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A Jordan form for certain infinite-dimensional operators

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We derive several theorems about invariant subspaces of operator algebras of finite strict multiplicity. We generalize a result of EMBRY [3] to show that such algebras have maximal invariant subspaces (Theorem 2), and we prove some related theorems. The main results are Theorems 5 and 6, which give a "Jordan form" for operators which inherit finite strict multiplicity.

The first theorem is a slight sharpening of its corollary, which is due to HERRERO [7]. Herrero's result generalizes LAMBERT's result for strictly cyclic algebras [11] to algebras of finite strict multiplicity. Our proof combines ideas from HERRERO [7] and from RADJAVI and ROSENTHAL [18].

We will use the following notation throughout this article. We use $\mathfrak{H}$ to denote a separable Hilbert space, and $\mathscr{B}(\mathfrak{H})$ is the algebra of all bounded, linear operators on $\mathfrak{H}$. If $\mathscr{A}$ is a subalgebra of $\mathscr{B}(\mathfrak{H})$ or if T is an operator in $\mathscr{B}(\mathfrak{H})$, Lat $\mathscr{A}$ or Lat T denotes the lattice of invariant subspaces of $\mathscr{A}$ or of T. $\mathscr{A}(T)$ will be used for the subalgebra of $\mathscr{B}(\mathfrak{H})$ generated by T and the identity. Finally, if $\mathfrak{M}$ and $\mathfrak{N}$ are subsets of $\mathfrak{H}$, $\mathfrak{M}\vee\mathfrak{N}$ is the closed linear span of $\mathfrak{M}$ and $\mathfrak{N}$.

Recall that a subalgebra $\mathscr{A}$ of $\mathscr{B}(\mathfrak{H})$ has *finite strict multiplicity* if there is a finite collection of vectors $\{x_1, x_2, \ldots, x_n\}$ such that

$$\{A_1x_1+A_2x_2+\ldots+A_nx_n\colon A_i\in\mathscr{A}\} = \mathfrak{H}.$$

In that case, $\{x_1, x_2, \ldots, x_n\}$ is called an *FSM set* for $\mathscr{A}$, and the minimal cardinality of all such sets of vectors is called the *strict multiplicity* of $\mathscr{A}$. If $\mathscr{A}$ has strict multiplicity 1, $\mathscr{A}$ is said to be *strictly cyclic*. The operator T has *finite strict multiplicity* if $\mathscr{A}(T)$ does, and T is *strictly cyclic* if $\mathscr{A}(T)$ is. $\mathscr{A}$ is said to *inherit finite strict multiplicity* if the uniform closure of its restriction to every invariant subspace has finite strict multiplicity, and $\mathscr{A}$ is said to be *hereditarily strictly cyclic* if the uniform closure of its restriction to every invariant subspace is strictly cyclic. We will reserve the terms strictly cyclic and finite strict multiplicity for infinite-dimensional operators

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(unless we are talking about the restriction of an infinite-dimensional operator to a finite-dimensional invariant subspace).

Theorem 1. *Let* $\mathscr{A}$ *be a uniformly closed subalgebra of* $\mathscr{B}(\mathfrak{H})$ *containing the identity, and let* $\mathfrak{M} \in \operatorname{Lat} \mathscr{A}$ *be such that* $\mathscr{A}|\mathfrak{M}$ *has finite strict multiplicity; assume that* $\{x_1, x_2, \ldots, x_n\}$ *is an FSM set for* $\mathscr{A}|\mathfrak{M}$. *Then every invariant linear manifold of* $\mathscr{A}|\mathfrak{M}$ *whose closure contains the vector* $x_1+x_2+\ldots+x_n$ *is closed.*

Proof. Let $\tilde{\mathscr{A}}$ be the (uniformly closed) algebra of all $n\times n$ matrices with entries from $\mathscr{A}$, and define $\varphi\colon \tilde{\mathscr{A}} \to \mathfrak{M}^{(n)}$ by

$$\varphi(A_{ij}) = (A_{ij})\bar{x}$$

where $\bar{x}=(x_1, x_2, \ldots, x_n)$. Then φ is obviously bounded, and φ is onto since $\mathscr{A}$ has finite strict multiplicity on $\mathfrak{M}$.

