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On the structure of standard regular semigroups

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We give a structure theorem for a class of regular semigroups and determine the smallest inverse semigroup congruence for this class of semigroups. Let S be a regular semigroup, let T denote the union of the maximal subgroups of S, and let $E(T)$ denote the set of idempotents of T. Assume T is a semigroup (equivalently T is a semilattice Y of completely simple semigroups $(T_y\colon y\in Y)$). If Y has a greatest element and $e, f, g\in E(T)$, $e\geqq f$, and $e\geqq g$ imply $fg=gf$, we term S a *standard regular semigroup*. The structure of S is given modulo standard inverse semigroups and standard completely regular semigroups by means of an explicit multiplication. In the case $|Y|=1$, our structure theorem reduces to the Rees theorem for completely simple semigroups. A structure theorem for standard completely regular semigroups is also given. The minimum inverse semigroup congruence on a standard regular semigroup is described.

Let us first state our structure theorem for standard regular semigroups. Let $(V, \circ)$ be a standard inverse semigroup with semilattice of idempotents Y, and let $(T, *)$ be a standard semilattice Y of completely simple semigroups $(T_y\colon y\in Y)$ with $y=y*y\in T_y$. Suppose $T_y\cap V=H_y$ for $y\in Y$ and $(H_y, \circ)$ $[(H_y, *)]$ is the maximal subgroup of $(V, \circ)$ $[(T, *)]$ containing y and assume $a*b=a\circ b$ for $a, b\in \bigcup(H_y\colon y\in Y)$. Let I_y $[J_y]$ denote the maximal left zero [right zero] subsemigroup of T_y containing y. Let (Y, T, V) denote $\{(i, b, j)\colon b\in V,\ i\in I_{b\circ b^{-1}},\ j\in J_{b^{-1}\circ b}\}$ under the multiplication $(i, b, j)(r, c, s)=(i*u,\ b\circ(j*r)\circ c,\ v*s)$ where $u\in I_{(b\circ c)\circ(b\circ c)^{-1}}$ and $v\in J_{(b\circ c)^{-1}\circ(b\circ c)}$. We show (Theorem 1.9) that (Y, T, V) is a standard regular semigroup and, conversely, every standard regular semigroup is isomorphic to some (Y, T, V).

In [4, Theorem 3.14], we gave a different structure theorem for standard regular semigroups.

The structure of standard inverse semigroups is clarified by [4, Theorem 5.5].

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In Section 1, we prove our structure theorem for standard regular semigroups (Theorem 1.9) and give some specializations of this theorem (Remarks 1.21 and 1.22). In Section 2, we describe standard completely regular semigroups in terms of groups by means of a "Rees type" multiplication (Theorem 2.1). In Section 3, we give the following description of the minimum inverse semigroup congruence on a standard regular semigroup $S=(Y, T, V)$. Let N denote the collection of all finite products of elements of the form $a^{-1}\circ s\circ a$ where $a\in V$ and s or $s^{-1}\in(\cup(J_y\colon y\in Y))*(\cup(I_y\colon y\in Y))$. Let $N_y=N\cap H_y$ for $y\in Y$. Let

$$\delta_N=\{((i,a,j),(p,b,q))\in S\times S\colon N_y\circ a=N_y\circ b \quad \text{where} \quad y=a\circ a^{-1}=b\circ b^{-1}\}.$$

Then, δ_N is the minimum inverse semigroup congruence on S.

We will use the definitions and notation of CLIFFORD and PRESTON [1, 2] unless otherwise specified. The terms mainly used are: Green's relations ($\mathcal{R}$, $\mathcal{L}$, $\mathcal{H}$, and $\mathcal{D}$), $\mathcal{R}$-class, regular semigroup, bisimple semigroup, inverses, inverse semigroup, left (right) zero semigroup, right group, idempotent, natural partial order of idempotents, semilattice, completely simple semigroup, semilattice of completely simple semigroups [groups, left (right) zero semigroups], maximal subgroup, congruence, and kernel of a homomorphism.

A semigroup is termed *completely regular* if it is a union of its subgroups. If X is a semigroup, $E(X)$ will denote the set of idempotents of X. A regular semigroup X is termed *locally inverse* if $e,f,g\in E(X)$, $e\geqq f$ and $e\geqq g$ imply $fg=gf$. (See [4] for an explanation of terminology.) A congruence ϱ on a semigroup X such that X/ϱ is an inverse semigroup is termed an inverse semigroup congruence on X. "Structure homomorphisms" are defined and discussed in [4, Section 1].

1. Standard regular semigroups. In this section, we establish our new structure theorem for standard regular semigroups (Theorem 1.9).

