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A note on integral operators

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Let (X, m) be a separable σ-finite measure space which is not purely atomic (it may include some atoms). A bounded linear operator T on $L^2(X)$ is called an integral operator if there exists a measurable function k on $X \times X$ such that $(Tf)(x) = \int k(x, y) f(y) m(dy)$ almost everywhere. It is known ([7], p. 35) that every Hilbert—Schmidt operator is an integral operator. It is also known that there are integral operators which are not Hilbert—Schmidt or even compact. For example, if k is the characteristic function of the set $\bigcup_{n=0}^{\infty} ([n, n+1] \times [n, n+1])$, the operator induced by k on $L^2(0, \infty)$ is a projection of infinite rank. (This example is in HALMOS [3].) However, KOROTKOV [6] proved that every operator unitarily equivalent to T is an integral operator if and only if T is a Hilbert—Schmidt operator. The purpose of this note is to give a proof of Korotkov's theorem which seems to be conceptually simpler than the original. Unlike the proof in [6], we do not use any results about Fourier series. Our techniques are more operator-theoretic.

We start by establishing notation. Let $\mathfrak{H}$ be a separable infinite-dimensional Hilbert space, and let $\mathscr{B}(\mathfrak{H})$ be the algebra of bounded operators on $\mathfrak{H}$. If $T \in \mathscr{B}(\mathfrak{H})$, and if $\mathfrak{M}$ is a (closed) subspace of $\mathfrak{H}$, then the *compression* of T to $\mathfrak{M}$ is the operator $PTP|\mathfrak{M}$, where P is the projection onto $\mathfrak{M}$. We will always assume that $\mathfrak{M}$ is a "half" of $\mathfrak{H}$, that is, both $\mathfrak{M}$ and $\mathfrak{M}^\perp$ are infinite-dimensional.

If K is a compact operator, then the sequence of *s-numbers* of K is the sequence $s_1 \geqq s_2 \geqq \ldots$ of nonzero eigenvalues of the compact positive operator $(K^* K)^{1/2}$, each repeated according to its multiplicity. A compact operator is called *Hilbert—Schmidt* if its sequence of s-numbers is square summable. For a detailed discussion of ideals, s-numbers and related concepts see pp. 25—27 of [7]. Here we need only the following fact: If two compact operators have the same sequence of s-numbers, then they must belong to the same two-sided ideals ([7, p. 26]).

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Lemma 1. *If K is a compact operator, then* $K=\begin{bmatrix} K_1 & S_2 \\ S_3 & S_4 \end{bmatrix}$, *where every S_i is a Hilbert—Schmidt operator. Consequently, there is a Hilbert—Schmidt operator S such that* $\begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix}$ *is unitarily equivalent to* $K+S$.

Proof. Let $\{e_n\}$ be an orthonormal basis for the underlying Hilbert space, then $\|Ke_n\|\to 0$ and $\|K^*e_n\|\to 0$. Choose a subsequence $\{f_n\}$ of $\{e_n\}$ such that $\sum\|Kf_n\|^2<\infty$ and $\sum\|K^*f_n\|^2<\infty$. Let $\mathfrak{M}$ be the orthogonal complement of the span of $\{f_n\}$. (By passing to a subsequence of $\{f_n\}$, if necessary, we can assume that $\mathfrak{M}$ is infinite-dimensional.) The matrix of K relative to the decomposition $\mathfrak{M}\oplus\mathfrak{M}^\perp$ has the required form. The second assertion of the Lemma follows easily from the first.

Lemma 2. *Let $\mathfrak{A}$ be a linear space of compact operators, and assume that $\mathfrak{A}$ is closed under unitary equivalence and under compression and that it contains every Hilbert—Schmidt operator. Then $\mathfrak{A}$ is a two-sided ideal in $\mathscr{B}(\mathfrak{H})$.*

Proof. Assume that $K\in\mathfrak{A}$, and apply Lemma 1 to conclude that the operator $T=\begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix}$ belongs to $\mathfrak{A}$. Each of the following operators is unitarily equivalent to T and hence belongs to $\mathfrak{A}$:

$$T_1=\begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix},\quad T_2=\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}=\frac{1}{2}\begin{pmatrix} K & K \\ K & K \end{pmatrix},$$

$$T_3=\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} K & K \\ K & K \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}=\frac{1}{2}\begin{pmatrix} K & -iK \\ iK & K \end{pmatrix}.$$

By taking an appropriate linear combination, we see that $\begin{pmatrix} 0 & 0 \\ K & 0 \end{pmatrix}$ is also in $\mathfrak{A}$, and so is the operator

$$\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}\begin{pmatrix} 0 & 0 \\ K & 0 \end{pmatrix}\begin{pmatrix} U^* & 0 \\ 0 & V^* \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ VKU^* & 0 \end{pmatrix}$$

for any unitary operators U and V. Since every operator can be written as a linear combination of four unitary operators ([1, p. 4]), the operator $\begin{pmatrix} 0 & 0 \\ AKB & 0 \end{pmatrix}$ belongs to $\mathfrak{A}$ for any operators A and B. By unitary equivalence, the following operator also belongs to $\mathfrak{A}$

$$\frac{1}{2}\begin{pmatrix} AKB & AKB \\ -AKB & -AKB \end{pmatrix}.$$

Consequently AKB belongs to $\mathfrak{A}$. Therefore $\mathfrak{A}$ is a two-sided ideal.