Let $\mathfrak{N}$ be an invariant linear manifold of $\mathscr{A}|\mathfrak{M}$ whose closure contains x_1+ $+x_2+\ldots+x_n$, let $\tilde{\mathscr{B}}=\varphi^{-1}(\overline{\mathfrak{N}}^{(n)})$, and let $\tilde{\mathscr{N}}=\varphi^{-1}(\mathfrak{N}^{(n)})$. If $\tilde{\mathscr{N}}=\tilde{\mathscr{B}}$, then $\mathfrak{N}=\overline{\mathfrak{N}}$ since $\varphi(\tilde{\mathscr{N}})=\varphi(\tilde{\mathscr{B}})=\overline{\mathfrak{N}}^{(n)}$. We show that $\tilde{\mathscr{N}}=\tilde{\mathscr{B}}$ by assuming that $\tilde{\mathscr{N}}\neq\tilde{\mathscr{B}}$ and finding a contradiction.

Since we are assuming $\tilde{\mathscr{N}}\neq\tilde{\mathscr{B}}$, the invariance of $\mathfrak{N}$ implies that $\tilde{\mathscr{N}}$ is a proper left ideal in $\tilde{\mathscr{B}}$. Also, $1\in\tilde{\mathscr{B}}$ since $\sum x_i \in \overline{\mathfrak{N}}$; i.e.,

$$1\bar{x} = \left(\sum x_i, \sum x_i, \sum x_i, \ldots, \sum x_i\right),$$

so $1\bar{x}\in\overline{\mathfrak{N}}^{(n)}$. Since $\tilde{\mathscr{N}}$ is a proper ideal, 1 is not in the closure of $\tilde{\mathscr{N}}$, and so $\tilde{\mathscr{N}}$ is not dense in $\tilde{\mathscr{B}}$. Now, let $\mathscr{U}\subset\tilde{\mathscr{B}}$ be an open set such that $\mathscr{U}\cap\tilde{\mathscr{B}}=\emptyset$. Then $\varphi(\mathscr{U})$ is open by the open mapping theorem, and $\varphi(\mathscr{U})\cap\mathfrak{N}^{(n)}=\emptyset$. But then $\mathfrak{N}^{(n)}$ is not dense in $\overline{\mathfrak{N}}^{(n)}$. But of course $\overline{\mathfrak{N}^{(n)}}=\overline{\mathfrak{N}}^{(n)}$, giving a contradiction.

The following special case of this theorem is the basis for many important known results, some of which are listed below.

Corollary 1. (Herrero [7]) *A uniformly closed algebra of finite strict multiplicity has no dense invariant linear manifolds other than* $\mathfrak{H}$.

Corollary 2. (Embry [3]) *If* $\mathscr{A}$ *is a uniformly closed algebra of finite strict multiplicity, and if* x_0 *is a cyclic vector for* $\mathscr{A}$, *then* x_0 *is a strictly cyclic vector.*

Proof. $\{Ax_0\colon A\in\mathscr{A}\}$ is dense since x_0 is a cyclic vector, and hence is all of $\mathfrak{H}$ by Corollary 1.

One of the best known unsolved problems in operator theory is the transitive algebra problem. Recall that an operator algebra is *transitive* if its only invariant subspaces are $\{0\}$ and $\mathfrak{H}$. The problem is whether $\mathscr{B}(\mathfrak{H})$ is the only (weakly closed) transitive algebra? Many partial results have been obtained, beginning with Arve-

SON's work [1]. An affirmative answer would imply that every operator has a nontrivial invariant subspace — see [18, Chapter 8]. Finding results such as the following appears to have been the main goal of LAMBERT [11], [13] and HERRERO [7], [8] in studying algebras of finite strict multiplicity.

Corollary 3. (HERRERO [7]) *The only weakly closed transitive algebra of finite strict multiplicity is* $\mathscr{B}(\mathfrak{H})$.

Proof. Let $\mathscr{A}$ be a transitive, weakly closed algebra of finite strict multiplicity. Since $\mathscr{A}$ is transitive, every invariant linear manifold of $\mathscr{A}$ (other than $\{0\}$) must be dense in $\mathfrak{H}$. Hence, by Corollary 1, $\{0\}$ and $\mathfrak{H}$ are the only invariant linear manifolds of $\mathscr{A}$. The Rickart—Yood theorem (cf. [18, Corollary 8.5]) then implies that $\mathscr{A}=\mathscr{B}(\mathfrak{H})$.

