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# Almost all algebras with triply transitive automorphism groups are functionally complete

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## 1. Introduction

The present work is a continuation of a series of results on the functional completeness of algebras with high symmetry. It is also a contribution to the solution of Problem 20 in GRÄTZER [4]. WERNER [14] proved that every finite algebra $\langle A; t\rangle$ where $t$ is Pixley's ternary discriminator function on $A$ is functionally complete. Recently, FRIED and PIXLEY [2] showed that for $3 \leqq |A| < \aleph_0$, the algebra $\langle A; d\rangle$ with $d$ the dual discriminator function on $A$ is also functionally complete. A considerable generalization of these results was found by CSÁKÁNY [1] who proved that, up to equivalence, except for six algebras every non-trivial finite algebra whose automorphism group is the full symmetric group is functionally complete. Our contribution to this topic is the following theorem: an at least four element non-trivial finite algebra whose automorphism group is triply transitive is either functionally complete or equivalent to an affine space over the two element field. In the proof our main tool is Rosenberg's completeness criterium which provides a powerful method for checking functional completeness.

There is an interesting phenomenon which is worth being referred to in connection with our result. This is the connection of our theorem to the Słupecki type criteria for completeness due to SALOMAA [10] and SCHOFIELD [11], saying that any set $F$ of functions over a finite set $A$ $(|A| \geqq 4)$ which contains a function satisfying the Słupecki condition and a triply transitive group of permutations of $A$, generates the set of all functions on $A$, except for the case when all functions in $F$ are linear in each variable, relative to some representation of $A$ as a vector space over the two element field. Making use of Rosenberg's completeness criterium, this theorem can be further improved to doubly transitive permutation groups and then the excep-

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tions are exactly those sets of functions that are linear with respect to a vector space over an arbitrary prime field (see ROSENBERG [8], also KNOEBEL [5]). It would be worthwhile to find out whether our theorem could be generalized for finite algebras with doubly transitive automorphism groups.

## 2. Preliminaries

Let $A$ be a non-empty set. By an operation we always mean a finitary operation. The set of $n$-ary operations on $A$ will be denoted by $\mathbf{O}_A^{(n)}$ $(n \geqq 1)$. Furthermore, we set $\mathbf{O}_A = \bigcup_{n=1}^{\infty} \mathbf{O}_A^{(n)}$. An operation $f \in \mathbf{O}_A^{(n)}$ is said to depend on its $i$'th variable $(1 \leqq i \leqq n)$ if there exist elements $a_1, \ldots, a_n, a_i' (\neq a_i)$ in $A$ such that

$$f(a_1, \ldots, a_n) \neq f(a_1, \ldots, a_{i-1}, a_i', a_{i+1}, \ldots, a_n).$$

$f$ is called *essentially k-ary*, if it depends on exactly $k$ of its variables. $f$ is termed *idempotent*, if for every $a \in A$, we have $f(a, \ldots, a) = a$. $f$ is called *non-trivial* if it is not a projection.

We adopt the terminology of [4] except that polynomials and algebraic functions are called *term functions* and *polynomial functions*, respectively. Accordingly, the set of polynomial functions and the set of term functions of an algebra $\mathfrak{A}$ are denoted by $\mathbf{P}(\mathfrak{A})$ and $\mathbf{T}(\mathfrak{A})$, respectively. Two algebras (with a common base set) are said to be *equivalent* if they have the same term functions. By a *clone* we mean a subset $\mathbf{C}$ of $\mathbf{O}_A$ for some set $A (\neq \emptyset)$, which contains the projections and is closed with respect to superposition. In particular, both $\mathbf{P}(\mathfrak{A})$ and $\mathbf{T}(\mathfrak{A})$ are clones for any algebra $\mathfrak{A}$. An algebra $\mathfrak{A} = \langle A; F \rangle$ is called *functionally complete* if $\mathbf{P}(\mathfrak{A}) = \mathbf{O}_A$ and *trivial* if $\mathbf{T}(\mathfrak{A})$ contains projections only. An algebra is said to be *idempotent* if its fundamental operations (and hence all term functions) are idempotent. For a field $K$, an *affine space over* $K$ is defined to be an algebra $\langle A; I \rangle$ where $I$ is the set of all idempotent term functions of a vector space over $K$ with base set $A$.