Let S be a standard regular semigroup and let T denote the union of the maximal subgroups of S. Hence, T is a semilattice Y' of completely simple semigroups $(T_y\colon y\in Y')$ [1, Theorem 4.6] where Y' has a greatest element y_0. Let $\{\zeta_{y,z}\colon y,z\in Y\}$ denote the set of structure homomorphisms of T [4, Section 1]. Let $E_y=E(T_y)$. Select and fix $e_{y_0}\in E(T_{y_0})$. For each $y\in Y'$, define $e_y=e_{y_0}\zeta_{y_0,y}$. Let $S_0=e_{y_0}Se_{y_0}$. Let I_{e_y} [J_{e_y}] denote the set of idempotents of the $\mathcal{L}$-class [$\mathcal{R}$-class] of T_y containing e_y. Let H_{e_y} denote the $\mathcal{H}$-class of S containing e_y.

Lemma 1.1. ([4, Lemma 2.2]) *$y\to e_y$ defines an isomorphism of Y' onto $E(S_0)$.*

Lemma 1.2. *$H_{e_y}=T_y\cap S_0$ for $y\in Y'$.*

Proof. Utilize [4, Theorem 2.3].

Let $E(S_0)=Y$ and let $\mathcal{I}(a)$ denote the collection of inverses of a.

Lemma 1.3. (a) *T is a semilattice Y of completely simple semigroups* $(T_y\colon y\in Y)$ *where* $y^2=y\in T_y$. (b) $I=\bigcup(I_y\colon y\in Y)$ $[J=\bigcup(J_y\colon y\in Y)]$ *is the semilattice Y of left zero semigroups [right zero semigroups]* $(I_y\colon y\in Y)$ $[(J_y\colon y\in Y)]$.

Proof. (a) Let $T_{e_y}=T_y(y\in Y')$. Then, using Lemma 1.1, $T_{e_y}T_{e_z}=T_yT_z\subseteq T_{yz}==T_{e_{yz}}=T_{e_ye_z}$. (b) Utilize the proof of [4, Lemma 2.4] and its dual.

Lemma 1.4. *Every element of S may be uniquely expressed in the form* $x=gbh$ *where* $b\in S_0$, $g\in I_{bb^{-1}}$, *and* $h\in J_{b^{-1}b}$.

Proof. Let $a\in S$. Hence, $a\in R_e\cap L_f$ for some $e,f\in E(S)$. Suppose $e\in T_y$ and $f\in T_z$ $(y,z\in Y')$. Let r_e $[l_f]$ denote the $\mathcal{R}$-class [$\mathcal{L}$-class] of T_y $[T_z]$ containing e $[f]$. Using [1, Theorem 2.51], $r_e\cap I_{e_y}\neq\square$ and $l_f\cap J_{e_z}\neq\square$. Let $g\in r_e\cap I_{e_y}$ and $h\in l_f\cap J_{e_z}$. Hence, $g(e_{y_0}ae_{y_0})h=(ge_{y_0})a(e_{y_0}h)=(ge_ye_{y_0})a(e_{y_0}e_zh)=gah=a$. By the proof of [1, Theorem 2.18], since $a\in R_g\cap L_h$, there exists a unique $a^{-1}\in R_h\cap L_g\cap\mathcal{I}(a)$ such that $aa^{-1}=g$ and $a^{-1}a=h$. Thus, $(e_{y_0}ae_{y_0})(e_{y_0}a^{-1}e_{y_0})(e_{y_0}ae_{y_0})=e_{y_0}ae_{y_0}ha^{-1}ge_{y_0}ae_{y_0}==e_{y_0}ae_{y_0}$, and similarly, $(e_{y_0}a^{-1}e_{y_0})(e_{y_0}ae_{y_0})(e_{y_0}a^{-1}e_{y_0})=e_{y_0}a^{-1}e_{y_0}$. Thus, if $b=e_{y_0}ae_{y_0}$, $b^{-1}=e_{y_0}a^{-1}e_{y_0}$. Hence, as above, $bb^{-1}=e_y$ and $b^{-1}b=e_z$. Hence, every element of S may be expressed in the form gbh where $b\in S_0$, $g\in I_{bb^{-1}}$, and $h\in J_{b^{-1}b}$. We next show $gbh\in R_g\cap L_h$. Since $gbhb^{-1}bb^{-1}=g$, $g\in gbhS$. Thus, since $gbh\in gS$, $gbh\in R_g$. Similarly, $gbh\in L_h$. We are now in a position to establish uniqueness. Let $x=gbh=wcz$ where $c\in S_0$, $w\in I_{cc^{-1}}$, and $z\in J_{c^{-1}c}$. Hence, $g\mathcal{R}x\mathcal{R}w$ and, similarly, $h\mathcal{L}z$. Since $gw=w$, $wg=g$, and S_0 is an inverse semigroup, using [1, Theorem 1.17], $cc^{-1}=bb^{-1}cc^{-1}=cc^{-1}bb^{-1}=bb^{-1}$. Thus, $g=w$. Similarly $b^{-1}b=c^{-1}c$ and $h=z$. Hence, $b=bb^{-1}bb^{-1}b=bb^{-1}gbhb^{-1}b=cc^{-1}wczc^{-1}c==cc^{-1}cc^{-1}c=c$. Q.E.D.

Using Lemma 1.2, H_{e_y} is the $\mathcal{H}$-class of S_0 $[T_{e_y}]$ containing e_y.