Lemma 1. *If $\mathscr{A}$ is an algebra of strict multiplicity n, and if $\mathfrak{M}\in\operatorname{Lat}\mathscr{A}$, then the compression of $\mathscr{A}$ to $\mathfrak{M}^\perp$ has strict multiplicity at most n.*

Proof. Let P be the projection onto $\mathfrak{M}^\perp$, and let $e_1, e_2, e_3, \ldots, e_n$ be vectors such that

$$\{A_1e_1+A_2e_2+\ldots+A_ne_n\colon A_i\in\mathscr{A}\} = \mathfrak{H}.$$

Let $\tilde{\mathscr{A}}=\{PA\colon A\in\mathscr{A}\}$, and let $f_i=Pe_i$, $m_i=e_i-f_i$. It suffices to show that

$$\{\tilde{A}_if_1+\tilde{A}_2f_2+\ldots+\tilde{A}_nf_n\colon \tilde{A}_i\in\tilde{\mathscr{A}}) = \mathfrak{M}^\perp.$$

Note that $PAm_i=0$ for every $A\in\mathscr{A}$ since $\mathfrak{M}\in\operatorname{Lat}\mathscr{A}$. Given $x\in\mathfrak{M}^\perp$, choose $\{A_1, A_2, \ldots, A_n\}\subset\mathscr{A}$ such that $x=A_1e_1+A_2e_2+\ldots+A_ne_n$. Then

$$PA_1f_1+\ldots+PA_nf_n = PA_1(f_1+m_1)+\ldots+PA_n(f_n+m_n) = Px = x.$$

In the above proof we found n vectors to prove that $\tilde{\mathscr{A}}$ had finite strict multiplicity. Some of these vectors might be 0. In the strictly cyclic case, since the strict multiplicity does not increase, the compression algebra will also be strictly cyclic. This proves

Corollary. *If $\mathscr{A}$ is a strictly cyclic algebra, and if $\mathfrak{M}\in\operatorname{Lat}\mathscr{A}$, the compression of $\mathscr{A}$ to $\mathfrak{M}^\perp$ is strictly cyclic.*

EMBRY [3, Theorem 2] proves that every intransitive strictly cyclic algebra has a maximal invariant subspace. The next theorem is a generalization of Embry's theorem.

Theorem 2. *If $\mathscr{A}$ is an algebra of finite strict multiplicity, then every (proper) invariant subspace of $\mathscr{A}$ is contained in a (proper) maximal invariant subspace.*

Proof. Since the uniform closure of $\mathscr{A}$ has the same invariant subspace lattice as $\mathscr{A}$, we can assume that $\mathscr{A}$ is closed. Let $\mathfrak{M}\in\operatorname{Lat}\mathscr{A}$ with $\mathfrak{M}\neq\mathfrak{H}$. By the Hausdorff Maximality Principle there exists a maximal chain $\{\mathfrak{M}_\alpha\}$ of proper invariant subspaces containing $\mathfrak{M}$. Choose a countable dense subset $\{x_i\colon i=1,2,\dots,\infty\}$ of $\bigcup_\alpha \mathfrak{M}_\alpha$, and choose $\mathfrak{M}_{\alpha_i}$ so that $x_i\in\mathfrak{M}_{\alpha_i}$. Then

$$\overline{\bigcup\mathfrak{M}_{\alpha_i}} = \overline{\{x_i\}} = \overline{\bigcup\mathfrak{M}_\alpha}.$$

If $\overline{\bigcup\mathfrak{M}_{\alpha_i}}=\mathfrak{H}$, $\bigcup\mathfrak{M}_{\alpha_i}$ is dense in $\mathfrak{H}$. By Corollary 1 to Theorem 1, $\bigcup\mathfrak{M}_{\alpha_i}=\mathfrak{H}$. By the Baire Category Theorem, some $\mathfrak{M}_{\alpha_i}=\mathfrak{H}$, which is impossible. Thus, $\overline{\bigcup\mathfrak{M}_{\alpha_i}}\neq\mathfrak{H}$, and so $\overline{\bigcup\mathfrak{M}_\alpha}$ is a maximal invariant subspace containing $\mathfrak{M}$.

For a large class of algebras of finite strict multiplicity, maximal invariant subspaces have co-dimension 1.

Theorem 3. *If $\mathscr{A}$ is an algebra of finite strict multiplicity such that for every $\mathfrak{M}\in\operatorname{Lat}\mathscr{A}$, the compression of $\mathscr{A}$ to $\mathfrak{M}^\perp$ is not strongly dense in $\mathscr{B}(\mathfrak{M}^\perp)$, then every maximal invariant subspace of $\mathscr{A}$ has co-dimension* 1.

Proof. Let $\mathfrak{M}$ be a maximal invariant subspace. If the co-dimension of $\mathfrak{M}$ is greater than 1, the compression of $\mathscr{A}$ to $\mathfrak{M}^\perp$ has a non-trivial invariant subspace $\mathfrak{N}$. This follows from Corollary 3 to Theorem 1 since the compression is not $\mathscr{B}(\mathfrak{M}^\perp)$ by hypothesis. If we show that $\mathfrak{N}\oplus\mathfrak{M}\in\operatorname{Lat}\mathscr{A}$, we will be done since this will contradict the maximality of $\mathfrak{M}$.