The automorphism group of an algebra $\mathfrak{A}$ is denoted by Aut $\mathfrak{A}$. If Aut $\mathfrak{A}$ is the full symmetric group then $\mathfrak{A}$ is called *homogeneous*.

Now we are going to formulate Rosenberg's Theorem [6, 7] which is our main tool in proving our theorem. First, however, we need some further definitions.

Let $A (\neq \emptyset)$ be a finite set, $k, n \geqq 1$, $f \in \mathbf{O}_A^{(n)}$ and $\varrho \subseteq A^k$ an arbitrary $k$-ary relation. $f$ is said to *preserve* $\varrho$ if $\varrho$ is a subalgebra of the $k$'th direct power of the algebra $\langle A; f \rangle$; in other words, $f$ preserves $\varrho$ if for any $n \times k$ matrix with entries in $A$, whose rows belong to $\varrho$, the row of column values of $f$ also belongs to $\varrho$. It is easy to verify that the set of operations preserving a relation $\varrho$ forms a clone, which will be denoted by Pol $\varrho$.

A $k$-ary relation $\varrho$ on $A$ is called *central* if $\varrho \neq A^k$ and there exists a non-void proper subset $C$ of $A$ such that

(a) $\langle a_1, \ldots, a_k \rangle \in \varrho$ whenever at least one $a_j \in C$ $(1 \leqq j \leqq k)$;

(b) $\langle a_1, \ldots, a_k \rangle \in \varrho$ implies $\langle a_{1\pi}, \ldots, a_{k\pi} \rangle \in \varrho$ for every permutation $\pi$ of the indices $1, \ldots, k$;

(c) $\langle a_1, \ldots, a_k \rangle \in \varrho$ if $a_i = a_j$ for some $i \neq j$ $(1 \leqq i, j \leqq k)$.

Let $2 < k \leqq |A|$ and $m \geqq 1$. A family $T = \{\Theta_1, \ldots, \Theta_m\}$ of equivalence relations on $A$ is termed *k-regular* if

(d) each $\Theta_j$ has $k$ equivalence classes $(j = 1, \ldots, m)$;

(e) the intersection $\bigcap_{i=1}^{m} \varepsilon_i$ of arbitrary equivalence classes $\varepsilon_i$ of $\Theta_i$ $(i = 1, \ldots, m)$ is non-empty.

The relation $\varrho$ *determined by* $T$ consists of all $\langle a_1, \ldots, a_k \rangle \in A^k$ having the property that for each $j$ $(j = 1, \ldots, m)$ at least two elements among $a_1, \ldots, a_k$ are equivalent modulo $\Theta_j$. Notice that $\varrho$ has properties (b) and (c).

We shall use the following version of Rosenberg's Theorem (see [9]):

Theorem. (ROSENBERG [6, 7]) *For a non-empty finite set $A$,* Pol $\varrho$ *is a maximal subclone of $\mathbf{O}_A$, provided $\varrho$ is one of the following relations on $A$:*

(α) *a bounded partial order;*

(β) *a binary relation $\{\langle a, a\pi \rangle | a \in A\}$ where $\pi$ is a permutation of $A$ with $|A|/p$ cycles of the same prime length $p$;*

(γ) *a quaternary relation $\{\langle a_1, a_2, a_3, a_4 \rangle \in A^4 | a_1 + a_2 = a_3 + a_4\}$ where $\langle A; + \rangle$ is an elementary abelian p-group (p is a prime number);*

(δ) *a non-trivial equivalence relation;*

(ε) *a central relation;*

(ζ) *a relation determined by a k-regular family of equivalence relations on $A$ $(k \geqq 3)$.*

*Moreover, every proper subclone of $\mathbf{O}_A$ is contained in at least one of the clones listed above.*

In the proof of our theorem we need two other results.

Lemma. (ŚWIERCZKOWSKI [12]; see also [1; Lemma 4]). *If an at least quaternary operation turns into projection whenever we identify any two of its variables, then it always turns into the same projection.*

Theorem. (URBANIK [13; Lemma 9]) *Let $\mathfrak{A} = \langle A; F \rangle$ $(|A| \geqq 2)$ be an idempotent algebra which has essentially ternary term functions but has neither essentially binary nor essentially quaternary term functions. Then $\mathfrak{A}$ is equivalent to an algebra $\langle A; I \cup G \rangle$ where*

(i) *$\langle A; I \rangle$ is an affine space over the two element field* GF (2)*;*