Lemma 1.5. *If* $i\in I_{e_y}$ *and* $j\in J_{e_z}$, $ji\in H_{e_ye_z}$.

Proof. Apply the proof of [4, Lemma 2.11].

Lemma 1.6. *Let* $H=\bigcup(H_{e_y}\colon y\in Y')$. *Then H is the semilattice Y of groups* $(H_y\colon y\in Y)$. *Hence, $E(H)$ is contained in the center of H (*i.e. $eh=he$ *for all* $e\in E(H)$ *and* $h\in H$*).*

Proof. Utilize [4, Proposition 1.9], Lemma 1.2, and [1, Lemma 4.8].

Lemma 1.7. *Let* $b,c\in S_0$, $j\in J_{b^{-1}b}$, *and* $p\in I_{cc^{-1}}$. *Then* $(b(jp)c)(b(jp)c)^{-1}==(bc)(bc)^{-1}$ *and* $(b(jp)c)^{-1}b(jp)c=(bc)^{-1}bc$.

Proof. Using Lemmas 1.5 and 1.6, $(b(jp)c)(b(jp)c)^{-1}=b(jp)cc^{-1}(jp)^{-1}b^{-1}==bcc^{-1}(jp)(jp)^{-1}b^{-1}=bcc^{-1}b^{-1}=(bc)(bc)^{-1}$ and, similarly, $(b(jp)c)^{-1}(b(jp)c)==(bc)^{-1}bc$.

For $a, b \in S_0$, define $a \circ b = ab$. For $a, b \in T$, define $a * b = ab$.

Lemma 1.8. *Let* $b, c \in S_0$, $i \in I_{b\circ b^{-1}}$, $j \in J_{b^{-1}\circ b}$, $p \in I_{c\circ c^{-1}}$, *and* $q \in J_{c^{-1}\circ c}$. *Then* $(ibj)(pcq) = (i * x)(b \circ (j * p) \circ c)(y * q)$ *where* $x \in I_{(b\circ c)\circ(b\circ c)^{-1}}$ *and* $y \in J_{(b\circ c)^{-1}\circ(b\circ c)}$. *Hence,* $S \cong \{(i, b, j) \colon b \in S_0,\ i \in I_{b\circ b^{-1}},\ j \in J_{b^{-1}\circ b}\}$ *under the multiplication* $(i, b, j)(p, c, q) = (i * x, b \circ (j * p) \circ c, y * q)$.

Proof. Utilizing Lemma 1.7, $(ibj)(pcq) = i(b \circ (j * p) \circ c)q = (i * ((b \circ c) \circ (b \circ c)^{-1}))(b \circ (j * p) \circ c)(((b \circ c)^{-1} \circ (b \circ c)) * q)$. Let $b \circ b^{-1} = e_r$ and $(b \circ c) \circ (b \circ c)^{-1} = e_w$. Thus, $i * ((b \circ c) \circ (b \circ c)^{-1}) = i\zeta_{r,w} = i * x$ and, similarly, $((b \circ c)^{-1} \circ (b \circ c)) * q = y * q$. Hence, using Lemmas 1.4, 1.3, 1.5, 1.2, and 1.7 the last sentence of the lemma is established.

Theorem 1.9. *(Y, T, V) is a standard regular semigroup, and, conversely, every standard regular semigroup is isomorphic to some (Y, T, V).*

Proof. The converse is a consequence of Lemmas 1.1, 1.6, 1.3, 1.2, and 1.8. We next establish the direct part of Theorem 1.9. Let $S = (Y, V, T)$.

Lemma 1.10. *S is a groupoid.*

Proof. Let $(i, b, j), (r, c, s) \in S$. Let $\{\zeta_{y,z} \colon y, z \in Y\}$ denote the set of structure homomorphisms of $(T, *)$. Suppose $y \geqq z$. Hence, $z = y * z = y\zeta_{y,z} * z = z * y = z * y\zeta_{y,z}$ or $z \leqq y\zeta_{y,z}$. Thus, $y\zeta_{y,z} = z$. Hence, $i\zeta_{b\circ b^{-1},(b\circ c)\circ(b\circ c)^{-1}} \mathscr{L} (b \circ c) \circ (b \circ c)^{-1}$, since $i \mathscr{L} b \circ b^{-1}$. Thus $i\zeta_{b\circ b^{-1},(b\circ c)\circ(b\circ c)^{-1}} \in I_{(b\circ c)\circ(b\circ c)^{-1}}$. Hence, $i * x = i\zeta_{b\circ b^{-1},(b\circ c)\circ(b\circ c)^{-1}}$ for $x \in I_{(b\circ c)\circ(b\circ c)^{-1}}$ and, similarly, $s\zeta_{c^{-1}\circ c,(b\circ c)^{-1}\circ b\circ c} \in J_{(b\circ c)^{-1}\circ b\circ c}$ and $y * s = s\zeta_{c^{-1}\circ c,(b\circ c)^{-1}\circ(b\circ c)}$ for $y \in J_{(b\circ c)^{-1}\circ(b\circ c)}$. Thus, $(i, b, j)(r, c, s)$ is independent of the choice of u and v. Furthermore, as in the proof of [2, Theorem 2.11], $j \in J_z$ and $i \in I_y$ implies $j * i \in H_{yz}$. Let $H = \bigcup (H_y \colon y \in Y)$. Then, Lemma 1.6 is valid for H. Thus, as in the proof of Lemma 1.7, $(b \circ (j * r) \circ c) \circ (b \circ (j * r) \circ c)^{-1} = (b \circ c) \circ (b \circ c)^{-1}$, and, similarly, $(b \circ (j * r) \circ c)^{-1} \circ (b \circ (j * r) \circ c) = (b \circ c)^{-1} \circ (b \circ c)$.