Lemma 3. *Let (X, m) be a separable σ-finite measure space and Y a Borel subset of X such that $L^2(Y)$ and $L^2(X\setminus Y)$ are both infinite dimensional, and let $\mathfrak{H}$ be a separable Hilbert space, $\mathfrak{M}$ a subspace of $\mathfrak{H}$ with* $\dim\mathfrak{M}=\dim\mathfrak{M}^\perp=\infty$, *$T$ an* *operator on $\mathfrak{H}$ and A the compression of T to $\mathfrak{M}$. If every operator on $L^2(X)$ which is unitarily equivalent to T is an integral operator, then every operator on $L^2(Y)$ which is unitarily equivalent to A is an integral operator.*

Proof. Let $V: \mathfrak{M} \to L^2(Y)$ be a unitary operator and let W be any unitary mapping $\mathfrak{M}^{\perp}$ onto $L^2(X \setminus Y)$, and let $U = V \oplus W$. Thus UTU^* is an integral operator on $L^2(X)$. If k is the kernel of the latter, then VAV^* is an integral operator whose kernel is the restriction of k to $Y \times Y$.

Lemma 4. *Let T be a bounded operator on $\mathfrak{H}$ such that UTU^* is an integral operator for every unitary operator U mapping $\mathfrak{H}$ onto $L^2(X)$. Then T is compact.*

Proof. First we show that every non-compact operator has a compression (to an infinite dimensional subspace) which equals the sum of a non-zero scalar and a Hilbert—Schmidt operator. Let T be a non-compact operator and let $T = T_1 + iT_2$ where T_1 and T_2 are self-adjoint. One of the operators T_1 and T_2 (say T_1) is not compact. Let E be the spectral measure of T_1. Then there is a real number $\lambda \neq 0$ such that $\dim (E(\Delta)\mathfrak{H}) = \infty$ for every open set Δ containing λ. Consequently, there is a compression $PT_1P|P\mathfrak{H}$ of T_1 (to an infinite dimensional subspace) which is equal to λ+a Hilbert—Schmidt operator. Since PT_2P is self-adjoint, the same argument shows that there is an infinite dimensional projection $Q \leqq P$ such that $QT_2Q|Q\mathfrak{H}$ is a scalar+a Hilbert—Schmidt operator (this scalar may be zero). Thus

$$QTQ|Q\mathfrak{H} = \mu + S$$

where $\mu \neq 0$ and S is a Hilbert—Schmidt operator. (This proof is due to the referee.)

Let Y be a non-atomic "half" of X. If T is non-compact and is always integral on X, then by Lemma 3, the compression $\mu + S$ is always integral on Y. It follows that the identity on $L^2(Y)$ is an integral operator, which is impossible (see [5, problem 134]). So T must be compact.

For clarity of exposition, we will prove the main result first when $X = [0, 1]$.

Theorem 1. *Let T be a bounded operator on $\mathfrak{H}$. Then UTU^* is an integral operator for every unitary operator U mapping $\mathfrak{H}$ onto $L^2(0, 1)$ if and only if T is a Hilbert—Schmidt operator.*

Proof. The "if" part is easy. To prove the converse, let $\mathscr{I}$ be the set of all operators T on $\mathfrak{H}$ with the property that UTU^* is an integral operator for every unitary $U: \mathfrak{H} \to L^2(0, 1)$. It is easy to see that $\mathscr{I}$ is a linear space and is closed under unitary equivalence. It is also closed under compression since if $T \in \mathscr{I}$ and A is a compression of T, then A is always integral on $\left(0, \frac{1}{2}\right)$ and hence is always integral on $(0, 1)$. By Lemma 2, $\mathscr{I}$ is a two-sided ideal. Let $K \in \mathscr{I}$ and let $\{\lambda_n\}$ be the sequence of s-numbers of K. Since $\mathscr{I}$ is an ideal, every operator on $L^2(0, 1)$ with the same sequence of s-numbers is an integral operator. We will now construct an operator on $L^2(0, 1)$ with s-numbers $\{\lambda_n\}$.