(ii) *either* $G=\emptyset$ *or there exists an integer* $r\geqq 5$ *such that* $G$ *contains an r-ary operation depending on every variable, furthermore, every* $g\in G$ *depends on at least* $r$ *variables and satisfies the equation* $g(x_1, \ldots, x_n)=x_1$ *whenever the elements* $x_1, \ldots, x_n$ *belong to a subalgebra of* $\langle A; I\rangle$ *generated by less than r elements.*

## 3. Results

Our main theorem was inspired by the following

Theorem. (Csákány [1]) *A non-trivial finite homogeneous algebra is functionally complete unless it is equivalent to one of the following algebras:*

$\langle \mathbf{2}; n\rangle$ *with* $n(x)\equiv x+1 \pmod 2$;

$\langle \mathbf{2}; s\rangle$ *with* $s(x, y, z)\equiv x+y+z \pmod 2$ (i.e. *the two element affine space over* GF (2));

$\langle \mathbf{2}; \bar{s}\rangle$ *with* $\bar{s}(x, y, z)\equiv x+y+z+1 \pmod 2$;

$\langle \mathbf{2}; d\rangle$ *with* $d(x, y, z)\equiv xy+yz+xz \pmod 2$;

$\langle \mathbf{3}; \circ\rangle$ *with* $x\circ y\equiv 2x+2y \pmod 3$;

*the four element affine space over* GF (2).

The proof of this result in [1] depends upon the Słupecki criterium. Trying to prove it by means of Rosenberg's Theorem, the first author noticed that it suffices to require Aut $\mathfrak{A}$ to be quadruply transitive. Moreover, the major part of his proof used 3-fold transitivity only. This observation led us to the following

Theorem. *An at least four element non-trivial finite algebra with triply transitive automorphism group is either functionally complete or equivalent to an affine space over* GF (2).

Remark. Examining the proof presented in the next section one can observe that the hypotheses of this theorem can be slightly weakened so that the conclusion still remain valid. Namely, it suffices to assume that the endomorphism monoid be *weakly triply transitive* in the sense that any three distinct elements of the algebra can be sent into any other tree distinct elements by an endomorphism.

It is easy to check that a more than four element affine space over GF (2) has a triply but not quadruply transitive automorphism group. Hence we get

Corollary 1. *An at least four element non-trivial finite algebra with quadruply transitive automorphism group is functionally complete unless it is equivalent to the four element affine space over* GF (2).

Corollary 2. *An at least four element non-trivial finite algebra with triply transitive automorphism group is simple or equivalent to an affine space over* GF (2).

This is a sharpening of a result of GANTER, PŁONKA and WERNER [3] on the simplicity of finite homogeneous algebras.

Corollary 3. *An at least four element finite simple algebra with triply transitive automorphism group is functionally complete.*

## 4. Proof of the main theorem

We start with two simple observations.

Proposition 1. *A finite algebra* $\mathfrak{A}=\langle A; F\rangle$ *is either functionally complete or* $\mathbf{P}(\mathfrak{A})\subseteq \operatorname{Pol}\varrho$ *for a relation* $\varrho$ *(on* $A$*) of type* $(\alpha)$, $(\gamma)$, $(\delta)$, $(\zeta)$ *or*
$(\varepsilon')$ *an at least binary central relation.*

Proof. Notice that if $\varrho$ is a unary central relation or a relation of type $(\beta)$ then $\operatorname{Pol}\varrho$ fails to contain all constant functions on $A$. Thus the statement follows from Rosenberg's Theorem.

Proposition 2. *Let* $\mathfrak{A}$ *be an at least four element finite algebra whose automorphism group is triply transitive. Then any non-trivial term function of* $\mathfrak{A}$ *is at least ternary. In particular,* $\mathfrak{A}$ *is idempotent.*

Proof. Let $f\in\mathbf{T}(\mathfrak{A})$, $f$ binary, and $a\neq b$ arbitrary elements in the base set $A$ of $\mathfrak{A}$. Then $f(a,b)\in\{a,b\}$, else there would exist $\pi\in\operatorname{Aut}\mathfrak{A}$ with $a\pi=a$, $b\pi=b$ and $f(a,b)\pi=c\notin\{a,b,f(a,b)\}$, implying that $f(a,b)=f(a\pi,b\pi)=c$ which contradicts the choice of $c$. Similarly, $g(x)\in\{x\}$ for any unary $g\in\mathbf{T}(\mathfrak{A})$ and $x\in A$. Thus $f(x,x)=x$ for any $x\in A$. Furthermore, if, say, $f(a,b)=a$ then by the 2-fold transitivity of $\operatorname{Aut}\mathfrak{A}$, $f(x,y)=x$ for any distinct $x,y\in A$. Hence $f$ is a projection, what was to be proved.