Lemma 1.11. *S obeys the associative law.*

Proof. Let $\alpha = (i, b, j)$, $\beta = (r, c, s)$, $\gamma = (w, d, z)$ be elements of S. Let $\alpha_1 = i$, $\alpha_2 = b$, and $\alpha_3 = j$. Then, $((\alpha\beta)\gamma)_1 = i\zeta_{b\circ b^{-1},(b\circ c\circ d)\circ(b\circ c\circ d)^{-1}} = (\alpha(\beta\gamma))_1$, and, similarly, $((\alpha\beta)\gamma)_3 = (\alpha(\beta\gamma))_3$. Furthermore, $((\alpha\beta)\gamma)_2 = b \circ (j * r) \circ c \circ ((v * s) * w) \circ d$ where $v \in J_{(b\circ c)^{-1}(b\circ c)}$. However, $(v * s) * w = (((b \circ c)^{-1} \circ (b \circ c)) * s) * w = (c^{-1} \circ b^{-1} \circ b \circ c) \circ (s * w)$. Hence, $((\alpha\beta)\gamma)_2 = b \circ (j * r) \circ c \circ c^{-1} \circ b^{-1} \circ b \circ c \circ (s * w) \circ d = b \circ (j * r) \circ c \circ (s * w) \circ d$. Similarly, $(\alpha(\beta\gamma))_2 = b \circ (j * r) \circ c \circ (s * w) \circ d$. Q.E.D.

Lemma 1.12. $(b^{-1} \circ b, b^{-1}, b \circ b^{-1}) \in \mathscr{I}((i, b, j))$. *Hence, S is a regular semigroup.*

Proof. This lemma follows from a straightforward calculation.

Lemma 1.13. (a) $(i, b, j)\mathscr{R}(p, z, q)$ *if and only if* $i=p$. (b) $(i, b, j)\mathscr{L}(p, z, q)$ *if and only if* $j=q$. (c) $(i, b, j)\mathscr{H}(p, z, q)$ *if and only if* $i=p$ *and* $j=q$.

Proof. (a) First assume $i=p$ (hence, $b\circ b^{-1}=z\circ z^{-1}$). Thus, $(i, b, j)(b^{-1}\circ b, b^{-1}\circ z, q)=(i, z, q)$ and $(i, z, q)(z^{-1}\circ z, z^{-1}\circ b, j)=(i, b, j)$. If $(i, b, j)\mathscr{R}(p, z, q)$, there exist $x, y\in I$ such that $i*x=p$ and $p*y=i$. Thus, $i*p=p$ and $p*i=i$. Hence, $b\circ b^{-1}=z\circ z^{-1}$ and $i=p$.

Lemma 1.14. $(i, b, j)\mathscr{D}(p, z, w)$ *if and only if* $b\mathscr{D}z(\in V)$. *Hence, S is bisimple if and only if V is bisimple.*

Proof. Suppose $b\mathscr{D}z$ (in V). Hence, there exists $x\in V$ such that $b\circ b^{-1}=x\circ x^{-1}$ and $x^{-1}\circ x=z^{-1}\circ z$. Thus, $(i, b, j)\mathscr{R}(i, x, w)\mathscr{L}(p, z, w)$. Conversely, suppose $(i, b, j)\mathscr{D}(p, z, w)$. Hence, $(i, b, j)\mathscr{R}(u, x, v)\mathscr{L}(p, z, q)$, say. Thus, $b\circ b^{-1}=x\circ x^{-1}$ and $x^{-1}\circ x=z^{-1}\circ z$ or $b\mathscr{D}z$.

Lemma 1.15. $E(S)=\{(i, b, j)\colon j*i=b^{-1},\ i\in I_y, j\in J_y, y\in Y\}$.

Proof. Suppose $(i, b, j)(i, b, j)=(i, b, j)$. Hence, $b\circ(j*i)\circ b=b$. Thus, $(b^{-1}\circ b)\circ(j*i)\circ(b\circ b^{-1})=b^{-1}$. Hence, $b^{-1}\in H$ and $b\circ b^{-1}=b^{-1}\circ b$. Hence, $j*i\in H_{b\circ b^{-1}}$ and $j*i=b^{-1}$. Conversely, $(i, (j*i)^{-1}, j)(i, (j*i)^{-1}, j)=(i*((j*i)^{-1}\circ (j*i)), (j*i)^{-1}, ((j*i)\circ(j*i)^{-1})*j)=(i, (j*i)^{-1}, j)$.

