\varkappa$. Thus by $\tau_\varkappa\subseteqq\tau'$ there is an epimorphism of U_λ onto T', relative w.r.t. S, hence respecting identities.

c) If $\chi(S)\in\chi(K)$, then there is an identity extension T of S with $\chi(T)=$ $=\chi(S)$ by Thm. 9 (or, likewise, $U_{\chi(S)}$ of b) just proved). If T is not strict, there exists an epimorphism of T onto a strict identity extension T_0 of S. Using (11) and Prop. 6a) we obtain $\chi(S)\supseteqq\chi(T_0)\supseteqq\chi(T)=\chi(S)$, as we have to show.

Each semiring S has a unique maximal epimorphic image S^* of S which is embeddable into a semiring with identity, and φ^*: $S\to S^*$ is universal with respect to this property (cf. [2]). Hence the universal identity extension U^* of S^*, together with φ^*: $S \to U^*$, satisfies that each homomorphism φ: $S \to T''$ of S into any semiring T'' with identity has a (unique) decomposition

$$\varphi: S \xrightarrow{\varphi^*} U^* \xrightarrow{\psi} T'',$$

i.e. U^* is the *universal semiring with identity of S* as introduced first in [4]. Hence the results of this section, applied to S^* instead of S, provide also information on an arbitrary semiring S. For instance: *The universal semiring with identity U^* of S has characteristic* $\chi(U^*)=(\infty, 0)$.

Finally we note: *A mult. can. semiring S is always embeddable into one with identity, and has a unique strict identity extension T_0. It is the only identity extension of S being again mult. can.* ([2], Thm. 12; for semirings with commutative addition cf. [6], Satz 2). Moreover, by Thm. 11c), *the characteristic $\chi(T_0)$ of T_0 coincides with $\chi(S)$.* For similar results on identity extensions of rings compare [7].

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