Lemma 1. *Let* $A$ *be a finite set,* $|A|\geqq 4$, *and* $f$ *a non-trivial ternary operation on* $A$ *such that the algebra* $\langle A;f\rangle$ *is functionally incomplete and has a triply transitive automorphism group. Then*

(i) $f$ *is a minority function, i.e.* $f(x,y,y)=f(y,x,y)=f(y,y,x)=x$ *for all* $x,y\in A$, *and for any distinct elements* $a,b,c\in A$, $f(a,b,c)\notin\{a,b,c\}$;

(ii) $\mathbf{P}(\langle A;f\rangle)\nsubseteq\operatorname{Pol}\varrho$ *if* $\varrho$ *is a relation of type* $(\alpha)$, $(\gamma)$ *with* $p>2$, $(\varepsilon')$ *or* $(\zeta)$.

Proof. Recall that $f$ turns into projection if we identify any two of its variables. Suppose that there exist distinct elements $a,b,c\in A$ such that $f(a,b,c)\in\{a,b,c\}$, say, $f(a,b,c)=a$. Then the 3-fold transitivity of $\operatorname{Aut}\langle A;f\rangle$ implies $f(x,y,z)=x$ for any distinct $x,y,z\in A$. Hence the algebra $\langle A;f\rangle$ is homogeneous, so that by Csákány's Theorem $\langle A;f\rangle$ must be equivalent to the four element affine space over

GF (2), else it would be functionally complete. However, then $f$ is necessarily the "parallelogram operation" $x+y+z$, which does not satisfy our assumption on $f$. This contradiction shows that $a, b, c, f(a, b, c)$ are pairwise different provided the first three of them are such. Let $a, b, c \in A$ be pairwise different. Then there exists an automorphism $\pi$ of $\langle A; f\rangle$ which sends $a$, $b$ and $f(a, b, c)$ into $a$, $b$ and $c$, respectively. Hence $f(a, b, c\pi)=c$. Consequently,

($*$) for any distinct elements $a, b, c \in A$ there exists $d \in A$ such that $f(a, b, d)=c$.

Now we are going to show that $\mathbf{P}(\langle A; f\rangle) \subsetneqq \operatorname{Pol} \varrho$ if $\varrho$ is a relation of type ($\alpha$), ($\varepsilon'$) or ($\zeta$). We do this by constructing matrices with entries in $A$ such that each row belongs to $\varrho$ but the row of column values of $f$ fails to belong to $\varrho$. We have to construct various matrices according to the various possibilities for the behaviour of $f$ when identifying two of its variables. Consider first a partial order $\leqq$ with lower bound 0 and upper bound 1, further, let $0 < a < 1$ ($a \in A$). Owing to ($*$), we can choose $d \in A$ such that $f(0, a, d)=1$. As regards the behaviour of $f$ when identifying two of its variables, by symmetry, it suffices to deal with the following two cases: $f(x, y, y)=x$ for all $x, y \in A$ or $f$ is a majority function (i.e. $f(x, y, y)=$ $=f(y, x, y)=f(y, y, x)=y$ for all $x, y \in A$). Accordingly, the two matrices disproving $\mathbf{P}(\langle A; f\rangle) \subseteqq \operatorname{Pol} \leqq$ are

$$\begin{array}{cc} 0 & 0 \\ a & 1 \\ d & 1 \\ \hline 1 & 0 \end{array} \quad \text{and} \quad \begin{array}{cc} 0 & a \\ a & a \\ d & d \\ \hline 1 & a \end{array}$$