Lemma 1.16. $T'=\{(i, b, j)\colon b\in H_y, i\in I_y, j\in J_y, y\in Y\}$ *is the union of the maximal subgroups of S.*

Proof. Let T' denote the union of the maximal subgroups of S. Hence, $(i, b, j)\in T'$ if and only if $(i, b, j)\mathscr{H}(p, c, q)\in E(S)$. Suppose $(i, b, j)\mathscr{H}(p, c, q)\in E(S)$. Using Lemmas 1.13 and 1.15, $c=(q*p)^{-1}\in H_y$, say, $i\in I_y, j\in J_y$, and $b\in H_y$. Suppose $i\in I_y$, $b\in H_y$, and $j\in J_y$. Hence, $(i, b, j)\mathscr{H}(i, (j*i)^{-1}, j)\in E(S)$. Q.E.D.

Lemma 1.17. *Let* $T'_y=\{(i, g, j)\colon g\in H_y, i\in I_y, j\in J_y\}$. *Then* T' *is the semilattice* Y *of completely simple semigroups* $(T'_y\colon y\in Y)$.

Proof. Let $(i, g, j), (p, h, q)\in T'_y$. Hence, $(i, g, j)(p, h, q)=(i, g\circ(j*p)\circ h, q)$. Hence, T'_y is completely simple. Let $(i, g, j)\in T'_y$ and $(p, h, q)\in T'_z$. Hence, $(i, g, j)(p, h, q)=(i*(y\circ z), g\circ(j*p)\circ h, (y\circ z)*q)\in T'_{yz}$.

Lemma 1.18. *Every element of* T_y *may be uniquely expressed in the form* $x=i*g*j$ *where* $i\in I_y$, $g\in H_y$, *and* $j\in J_y$.

Proof. Suppose $T_y=\mathscr{M}(G; M, K; P)$ (notation of [1]). Let $e_y=(p_{11}^{-1})_{11}$. Hence, $I_y=\{(p_{1i}^{-1})_{i1}\colon i\in M\}$, $J_y=\{(p_{j1}^{-1})_{1j}\colon j\in K\}$, and $H_y=\{(g)_{11}\colon g\in G\}$. Hence, $(g)_{ij}=(p_{1i}^{-1})_{i1}(x)_{11}(p_{j1}^{-1})_{1j}$ where $x=p_{11}^{-1}p_{1i}gp_{j1}p_{11}^{-1}$.

Lemma 1.19. *Let* $(i*g*j)\theta=(i,g,j)$ $(i\in I_y,\ g\in H_y,\ j\in J_y)$. *Then* θ *defines an isomorphism of* T *onto* T'. *Hence,* T' *is locally inverse.*

Proof. If $j\in H_y$ and $p\in I_z$, $j*p\in H_{y\circ z}$. Let $i*g*j\in T_y$ and $p*h*q\in T_z$ $(i\in I_y,\ g\in H_y,\ j\in J_y,\ p\in I_z,\ h\in H_z,\ q\in J_z)$. Hence, $((i*g*j)*(p*h*q))\theta= =(i*g*(j*p)*h*q)\theta=(i*(y\circ z)*g*(j*p)*h*(y\circ z)*q)\theta=(i*(y\circ z), g\circ(j*p)\circ h, (y\circ z)*q)=(i,g,j)(p,h,q)=(i*g*j)\theta(p*h*q)\theta$.

Remark 1.20. The isomorphism $g\to(g\circ g^{-1}, g, g^{-1}\circ g)$ embeds $(V,\circ)$ into (Y,T,V). In fact, $\{(g\circ g^{-1}, g, g^{-1}\circ g)\colon g\in V\}=(y_0,y_0,y_0)(Y,T,V)(y_0,y_0,y_0)$ where y_0 is the greatest of Y.

The terms standard regular semigroup of type ωY, ωY inverse semigroup, locally inverse semigroup, rectangular group, orthodox semigroup, standard orthodox semigroup and standard $\mathscr{L}$-unipotent semigroup are defined in [4, pp. 540—542].

Remark 1.21. Using Lemmas 1.14—1.17, (Y,T,V) is a standard regular semigroup of type ωY if and only if V is an ωY inverse semigroup.

Remark 1.22. Let $(Y,T,V)_0$ denote (Y,T,V) with "completely simple semigroups" replaced by "rectangular groups" and "$b\circ(j*r)\circ c$" replaced by "$b\circ c$". Let $(Y,T,V)_{\mathscr{L}}$ denote $(Y,T,V)_0$ with "rectangular groups" replaced by "right groups". Then, $(Y,T,V)_0$ $[(Y,T,V)_{\mathscr{L}}]$ is a standard orthodox [standard $\mathscr{L}$-unipotent] semigroup, and conversely every standard orthodox [standard $\mathscr{L}$-unipotent] semigroup is isomorphic to some $(Y,T,V)_0$ $[(Y,T,V)_{\mathscr{L}}]$ (cf. [4, Theorems 5.1 and 5.3 and Remark 5.6]).