Since $\mathfrak{M}\in\operatorname{Lat}\mathscr{A}$, it is enough to show that if $y\in\mathfrak{N}$, then $Ay\in\mathfrak{N}\oplus\mathfrak{M}$ for every $A\in\mathscr{A}$. So let $P_\mathfrak{M}$ and $P_{\mathfrak{M}^\perp}$ be the projections onto $\mathfrak{M}$ and $\mathfrak{M}^\perp$, respectively. Then

$$Ay = (P_{\mathfrak{M}^\perp}+P_\mathfrak{M})Ay = P_{\mathfrak{M}^\perp}Ay+P_\mathfrak{M}Ay.$$

Note that $P_{\mathfrak{M}^\perp}\mathscr{A}$ is in the compression algebra, so $P_{\mathfrak{M}^\perp}Ay\in\mathfrak{N}$. And of course $P_\mathfrak{M}Ay\in\mathfrak{M}$.

Corollary. *If $\mathscr{A}$ is an Abelian algebra of finite strict multiplicity, then every invariant subspace of $\mathscr{A}$ is contained in an invariant subspace of co-dimension* 1.

Proof. This follows immediately from the previous two theorems since $\mathscr{A}$ being Abelian guarantees that the compression of $\mathscr{A}$ to $\mathfrak{M}^\perp$ is also Abelian and hence not strongly dense in $\mathscr{B}(\mathfrak{M}^\perp)$.

We can even say more about algebras generated by certain strictly cyclic operators.

Lemma 2. *If T is a strictly cyclic operator, and if $\sigma(T)$ is a singleton, then T has a unique maximal invariant subspace.*

Proof. The Corollary to Theorem 3 implies that T has a maximal invariant subspace. Suppose that T has two distinct maximal invariant subspaces $\mathfrak{M}_1$ and $\mathfrak{M}_2$. Let $\mathfrak{M}=\mathfrak{M}_1\cap\mathfrak{M}_2$, and choose unit vectors $e_1\in\mathfrak{M}_1^\perp\cap\mathfrak{M}_2$ and $e_2\in\mathfrak{M}_1\cap\mathfrak{M}_2^\perp$. Since $\mathfrak{M}_1$ and $\mathfrak{M}_2$ have co-dimension 1 by the last corollary, $\mathfrak{M}^\perp=\vee\{e_1, e_2\}$. Let

$$T=\begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

be the decomposition of T with respect to $\mathfrak{M}\oplus\mathfrak{M}^\perp$.

Now, T_3 has one-point spectrum since T does, and T_3 is strictly cyclic by Corollary 1 to Theorem 1. Since a strictly cyclic operator with one-point spectrum on a finite-dimensional space is similar to a unilateral shift, T_3 has a one-dimensional eigenspace. Thus e_1 or e_2 is not an eigenvector; suppose e_1 is not. Since $e_1\in\mathfrak{M}_2$, $Te_1\in\mathfrak{M}_2$; i.e.,

$$\begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}\begin{pmatrix} 0 \\ e_1 \end{pmatrix} = (T_2 e_1, T_3 e_1)\in\mathfrak{M}_2.$$

So $T_3 e_1\in\mathfrak{M}_2$, and $T_3 e_1\in\mathfrak{M}$; i.e., $T_3 e_1\in\mathfrak{M}_2\cap\mathfrak{M}^\perp$. But $\mathfrak{M}_2\cap\mathfrak{M}^\perp=\vee\{e_1\}$, which shows that e_1 is an eigenvector of T_3. This contradiction completes the proof.

Corollary. *If T is a strictly cyclic operator whose spectrum is a singleton, if $\mathfrak{M}\in\operatorname{Lat} T$ and is the unique maximal invariant subspace of T, and if $e\in\mathfrak{M}^\perp$, $e\neq 0$, then e is a strictly cyclic vector for $\mathscr{A}(T)$.*

Proof. Since e is not contained in any proper invariant subspace, it is a cyclic vector. By Corollary 2 to Theorem 1, e is a strictly cyclic vector.

Theorem 4. *Let T be a strictly cyclic operator with one-point spectrum. Let $\mathfrak{M}$ be its maximal invariant subspace, let $e\in\mathfrak{M}^\perp$, $e\neq 0$, and let $B\in\mathscr{A}(T)$. Then*

(i) *$Be\in\mathfrak{M}$ if and only if $\mathfrak{R}(B)\subset\mathfrak{M}$.**)

(ii) *x is a strictly cyclic vector if and only if $(x, e)\neq 0$.*

(iii) *B is invertible if and only if $(Be, e)\neq 0$.*

(iv) *Every operator in $\mathscr{A}(T)$ has one-point spectrum.*

Proof. By the Corollary to Lemma 2, e is a strictly cyclic vector. Let $\mathscr{A}=\mathscr{A}(T)$.