Let $\varrho$ be a $k$-ary central relation ($k \geqq 2$) and select $\langle a_1, \ldots, a_k\rangle \in A^k - \varrho$. Furthermore, let $c \in C$, the centre of $\varrho$. By the definition of a central relation, $a_1$, $a_2$, $c$ are pairwise different, so that, by ($*$), there exists $d \in A$ such that $f(c, a_2, d)=a_1$. Now the matrices

$$\begin{array}{cccc} c & a_2 & a_3 \ldots & a_k \\ a_2 & a_2 & a_3 \ldots & a_k \\ d & d & d \ldots & d \\ \hline a_1 & a_2 & a_3 \ldots & a_k \end{array} \quad \text{and} \quad \begin{array}{cccc} a_1 & c & a_3 \ldots & a_k \\ c & c & a_3 \ldots & a_k \\ c & a_2 & a_3 \ldots & a_k \\ \hline a_1 & a_2 & a_3 \ldots & a_k \end{array}$$

show that $\mathbf{P}(\langle A; f\rangle) \subsetneqq \operatorname{Pol} \varrho$ whether $f(x, x, y)=x$ for all $x, y \in A$ or $f$ is a minority function. By symmetry, all other cases can be reduced to one of these. Similarly, if $\varrho$ is a $k$-ary relation of type ($\zeta$) ($k \geqq 3$) and $\langle a_1, \ldots, a_k\rangle \in A^k - \varrho$ then, $a_1$, $a_2$, $a_3$ being pairwise different (by property (c)), there is a $d \in A$ with $f(a_2, a_3, d)=a_1$. Hence the two matrices

$$\begin{array}{ccccc} a_2 & a_2 & a_3 & a_4 \ldots & a_k \\ a_3 & a_2 & a_3 & a_4 \ldots & a_k \\ d & d & d & d \ldots & d \\ \hline a_1 & a_2 & a_3 & a_4 \ldots & a_k \end{array} \quad \text{and} \quad \begin{array}{ccccc} a_1 & a_2 & a_2 & a_4 \ldots & a_k \\ a_2 & a_2 & a_2 & a_4 \ldots & a_k \\ a_2 & a_2 & a_3 & a_4 \ldots & a_k \\ \hline a_1 & a_2 & a_3 & a_4 \ldots & a_k \end{array}$$

meet our requirements if $f(x,x,y)=x$ for all $x,y\in A$ or $f$ is a minority function, respectively.

Suppose $f$ is not a minority function, say $f(x,x,y)=x$ for all $x,y\in A$. Then $\mathbf{P}(\langle A;f\rangle)\subsetneqq \mathrm{Pol}\,\varrho$ for any relation of type ($\gamma$) or ($\delta$). Indeed, assume first $\varrho$ is a non-trivial equivalence relation, $a\varrho b$, $a\neq b$ and $a\bar{\varrho}c$ (i.e. $\langle a,c\rangle\in A^2-\varrho$), $a,b,c\in A$. Then, by ($*$), there exists $d\in A$ such that $f(a,b,d)=c$, hence the matrix

$$\begin{array}{cc} a & a \\ a & b \\ d & d \\ \hline a & c \end{array}$$

proves $\mathbf{P}(\langle A;f\rangle)\subsetneqq \mathrm{Pol}\,\varrho$. If in turn $\varrho$ is a relation of type ($\gamma$), take into consideration that $f$ is essentially ternary, hence in particular, $f$ depends on the third variable, i.e. there exist elements $a,b,c,d\in A$, $c\neq d$ such that $f(a,b,c)\neq f(a,b,d)$. Then the matrix

$$\begin{array}{cccc} a & a & a & a \\ b & a & b & a \\ c & a & d & c-d+a \\ \hline f(a,b,c) & a & f(a,b,d) & a \end{array}$$

shows that $\mathbf{P}(\langle A;f\rangle)\subsetneqq \mathrm{Pol}\,\varrho$, what was to be proved. By Proposition 1, this contradicts the functional incompleteness of $\langle A;f\rangle$. Thus $f$ is a minority function.

It remains to verify that if $f$ is a minority function and $\mathbf{P}(\langle A;f\rangle)\subseteqq \mathrm{Pol}\,\varrho$ for a relation $\varrho$ of type ($\gamma$) then $p=2$. This is done by the following matrix:

$$\begin{array}{cccc} a & 0 & a & 0 \\ a & 0 & 0 & a \\ a & a & a & a \\ \hline a & a & 0 & 0 \end{array}$$

where 0 is the zero element of the abelian group $\langle A;+\rangle$, $a\in A$ is arbitrary and, by definition, $\langle a,a,0,0\rangle\in\varrho$ iff $a+a=0$.