Let $\{e_n\}$ be an orthonormal basis of $L^2(0, 1)$ consisting of unimodular function, that is $|e_n(x)|=1$. (For example, the usual exponentials $\exp(2\pi ikx)$, arranged in a sequence.) Let $\{\alpha_n\}$ be a sequence of positive numbers such that $\sum \alpha_n^2=1$, and let $\{E_n\}$ be a sequence of disjoint measurable subsets of $(0, 1)$ whose union is $(0, 1)$ and such that $m(E_n)=\alpha_n^2$. Let $\varphi_n=\alpha_n^{-1}\chi_n$, where χ_n is the characteristic function of E_n. Therefore $\{\varphi_n\}$ is an orthonormal set in $L^2(0, 1)$. Define an operator C on $L^2(0, 1)$ by the equations

$$C\varphi_n = \lambda_n e_n, \quad \text{and} \quad Cf = 0 \quad \text{if} \quad f\in\{\varphi_n\}^{\perp}.$$

It is easy to see that $C^*e_n=\lambda_n\varphi_n$ and $CC^*e_n=\lambda_n^2 e_n$, and so C is a compact operator whose sequence of s-numbers is $\{\lambda_n\}$.

By the foregoing, the operator C must be an integral operator. Let k be the kernel of C, so

$$(Cf)(x) = \int k(x, y)f(y)m(dy) \quad \text{a.e.}$$

By considering only functions in $L^2(E_n)$ for a fixed n, we see that

$$Cf = (f, \varphi_n)\lambda_n e_n \quad \text{for} \quad f\in L^2(E_n),$$

so

$$(Cf)(x) = \int \alpha_n^{-1}\lambda_n e_n(x)f(y)m(dy) \quad \text{for} \quad f\in L^2(E_n).$$

By the uniqueness of the kernel, we must have $k(x, y)=\alpha_n^{-1}\lambda_n e_n(x)$ when $y\in E_n$. For every $f\in L^2(0, 1)$, the function $|k(x, \cdot)f(\cdot)|$ must be integrable for almost every x. By taking $f=1$, we have $\int |k(x, y)|m(dy)<\infty$ for almost every x, so $\sum \alpha_n\lambda_n<\infty$. Since this is true for any (normalized) square-summable sequence $\{\alpha_n\}$, we must have $\{\lambda_n\}$ square-summable, and so K is a Hilbert—Schmidt operator.

Theorem 2. *Let (X, m) be a separable σ-finite measure space with no atoms, and let T be a bounded operator on $\mathfrak{H}$. Then UTU^* is an integral operator for every unitary operator U mapping $\mathfrak{H}$ onto $L^2(X)$ if and only if T is a Hilbert—Schmidt operator.*

Proof. This theorem is proved by slight modifications of the proof of Theorem 1. We only indicate the necessary changes.

As before, let $\mathscr{I}$ be the set of all operators T on $\mathfrak{H}$ with the property that UTU^* is an integral operator for every unitary U: $\mathfrak{H}\to L^2(X)$. Unlike the case $X=[0, 1]$, it is not immediately obvious that $\mathscr{I}$ is closed under compression. (If $T\in\mathscr{I}$ and A is a compression of T, then we only know that A is always integral on every half of X.) So we introduce the class $\mathscr{J}$ of all compressions of operators in $\mathscr{I}$, that is $A\in\mathscr{J}$ if and only if there exist operators B, C, D such that $\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in\mathscr{I}$. It is obvious that $\mathscr{J}$ is a linear space, and is closed under compression and under unitary equivalence, and so it is a two-sided ideal by Lemma 2. In view of Lemma 1, we need only show that every operator in $\mathcal{J}$ is Hilbert—Schmidt. Let Y be a "half" of X which has finite measure and we may assume that $m(Y)=1$. If $K\in\mathcal{J}$, then as before, every operator on $L^2(Y)$ with the same s-numbers as K must be an integral operator. An examination of the remainder of the proof of Theorem 1 shows that it depends on the two properties of Y which we now list and prove.

(i) There exists an orthonormal basis of $L^2(Y)$ consisting of unimodular functions.

Proof. There is an isomorphism of the measure algebra of (Y, m) onto the measure algebra of the unit interval [4, p. 173]. This isomorphism induces a linear map V of the linear space of equivalence classes of measurable functions on [0, 1] onto the space of equivalence classes of measurable functions on Y (see [2, pp. 252—254] for details). This map can be seen to carry the exponentials ($\exp 2\pi inx$) into a basis of $L^2(Y)$ consisting of unimodular functions.

(ii) If $\{\alpha_n\}$ is a sequence of positive numbers such that $\sum \alpha_n=1$, then there is a sequence of disjoint measurable subsets of Y whose union is Y and such that $m(E_n)=\alpha_n$.

Proof. Again this follows immediately from the isomorphism of the measure algebras.

This ends the proof of Theorem 2.

Corollary. *The conclusion of Theorem 2 is valid if X contains atoms but is not purely atomic.*

Proof. Let Y be a half of X which contains no atoms, and let T be an operator which is always integral on X. Every compression of T is always integral on Y, hence is Hilbert—Schmidt by Theorem 2. Therefore T must be Hilbert—Schmidt by Lemma 1.

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