Remark 1.23. If we specialize Theorem 1.9 to orthodox semigroups, we obtain the specialization of Yamada's structure theorem for generalized inverse semigroups [6] to standard regular semigroups.

2. Standard completely regular semigroups. In this section, we give a structure theorem for standard completely regular semigroups (Theorem 2.1).

Let Y be a semilattice with greatest element. Let I $[J]$ be a locally inverse semilattice Y of left zero [right zero] semigroups $(I_\alpha\colon \alpha\in Y)$ $[(J_\alpha\colon \alpha\in Y)]$ with structure homomorphisms $(\xi_{\alpha,\beta})$ $[(\zeta_{\alpha,\beta})]$. Let G be a semilattice Y of groups $(G_\alpha\colon \alpha\in Y)$ with structure homomorphisms $\{\varphi_{\alpha,\beta}\}$. Let $(j,i)\to p_{j,i}$ be a function of $J\times I$ into G such that

(1) if $j\in J_\alpha$ and $i\in I_\alpha$, $p_{j,i}\in G_\alpha$;

(2) if $j\in J_\alpha$ and $i\in I_\beta$, $p_{j,i}=p_{j\zeta_{\alpha,\alpha\beta},\, i\xi_{\beta,\alpha\beta}}$;

(3) if $j\in J_\alpha$ and $i\in I_\alpha$ and $\alpha\geqq\beta$, $p_{j,i}\varphi_{\alpha,\beta}=p_{j\zeta_{\alpha,\beta},\, i\xi_{\alpha,\beta}}$.

Let $(Y, I, J, G, \zeta, \xi, \varphi)$ denote $\{(i, g, j)\colon i \in I_\alpha, g \in G_\alpha, j \in J_\alpha \text{ and } \alpha \in Y\}$ under the multiplication

(4) $(i, g, j)(w, h, z) = (iw, gp_{j,w}h, jz)$.

Theorem 2.1. *$(Y, I, J, G, \zeta, \xi, \varphi)$ is a standard completely regular semigroup, and, conversely, every such semigroup is isomorphic to some $(Y, I, J, G, \zeta, \xi, \varphi)$.*

Proof. Let S be a standard completely regular semigroup. Hence, S is a semilattice Y of completely simple semigroups $(S_\alpha\colon \alpha \in Y)$. Let α_0 denote the greatest element of Y. Let $\{\delta_{\alpha,\beta}\}$ denote the set of structure homomorphisms of S. Let $e_{\alpha_0} \in E(S_{\alpha_0})$ and define $e_\alpha = e_{\alpha_0}\delta_{\alpha_0,\alpha}$. Hence, $e_\alpha e_\beta = e_{\alpha\beta}$. Let I_α $[J_\alpha]$ denote the set of idempotents of the $\mathscr{L}$-class [$\mathscr{R}$-class] of S_α containing e_α. Hence, I_α $[J_\alpha]$ is a left zero [right zero] semigroup. As in the proof of Lemma 1.3, $I = \bigcup(I_\alpha\colon \alpha \in Y)$ $[J = \bigcup(J_\alpha\colon \alpha \in Y)]$ is a semilattice Y of left zero [right zero] semigroups $(I_\alpha\colon \alpha \in Y)$ $[(J_\alpha\colon \alpha \in Y)]$. Let $\zeta_{\alpha,\beta} = \delta_{\alpha,\beta}|J$ and $\xi_{\alpha,\beta} = \delta_{\alpha,\beta}|I$. Thus I and J are locally inverse by [4, Theorem 1.6]. Let G_α denote the $\mathscr{H}$-class of S_α containing e_α. Hence, using [4, Proposition 1.9], $G = \bigcup(G_\alpha\colon \alpha \in Y)$ is the semilattice Y of groups $(G_\alpha\colon \alpha \in Y)$ with structure homomorphisms $\varphi_{\alpha,\beta} = \delta_{\alpha,\beta}|G$. As in the proof of Lemma 1.18, every element of S may be uniquely expressed in the form $x = igj$ where $i \in I_\alpha$, $g \in G_\alpha$, and $j \in J_\alpha$. Let $j \in J_\alpha$ and $i \in I_\beta$. Hence, $ji = j\zeta_{\alpha,\alpha\beta} i\xi_{\beta,\alpha\beta} \in G_{\alpha\beta}$. For $j \in J_\alpha$ and $i \in I_\beta$, define $p_{j,i} = ji$. Hence, $(j, i) \to p_{j,i}$ defines a function of $J \times I$ into G satisfying (1) and (2). (3) is verified by a straightforward calculation. Let $x = igj \in S_\alpha$ and $y = whz \in S_\beta$. Hence $xy = (igj)(whz) = i(gp_{j,w}h)z = (i\zeta_{\alpha,\alpha\beta})(gp_{j,w}h)(z\xi_{\beta,\alpha\beta}) = (iw)(gp_{j,w}h)(jz)$. Thus, $igj \to (i, g, j)$ defines an isomorphism of S onto $X = (Y, I, J, G, \zeta, \xi, \varphi)$ under (4).