Lemma 2. *Consider a finite set $A$, $|A|\geqq 4$, and an at least quaternary nontrivial operation $f$ on $A$ such that the algebra $\langle A;f\rangle$ has a triply transitive automorphism group and $f$ turns into projection whenever we identify any two of its variables. Then $\langle A;f\rangle$ is functionally complete.*

Proof. Suppose $f$ is $n$-ary, $n\geqq 4$. By Świerczkowski's Lemma $f$ always turns into the same, say the first, projection if we identify any two of its variables. Since $f$ itself is not the first projection, there exist (necessarily distinct) elements $e_i$ ($1\leqq i\leqq n$)

such that $f(e_1, \ldots, e_n) \neq e_1$. Then $f(e_1, \ldots, e_n) \neq e_2$ or $e_3$. We can assume without loss of generality that $f(e_1, \ldots, e_n) \neq e_2$, i.e. $e_1$, $e_2$ and $f(e_1, \ldots, e_n)$ are pairwise different. Hence, 3-fold transitivity of Aut $\langle A; f\rangle$ implies

($* *$) for any distinct elements $a, b, c \in A$ there exist elements $d_3, \ldots, d_n$ such that $f(a, b, d_3, \ldots, d_n) = c$.

By Proposition 1, we are done if we show that $\mathbf{P}(\langle A; f\rangle)$ is not contained in Pol $\varrho$ for any relation $\varrho$ of type ($\alpha$), ($\gamma$), ($\delta$), ($\varepsilon'$) or ($\zeta$). To this end we have to construct matrices with entries in $A$ whose rows belong to $\varrho$ but the row of column values of $f$ fails to belong to $\varrho$. The five matrix schemes corresponding to the five types are the following:

$$(\alpha)\ \begin{array}{cc} 0 & 0 \\ b & 1 \\ d_3 & 1 \\ \vdots & \vdots \\ d_n & 1 \\ \hline 1 & 0 \end{array} \qquad (\gamma)\ \begin{array}{cccc} a & a & a & a \\ b & a & b & a \\ d_3 & a & a & d_3 \\ \vdots & \vdots & \vdots & \vdots \\ d_n & a & a & d_n \\ \hline c & a & a & a \end{array} \qquad (\delta)\ \begin{array}{cc} a & a \\ b & a \\ d_3 & d_3 \\ \vdots & \vdots \\ d_n & d_n \\ \hline c & a \end{array}$$

$$(\varepsilon')\ \begin{array}{cccc} c & a_2 & a_3 \ldots & a_k \\ a_2 & a_2 & a_3 \ldots & a_k \\ d_3 & d_3 & d_3 \ldots & d_3 \\ \vdots & \vdots & \vdots & \vdots \\ d_n & d_n & d_n \ldots & d_n \\ \hline a_1 & a_2 & a_3 \ldots & a_k \end{array} \quad (k \geqq 2) \qquad (\zeta)\ \begin{array}{ccccc} a_2 & a_2 & a_3 & a_4 \ldots & a_k \\ a_3 & a_2 & a_3 & a_4 \ldots & a_k \\ d_3 & d_3 & d_3 & d_3 \ldots & d_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ d_n & d_n & d_n & d_n \ldots & d_n \\ \hline a_1 & a_2 & a_3 & a_4 \ldots & a_k \end{array} \quad (k \geqq 3).$$

If $\varrho$ is a partial order, 0 and 1 denote the lower and upper bounds, respectively, $b$ is another element, $b \neq 0, 1$. The existence of the elements $d_3, \ldots, d_n$ is ensured by ($* *$). Similar argument can be applied in the other cases, too. In case ($\gamma$) $a, b, c$ are arbitrary distinct elements of $A$ while in case ($\delta$) $a$, $b$ and $c$ are selected such that $a \neq b$, $a\varrho b$ and $a\bar{\varrho}c$ (i.e. $\langle a, c\rangle \in A^2 - \varrho$); $d_3, \ldots, d_n$ is chosen according to ($* *$). Finally, if $\varrho$ is of type ($\varepsilon'$) or ($\zeta$) then we fix a $k$-tuple $\langle a_1, \ldots, a_k\rangle \in A^k - \varrho$. By definition, its components are necessarily pairwise different, moreover, if $\varrho$ is a central relation, none of them belong to the centre $C$. Thus $c(\in C)$, $a_2$, $a_1$, resp. $a_2$, $a_3$, $a_1$ are pairwise different, hence ($* *$) implies the existence of the elements $d_3, \ldots, d_n \in A$ completing the first columns of the corresponding matrices.