(i) If $\mathfrak{R}(B)\subset\mathfrak{M}$, trivially $Be\in\mathfrak{M}$. If $Be=x\in\mathfrak{M}$, since $\mathscr{A}e=\mathfrak{H}$, we have $\mathfrak{R}(B)=B\mathscr{A}e=\mathscr{A}Be=\mathscr{A}x\subset\mathfrak{M}$.

(ii) If x is a strictly cyclic vector, $x\notin\mathfrak{M}$, so $(x, e)\neq 0$. If $(x, e)\neq 0$, $\mathscr{A}x\not\subset\mathfrak{M}$, so $\mathscr{A}x=\mathfrak{H}$ (since every element of $\operatorname{Lat}\mathscr{A}$ is contained in $\mathfrak{M}$).

(iii) If B is invertible, $\mathfrak{R}(B)\not\subset\mathfrak{M}$, so $(Be, e)\neq 0$ by (i).

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*) $\mathfrak{R}$ denotes "range".

If $(Be, e) \neq 0$, $\mathfrak{R}(B) \not\subset \mathfrak{M}$. Thus $\mathfrak{R}(B)$ is dense in $\mathfrak{H}$, and $\mathfrak{R}(B)$ is invariant under $\mathfrak{A}$. Hence $\mathfrak{R}(B) = \mathfrak{H}$. By a theorem of LAMBERT [11, Lemma 3.1] every point in the spectrum of B is compression spectrum. Thus B must be invertible.

(iv) Let $Be = \alpha e + m$, where $m \in \mathfrak{M}$. If $\lambda \neq \alpha$, then $((B-\lambda)e, e) \neq 0$. Thus, $B-\lambda$ is invertible, so $\lambda \notin \sigma(B)$; i.e., $\sigma(B) = \{\alpha\}$.

LAMBERT [11] proved that a unilateral shift whose weights are p-summable and decrease monotonically to 0 is strictly cyclic. Since such an operator has Donoghue lattice, and since that property is trivially inherited, such an operator is hereditarily strictly cyclic. The class of hereditarily strictly cyclic operators is much wider than this. For example, if S is any quasinilpotent hereditarily strictly cyclic shift with Donoghue lattice, let $T = S \oplus (S+1)$. Since the full spectra of S and $S+1$ are disjoint, $\mathscr{A}(T) = \mathscr{A}(S) \oplus \mathscr{A}(S+1)$. (Consider the Riesz decomposition.) Thus if e is a strictly cyclic vector for $\mathscr{A}(S)$, $e \oplus e$ is obviously a strictly cyclic vector for $\mathscr{A}(T)$.

An example of HEDLUND [6] shows that even for $\mathscr{A} = \mathscr{A}(T)$ where T is a unilateral weighted shift, $\mathscr{A}$ being strictly cyclic does not in general imply that $\mathscr{A}$ is hereditarily strictly cyclic. Thus, in the theorems that follow, we cannot remove the "hereditary" part of the hypothesis.

In the case of an hereditarily strictly cyclic operator with one-point spectrum, we can describe its invariant subspaces in some detail. This is done in the following theorem, which generalizes the well-known fact that an operator on a finite-dimensional space is unicellular if and only if it is cyclic and has one-point spectrum (see [18, Theorem 4.7]). There are operators which are unicellular but have spectra containing more than one point — see [4].

Theorem 5. *Let T be strictly cyclic. If T is unicellular, then $\sigma(T)$ is one point. Conversely, if T is hereditarily strictly cyclic and if $\sigma(T)$ is one point, then T has Donoghue lattice* (i.e. there is an orthonormal basis $\{e_n\}_{n=0}^{\infty}$ such that the non-trivial invariant subspaces of T are the subspaces $\mathfrak{M}_k = \bigvee_{j=k}^{\infty} e_j$ for positive integers k).

Proof. Every point in $\sigma(T^*)$ is an eigenvalue of T^* since every point in $\sigma(T)$ is compression spectrum. If T^* had two linearly independent eigenvectors, T would have two non-comparable invariant subspaces. Thus T unicellular implies that $\sigma(T)$ is a singleton.

Conversely, suppose that T is hereditarily strictly cyclic, and that $\sigma(T) = \{\lambda_0\}$. Let $S = T - \lambda_0$. Since $\operatorname{Lat} S = \operatorname{Lat} T$, it suffices to show that S has Donoghue lattice.