Next, we show $X = (Y, I, J, G, \zeta, \xi, \varphi)$ is a standard completely regular semigroup. Closure is a consequence of (1) and (2). For $\alpha \in Y$, let $T_\alpha = \{(i, g, j)\colon i \in I_\alpha, g \in G_\alpha, j \in J_\alpha\}$. Let $x = (i, g, j) \in T_\alpha$, $y = (m, h, n) \in T_\beta$, and $w = (c, z, d) \in T_\gamma$. Using (2) and (3), $p_{j,m}\varphi_{\alpha\beta,\alpha\beta\gamma} = p_{j\zeta_{\alpha,\alpha\beta}, m\xi_{\beta,\alpha\beta}}\varphi_{\alpha\beta,\alpha\beta\gamma} = p_{j\zeta_{\alpha,\alpha\beta\gamma}, m\xi_{\beta,\alpha\beta\gamma}} = p_{j\zeta_{\alpha,\alpha\beta\gamma}, m\xi_{\beta,\alpha\beta\gamma}c\xi_{\gamma,\alpha\beta\gamma}} = p_{j\zeta_{\alpha,\alpha\beta\gamma}, (mc)\xi_{\beta\gamma,\alpha\beta\gamma}} = p_{j,mc}$. Similarly, $p_{n,c}\varphi_{\beta\gamma,\alpha\beta\gamma} = p_{jn,c}$. Thus, $(xy)w = (imc, g\varphi_{\alpha,\alpha\beta\gamma}p_{j,m}\varphi_{\alpha\beta,\alpha\beta\gamma}h\varphi_{\beta,\alpha\beta\gamma}p_{jn,c}z\varphi_{\gamma,\alpha\beta\gamma}, jnd) = x(yw)$. Using (4), the Rees theorem [1, Theorem 3.5], (1) and (2), X is the semilattice Y of completely simple semigroups $(T_\alpha\colon \alpha \in Y)$. Hence, X is completely regular by [1, Theorem 4.6]. We next show X is locally inverse. Using (4), $E(X) = \{(i, p_{j,i}^{-1}, j)\colon i \in I_\alpha, j \in J_\alpha, \alpha \in Y\}$. Let $(i, p_{j,i}^{-1}, j) \in T_\alpha$ and $(a, p_{b,a}^{-1}, b) \in T_\beta$. Then, using (4), (3) and (2), $(i, p_{j,i}^{-1}, j) \geqq (a, p_{b,a}^{-1}, b)$ if and only if $\alpha \geqq \beta$, $i\xi_{\alpha,\beta} = a$, and $j\zeta_{\alpha,\beta} = b$. Thus, using (4), (3) and (2), S is locally inverse.

Remark 2.2. The structure of I, J, and G are given in terms of their respective structure homomorphisms (see [4, Section 1, especially Remark 1.7], [5, Theorem 1] and [1, Theorem 4.11]).

Remark 2.3. In [4], we used the term Cliffordian semigroup to describe a union of groups. In order not to conflict with the terminology of [3], we adopted our present terminology which appears to be the prevalent terminology.

3. The minimum inverse semigroup congruence. In this section, we describe the minimum inverse semigroup congruence on a standard regular semigroup $S=(Y, V, T)$. If φ is a homomorphism, $\ker\varphi$ will denote the kernel of φ.

Proposition 3.1. *Let ω be a homomorphism of V onto an inverse semigroup V^* such that $J*I\subseteqq\bigcup\ker\omega$. Then, $(i,b,j)\theta=b\omega$ defines a homomorphism of S onto V^*. Conversely, if θ is a homomorphism of S onto an inverse semigroup V^*, then $(i,b,j)\theta=b\omega$ where ω is a homomorphism of V onto V^* with $\bigcup\ker\omega\subseteqq J*I$.*

Proof. We first establish the direct part. Let (i,b,j), $(r,c,s)\in S$. Hence,

$$((i,b,j)(r,c,s))\theta=(b\circ(j*r)\circ c)\omega=b\omega\circ((b^{-1}\circ b)\circ(c\circ c^{-1}))\omega\circ c\omega=$$