Lemma 3. *Let* $\mathfrak{A} = \langle A; F\rangle$ ($|A| \geqq 4$) *be a functionally incomplete non-trivial finite algebra with a triply transitive automorphism group. Then*

(i) $\mathfrak{A}$ *has a unique non-trivial ternary term function* $m$. *It is a minority function and has the property that for any distinct elements* $a, b, c \in A$, $m(a, b, c) \notin \{a, b, c\}$;

(ii) *any non-trivial quaternary term function $h$ of $\mathfrak{A}$ satisfies the identities*

$$h(x, x, y, z) = m(x, y, z), \tag{1}$$

$$h(x, y, x, z) = m(x, y, z), \tag{2}$$

$$h(x, y, z, x) = m(x, y, z), \tag{3}$$

$$h(x, y, y, z) = z, \tag{4}$$

$$h(x, y, z, z) = y, \tag{5}$$

$$h(x, y, z, y) = z, \tag{6}$$

$$h(m(x, y, z), x, y, z) = m(x, y, z), \tag{7}$$

*or arises from such a term function by interchanging its variables.*

Proof. Let $n$ denote the minimum of the arities of non-trivial term functions of $\mathfrak{A}$. By Proposition 2, $n \geqq 3$. If $n \geqq 4$, arbitrary non-trivial $n$-ary term function $f$ turns into projection when we identify any two of its variables. Hence, by Lemma 2, $\langle A; f\rangle$ (consequently, also $\mathfrak{A}$) is functionally complete, contradicting our hypothesis. Thus $n=3$, i.e. $\mathfrak{A}$ has a non-trivial ternary term function. By Lemma 1, every such term function enjoys property (i).

In order to prove uniqueness we first show that for any non-trivial ternary term functions $f, g \in \mathbf{T}(\mathfrak{A})$, the following identity holds:

$$f(g(x, y, z), y, z) = x. \tag{8}$$

Indeed, $f(g(x, y, z), y, z)$, being a ternary term function of $\mathfrak{A}$, must be a minority function or a projection. Since by the identification $x=y$ we get $x$, the former case is excluded. Thus $f(g(x, y, z), y, z)=x$ or $y$. On the other hand, by the identification $x=z$ we also get $x$, so the proof of (8) is concluded. Taking into consideration that (8) holds for any $f, g \in \mathbf{T}(\mathfrak{A})$, in particular for $g=f$, too, we get the identity

$$g(x, y, z) = f(f(g(x, y, z), y, z), y, z) = f(x, y, z).$$

This completes the proof of (i).

Let $h$ be a non-trivial quaternary term function of $\mathfrak{A}$. If we identify any two of its variables, we either get a projection or the (unique) non-trivial ternary term function $m$. The latter must occur at least once, otherwise, by Lemma 2, the algebra $\langle A; h\rangle$ (hence also $\mathfrak{A}$) would be functionally complete. Suppose e.g. that $h(x, x, y, z)=m(x, y, z)$. Thus $h(x, x, z, z)=x$, so that $h(x, y, z, z)=x$ or $y$ (since neither $h(x, y, z, z)=z$ nor $h(x, y, z, z)=m(x, y, z)$ can hold). We can assume without loss of generality that $h(x, y, z, z)=y$. So far, we have (1) and (5). They imply

$$\text{(I)}\quad h(x, x, x, z) = z, \qquad \text{(II)}\quad h(x, x, y, x) = y,$$

$$\text{(III)}\quad h(x, y, y, y) = y, \qquad \text{(IV)}\quad h(x, y, x, x) = y.$$

By (II) and (III) $h(x, y, z, y) \neq x, y, m(x, y, z)$, which proves (6). Similarly, (I) and (IV) exclude all possibilities for $h(x, y, x, z)$ but (2). (4) follows from (I) and (III), while (3) from (II) and (IV). In order to verify (7) one has to check that $h(m(x, y, z), x, y, z)$ is a minority function, which is straightforward by the preceding identities. The proof is complete.

Now we are ready to prove our main result formulated in Section 3.