Let $\mathfrak{M}_0 = \mathfrak{H}$. If $\mathfrak{M}_k$ has been defined, let $\mathfrak{M}_{k+1}$ be the unique maximal invariant subspace of $S|\mathfrak{M}_k$. Since 0 is compression spectrum for $S|\mathfrak{M}_k$ for each k, $S(\mathfrak{M}_k) \subset$

$\subset \mathfrak{M}_{k+1}$. Choose a unit vector $e_0 \in \mathfrak{M}_1^\perp$. Then e_0 is a strictly cyclic vector, and $S^n e_0 \in \mathfrak{M}_n$.

Let $S^n e_0 = e_n + m_{n+1}$ where $e_n \in \mathfrak{M}_{n+1}^\perp$ and $m_{n+1} \in \mathfrak{M}_{n+1}$. Since S is strictly cyclic, $\{S^n e_0 : n=0, 1, 2, \ldots\}$ spans $\mathfrak{H}$. Hence, $\mathfrak{E} = \{e_n : n=0, 1, 2, \ldots\}$ is an orthogonal spanning set. Each $e_i \notin \mathfrak{M}_j$ if $j>i$, so $\mathfrak{E} \perp (\bigcap \mathfrak{M}_n)$; i.e., $\bigcap \mathfrak{M}_n = \{0\}$. This shows that Lat T contains a Donoghue lattice. To complete the proof, we must show that there are no other invariant subspaces.

Let $\mathfrak{M}$ be any invariant subspace of T, and let n be the largest index such that $\mathfrak{M} \subset \mathfrak{M}_n$. (Since $\mathfrak{M} \subset \mathfrak{M}_0$ and since $\bigcap \mathfrak{M}_i = \{0\}$, such an n exists). If $\mathfrak{M} \neq \mathfrak{M}_n$, $\mathfrak{M}_{n+1}$ is the unique maximal invariant subspace of $T|\mathfrak{M}_n$, and so $\mathfrak{M} \subset \mathfrak{M}_{n+1}$, a contradiction. So $\mathfrak{M} = \mathfrak{M}_n$, and we are done.

Shields' article [20] contains many of the known results about weighted shifts. It includes some discussion of strictly cyclic shifts. The invariant subspace lattice of every weighted shift obviously contains the Donoghue subspaces, but there may be other invariant subspaces. Shields defines a shift to be *strongly strictly cyclic* if its restriction to each of its Donoghue subspaces is strictly cyclic. This definition is somewhat weaker than hereditarily strictly cyclic. Shields proves [20, Prop. 38] that a quasinilpotent strongly strictly cyclic shift is unicellular. Although the statement of Theorem 5 does not include Shields' theorem, its proof obviously yields his result too. (The proof in [20] depends on calculations with the weights.)

Theorem 5 suggests that hereditarily strictly cyclic operators might serve as "generalized Jordan blocks", and that operators which inherit finite strict multiplicity may have a "Jordan form" in some sense. This is the case, and it is proven in Theorem 6 below. We require two lemmas.

Lemma 3. *Let T be a strictly cyclic operator whose spectrum is a singleton, and suppose that T inherits finite strict multiplicity. Then T is hereditarily strictly cyclic.*

Proof. Let $\sigma(t) = \{\lambda\}$ and let $S = T - \lambda$. It suffices to prove the theorem for S. By Lemma 2, S has a unique maximal invariant subspace $\mathfrak{M}_1$. As in the proof of Theorem 5, let e_0 be a unit vector orthogonal to $\mathfrak{M}_1$. Then e_0 is a strictly cyclic vector by Theorem 4, and $e_1 = Se_0$ is in $\mathfrak{M}_1$ since 0 is compression spectrum for S. Since S is strictly cyclic, $\{S^n e_0 : n=0, 1, 2, \ldots\}$ spans $\mathfrak{H}$, and so $\{S^n e_0 : \mathrm{n}=1, 2, 3, \ldots\} = \{S^n e_1 : n=0, 1, 2, \ldots\}$ spans $\mathfrak{M}_1$; i.e. e_1 is a cyclic vector for $S|\mathfrak{M}_1$. Hence, by Corollary 2 to Theorem 1, e_1 is a strictly cyclic vector since $S|\mathfrak{M}_1$ has finite strict multiplicity.

We now proceed as we did in the proof of Theorem 5 to construct a sequence of invariant subspaces $\mathfrak{M}_n$ such that $S|\mathfrak{M}_n$ is strictly cyclic. At each step, $S|\mathfrak{M}_n$ strictly cyclic implies the existence of a maximal invariant subspace $\mathfrak{M}_{n+1}$, and then $S|\mathfrak{M}_{n+1}$ will be strictly cyclic. To complete the proof, we must show that $\bigcap \mathfrak{M}_n =$

$=\{0\}$, and that S has no other invariant subspaces. But this follows exactly as in the proof of Theorem 5.