$$=b\omega\circ c\omega=(i,b,j)\theta(r,c,s)\theta.$$

Conversely, let θ be a homomorphism of S onto V^*. For $b\in V$, define $b\omega=(b\circ b^{-1},b,b^{-1}\circ b)\theta$. Thus, $b\omega c\omega=((b\circ b^{-1},b,b^{-1}\circ b)(c\circ c^{-1},c,c^{-1}\circ c))\theta=((b\circ c)\circ(b\circ c)^{-1},b\circ c,(b\circ c)^{-1}\circ(b\circ c))\theta=(b\circ c)\omega$. Hence, ω is a homomorphism of V into V^*. Let $(i,b,j)\in S$. Then, $(i,b,j)=(i,b\circ b^{-1},b\circ b^{-1})(b\circ b^{-1},b,b^{-1}\circ b)(b^{-1}\circ b,b^{-1}\circ b,j)$. Using Lemma 1.13 (b), $(i,b\circ b^{-1},b\circ b^{-1})\mathscr{L}(b\circ b^{-1},b\circ b^{-1},b\circ b^{-1})$. Hence, using Lemma 1.15, $(i,b\circ b^{-1},b\circ b^{-1})\theta=(b\circ b^{-1},b\circ b^{-1},b\circ b^{-1})\theta$. Similarly, $(b^{-1}\circ b,b^{-1}\circ b,j)\theta=(b^{-1}\circ b,b^{-1}\circ b,b^{-1}\circ b)\theta$. Thus, using Lemmas 1.15 and 1.13, $(i,b,j)\theta=((b\circ b^{-1},b\circ b^{-1},b\circ b^{-1})(b\circ b^{-1},b,b^{-1}\circ b)(b^{-1}\circ b,b^{-1}\circ b,b^{-1}\circ b))\theta=(b\circ b^{-1},b,b^{-1}\circ b)\theta=b\omega$. Let $c\in V^*$. Hence, $c=(i,d,j)\theta=d\omega$ for some $(i,d,j)\in S$. Thus, ω is a homomorphism of V onto V^*. Let $j\in J_y$ and $i\in I_z$. Since $(y,y,j)\theta(i,z,z)\theta=(y*z,j*i,y*z)\theta=(j*i)\omega=y\omega z\omega=(yz)\omega$, $j*i\in\bigcup\ker\omega$. Q.E.D.

Let N denote the collection of all finite products of elements of the form $a^{-1}\circ s\circ a$ where $a\in V$ and s or $s^{-1}\in J*I$. Since $\mathscr{H}$ is a congruence relation on V by [4, Lemma 2.13] and $J*I\subseteqq H=\bigcup(H_y\colon y\in Y)$, $a^{-1}\circ s\circ a\in H$. Thus, N is an inverse subsemigroup of V and H. Since $E(V)$ is contained in the center of H, it follows that $x^{-1}\circ N\circ x\subseteqq N$ for all $x\in V$. Let $N_y=H_y\cap N$. Then N is the semilattice Y of groups $(N_y\colon y\in Y)$. Let $\varrho_N=\{(a,b)\in V\times V\colon a\circ a^{-1},b\circ b^{-1},a\circ b^{-1}\in N_y$ for some $y\in Y\}$. Then, using [2, Theorem 7.54 and Lemma 7.48], ϱ_N is a congruence relation on V with kernel $\{N_y\colon y\in Y\}$.

Proposition 3.2. *V/ϱ_N is the maximal inverse semigroup homomorphic image of S under the homomorphism $(i,b,j)\theta_N=b\varrho_N^\#$ where $\varrho_N^\#$ is the natural homomorphism of V onto V/ϱ_N.*

Proof. Using Proposition 3.1, θ_N is a homomorphism of S onto V/ϱ_N. Let θ be a homomorphism of S onto an inverse semigroup V^*. Define $(x\theta_N)\gamma=x\theta$ for $x\in S$. We will show that γ is a homomorphism of V/ϱ_N onto V^*. Suppose that $(i,b,j)\theta_N=(p,c,q)\theta_N$. Hence, $b\varrho_N^\#=c\varrho_N^\#$ and $(b,c)\in\varrho_N$. Thus, using [2, Theorem 7.55], $b=nc$ for some $n\in N_{c\circ c^{-1}}$. By Proposition 3.1, $(i,b,j)\theta=b\omega$ for some homomorphism ω of V onto V^* with $\bigcup\ker\omega\subseteqq J*I$. We note that $n=(a_1^{-1}s_1a_1)\dots(a_n^{-1}s_na_n)$ where $a_i\in V$ and s_i or $s_i^{-1}\in J*I$. Thus, $s_i\omega\in E(V^*)$ and, hence, $n\omega\in E(V^*)$. Thus, since $n\mathcal{H}c\circ c^{-1}$, $n\omega=(c\omega)(c\omega)^{-1}$. Hence $b\omega=n\omega c\omega=$ $=c\omega(c\omega)^{-1}c\omega=c\omega$. Thus, $(i,b,j)\theta=(p,c,q)\theta$. Q.E.D.

Theorem 3.3. *Let $S=(Y,V,T)$ be a standard regular semigroup. Let N denote the collection of all finite products of elements of the form $a^{-1}\circ s\circ a$ where $a\in V$ and s or $s^{-1}\in J*I$. Let $N_y=N\cap H_y$ for $y\in Y$. Let $\delta_N=\{((i,a,j),(p,b,q))\in S\times S\colon N_y\circ a=$ $=N_y\circ b$ where $y=a\circ a^{-1}=b\circ b^{-1}\}$. Then, δ_N is the minimum inverse semigroup congruence on S.*

Proof. Utilize Proposition 3.2 and its proof.

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