Proof of the Theorem. Let $\mathfrak{A}=\langle A; F\rangle$ ($|A|\geqq 4$) be a non-trivial finite algebra which is functionally incomplete and has a triply transitive automorphism group. By Proposition 2, $\mathfrak{A}$ is idempotent and has no essentially binary term function. On the other hand, by Lemma 3, $\mathfrak{A}$ has an essentially ternary term function. We are going to prove that $\mathfrak{A}$ has no essentially quaternary term function. Suppose the contrary and choose an essentially quaternary $h \in \mathbf{T}(\mathfrak{A})$ such that it satisfy identities (1)—(7) in Lemma 3. Since $h$ depends on its first variable, there exist elements $a, b, c, d \in A$ such that $h(a, b, c, d) \neq h(b, b, c, d) = m(b, c, d)$. Then the matrix

$$\begin{array}{cccc} a & b & a & b \\ b & b & b & b \\ c & c & b & b \\ d & d & b & b \\ \hline h(a, b, c, d) & h(b, b, c, d) & b & b \end{array}$$

shows that $\mathbf{P}(\mathfrak{A}) \nsubseteq \operatorname{Pol} \varrho$ if $\varrho$ is a relation of type ($\gamma$) with $p=2$. By Lemma 1, $\mathbf{P}(\mathfrak{A}) \nsubseteq \operatorname{Pol} \varrho$ if $\varrho$ is a relation of type ($\alpha$), ($\gamma$) with $p>2$, ($\varepsilon'$) or ($\zeta$). Thus $\mathbf{P}(\mathfrak{A}) \subseteq \operatorname{Pol} \varrho$ where $\varrho$ is a non-trivial equivalence relation. Select distinct elements $a', b', c' \in A$ such that $a' \varrho b'$ but $a' \bar{\varrho} c'$ (i.e. $\langle a', c'\rangle \in A^2 - \varrho$). Assume first $h(a, b, c, d) \neq a$. Then, by (7)

$$h(a, b, c, d) \neq h(m(b, c, d), b, c, d) = m(b, c, d) \tag{9}$$

where $a$, $m(b, c, d)$ and $h(a, b, c, d)$ are pairwise different ($a=m(b, c, d)$ would imply equality in (9), contradicting the choice of $a$, $b$, $c$, $d$). Hence, by the 3-fold transitivity of $\operatorname{Aut} \mathfrak{A}$ there exists $\pi \in \operatorname{Aut} \mathfrak{A}$ which sends $m(b, c, d)$, $a$, $h(a, b, c, d)$ into $a'$, $b'$ and $c'$, respectively. Thus we have the matrix

$$\begin{array}{cc} a' & b' \\ b\pi & b\pi \\ c\pi & c\pi \\ d\pi & d\pi \\ \hline a' & c' \end{array}$$

with its rows belonging to $\varrho$ but $a' \bar{\varrho} c'$, contradicting the inclusion $\mathbf{P}(\mathfrak{A}) \subseteq \operatorname{Pol} \varrho$.

Assume now that $h(a, b, c, d)=a$. Then, by (1) and (7)

$$(10)\qquad a = h(a, b, c, d) \neq h(b, b, c, d) = m(b, c, d) = h(m(b, c, d), b, c, d).$$

Thus $a, b, m(b, c, d)$ are pairwise different. (10) implies immediately that $a \neq b$, $m(b, c, d)$. If $b=m(b, c, d)$ then by Lemma 3(i), $b$, $c$, $d$ are not distinct, so that, since $m$ is a minority function, we have $c=d$. However, then by (5), $h(a, b, c, d)= =b=h(b, b, c, d)$, which is impossible by (10). By the 3-fold transitivity of Aut $\mathfrak{A}$ there exists an automorphism $\pi$ sending $a, b, m(b, c, d)$ into $a'$, $b'$, $c'$, respectively. Hence we get the matrix

$$\begin{array}{cc} a' & b' \\ b' & b' \\ c\pi & c\pi \\ d\pi & d\pi \\ \hline a' & c' \end{array},$$

again contradicting the inclusion $\mathbf{P}(\mathfrak{A}) \subseteq \mathrm{Pol}\ \varrho$.

It follows from the foregoing argument that $\mathfrak{A}$ has no essentially quaternary term function. Thus $\mathfrak{A}$ satisfies the hypotheses of Urbanik's Theorem, so that $\mathfrak{A}$ is equivalent to an affine space over GF(2) or arises from such a space by adding new at least $r(\geqq 5)$-ary fundamental operations among which there is an essentially $r$-ary operation which turns into projection if we identify any two of its variables. However, by Lemma 2, the existence of such an operation would imply functional completeness. Hence $\mathfrak{A}$ is equivalent to an affine space over GF(2), what was to be proved.

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