Lemma 4. *Let $\{x, y\}$ be an FSM set for the operator T, and suppose that T inherits finite strict multiplicity, and that $\sigma(T)=\{0\}$. Let $\mathfrak{M}=\overline{\mathscr{A}(T)x}$ and $\mathfrak{N}=\overline{\mathscr{A}(T)y}$, and assume that $\mathfrak{M}$ is infinite-dimensional. Then either $\mathfrak{M}\cap\mathfrak{N}=\{0\}$ or there exists a finite-dimensional invariant subspace $\mathfrak{K}$ of T complementary to $\mathfrak{M}$ such that $\mathfrak{M}\vee\mathfrak{K}=$ $=\mathfrak{M}\vee\mathfrak{N}=\mathfrak{H}$.*

Remark. The assumption that $\mathfrak{M}$ is infinite-dimensional is for convenience. If both $\mathfrak{M}$ and $\mathfrak{N}$ are finite-dimensional, since $T|\mathfrak{M}$ will then be cyclic and nilpotent, the lemma reduces to a well-known finite-dimensional theorem (see [5, Theorem 1, 57]).

Proof. Note first that $\mathfrak{M}\vee\mathfrak{N}=\mathfrak{H}$ since every vector in $\mathfrak{H}$ has the form $Ax+By$ where $A, B\in\mathscr{A}(T)$. By Corollary 2 to Theorem 1 and by Lemma 3, $T|\mathfrak{M}$ is hereditarily strictly cyclic, and we may apply Theorem 5. So let $\mathfrak{M}_0, \mathfrak{M}_1, \mathfrak{M}_2, \ldots$ be the non-zero invariant subspaces of $T|\mathfrak{M}$ in decreasing order ($\mathfrak{M}_0=\mathfrak{M}$). If $\mathfrak{N}$ were finite-dimensional, then $\mathfrak{M}\cap\mathfrak{N}=\{0\}$ since $T|\mathfrak{M}$ has no finite-dimensional invariant subspaces. So assume that $\mathfrak{N}$ is infinite-dimensional and that the non-zero invariant subspaces of $T|\mathfrak{N}$ are $\mathfrak{N}_0, \mathfrak{N}_1, \mathfrak{N}_2, \ldots$ in decreasing order.

Now, if $\mathfrak{M}\cap\mathfrak{N}=\{0\}$, we are done. If not, $\mathfrak{M}\cap\mathfrak{N}=\mathfrak{M}_k=\mathfrak{N}_m$ for some k and m. Thus, $\mathfrak{M}\cap\mathfrak{N}$ has finite co-dimension α in $\mathfrak{H}$. We proceed by induction on α. If $\alpha=0$, then $\mathfrak{M}=\mathfrak{N}=\mathfrak{H}$, and $\mathfrak{K}=\{0\}$ does the trick. So assume true for $\alpha=n-1$, and consider $\alpha=n$.

Since $\mathfrak{M}\cap\mathfrak{N}=\mathfrak{N}_m$, if $\mathfrak{M}\cap\mathfrak{N}\not\subset\mathfrak{N}_1$, then $m=0$ and $\mathfrak{N}\subset\mathfrak{M}$, and again $\mathfrak{K}=\{0\}$ suffices. So assume $\mathfrak{M}\cap\mathfrak{N}\subset\mathfrak{N}_1$, and let y_1 be a unit vector in $\mathfrak{N}_1\ominus\mathfrak{N}_2$. Then y_1 is a cyclic vector for $T|\mathfrak{N}_1$. (In fact, y_1 is a strictly cyclic vector.) Thus, $\{x, y_1\}$ is an FSM set for $\mathfrak{M}\vee\mathfrak{N}_1$, and $\mathfrak{M}\cap\mathfrak{N}=\mathfrak{M}\cap\mathfrak{N}_1\subset\mathfrak{M}\vee\mathfrak{N}_1$. Moreover, the co-dimension of $\mathfrak{M}\cap\mathfrak{N}$ in $\mathfrak{M}\vee\mathfrak{N}_1$ is exactly $n-1$, and the inductive hypothesis applies. So choose a finite-dimensional invariant subspace $\mathfrak{K}_0$ of T complementary to such that $\mathfrak{M}\vee\mathfrak{K}_0=\mathfrak{M}\vee\mathfrak{N}_1$, and let $\mathfrak{K}=\{z\in\mathfrak{H}\colon Tz\in\mathfrak{K}_0\}$. We will show that $\mathfrak{K}$ has the desired properties.

First, if $z\in\mathfrak{M}$ and $z\neq 0$, then $Tz\notin\mathfrak{K}_0$ since $T|\mathfrak{M}$ has no finite-dimensional invariant subspaces. Hence, since the co-dimension of $\mathfrak{M}$ in $\mathfrak{H}$ is finite, $\mathfrak{K}$ is finite-dimensional, and since 0 is the only point in the spectrum of T, $T|\mathfrak{K}$ is nilpotent. Thus, the dimension of $\mathfrak{K}$ is greater than the dimension of $\mathfrak{K}_0$ since $T\mathfrak{K}\subset\mathfrak{K}_0$. Hence, the co-dimension of $\mathfrak{M}\vee\mathfrak{K}$ is less than the co-dimension of $\mathfrak{M}\vee\mathfrak{K}_0$, i.e. $\mathfrak{M}\vee\mathfrak{K}$ must be all of $\mathfrak{H}$. Finally, $T\mathfrak{K}\subset\mathfrak{K}_0\subset\mathfrak{K}$, so $\mathfrak{K}$ is an invariant subspace of T. The proof is complete.

The purpose of Lemma 4 is of course Theorem 6 below. Lemma 4 yields Theorem 6 fairly easily. But first, define the operator T to be a *Jordan operator* if T has n complementary invariant subspaces $\mathfrak{M}_1, \mathfrak{M}_2, \ldots, \mathfrak{M}_n$ such that $\mathfrak{H}$ is the (not necessarily orthogonal) direct sum of the $\mathfrak{M}_i$'s, and such that either the matrix of $T|\mathfrak{M}_i$ is a (finite-dimensional) Jordan block, or $T|\mathfrak{M}_i$ has Donoghue lattice.

Theorem 6. *Let T be an operator on $\mathfrak{H}$ which inherits finite strict multiplicity, and whose spectrum is finite. Then T is a Jordan operator.*

Proof. Let $\sigma(T)=\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$. Then T has k complementary invariant subspaces $\mathfrak{H}_1, \mathfrak{H}_2, \ldots, \mathfrak{H}_k$ whose span is all of $\mathfrak{H}$ such that $\sigma(T|\mathfrak{H}_i)=\{\lambda_i\}$. It suffices to prove the theorem for each $T|\mathfrak{H}_i$, so assume that $\mathfrak{H}_1=\mathfrak{H}$ and $\lambda_1=0$ (otherwise, consider $T-\lambda_1$).

We now proceed by induction on n the strict multiplicity of T. For $n=1$, Theorem 5 applies. So assume true for strict multiplicity $n-1$, and let $\{x_1, x_2, \ldots, x_n\}$ be an FSM set for T. Let $\mathfrak{N}_i=\overline{\mathscr{A}(T)x_i}$. By the inductive hypothesis, the theorem holds on $\bigvee_{i=1}^{n-1}\mathfrak{N}_i$. Let

$$\bigvee_{i=1}^{n-1}\mathfrak{N}_i=\bigvee_{j=1}^{m}\mathfrak{M}_j,$$

where the $\mathfrak{M}_j$'s are mutually complementary invariant subspaces of T, where $T|\mathfrak{M}_j$ has Donoghue lattice for $j<m$, and where $\mathfrak{M}_m$ is finite-dimensional. (We are thus throwing all of the finite-dimensional invariant subspaces of T into $\mathfrak{M}_m$. By the Jordan Canonical Form Theorem, this is equivalent to the above definition of Jordan operator.)

Now, if $\mathfrak{N}_n$ is finite-dimensional, since $T|\mathfrak{M}_j$ has Donoghue lattice for $j<m$, $\mathfrak{N}_n$ is necessarily complementary to $\mathfrak{M}_j$ for $j<m$. In that case, replacing $\mathfrak{M}_m$ by $\mathfrak{M}_m\vee\mathfrak{N}_n$ does the trick. If $\mathfrak{N}_n$ is infinite-dimensional, Lemma 4 applies to $\mathfrak{N}_n\vee\mathfrak{M}_1$. (If $m=1$, $\mathfrak{N}_n$ is complementary to $\mathfrak{M}_1$, and we are done.) So let $\mathfrak{N}_n\vee\mathfrak{M}_1=\mathfrak{K}\vee\mathfrak{M}_1$, where $\mathfrak{K}$ and $\mathfrak{M}$ are complementary, and where $\mathfrak{K}$ is a finite-dimensional invariant subspace of T. Then $\mathfrak{K}$ must be complementary to $\mathfrak{M}_1, \mathfrak{M}_2, \ldots, \mathfrak{M}_{m-1}$, and replacing $\mathfrak{M}_m$ by $\mathfrak{M}_m\vee\mathfrak{K}$ comples the proof.

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