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Mean ergodicity in G-semifinite von Neumann algebras

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Introduction. Let A be a von Neumann algebra in a complex Hilbert space H, and let G be a semigroup of normal endomorphisms of A. Denote by A^G the set of all elements of A which are invariant with respect to each element of G. If the identity I belongs to A^G, then A^G is a von Neumann algebra too, but if this isn't so, then A^G is 'only' an ultraweakly closed involutive subalgebra of A, and hence there exists a largest projection $P \neq I$ in A such that for every element T of A one has $PT=TP=T$ ([7], Chap. I. § 3, Théorème 2.).

Let Q denote the set of positive, normal, linear mappings of A into itself obtained from the elements of G by forming convex combinations. The operators in A of the form $V(T)$, where $V \in Q$ and $T \in A$ are called the means of the operator T. For any $T \in A$ let $K_0(T, G)$ denote the set of all means of T. The investigation of the 'behaviour' of the means is one of the subjects of mean ergodic theory ([9], Kap. 1, § 2.). Concerning von Neumann algebras we refer only to the classical results of J. Dixmier ([6]) and the paper of I. Kovács and J. Szűcs ([10]).

The purpose of this paper is to investigate a special class of von Neumann algebras.

§ 1 contains preliminary results without their proofs.

In § 2 we define the notion of 'weak ergodicity in means' to express a 'good behaviour' of the means of an operator. This section is devoted to establishing the simplest consequences of this definition.

Let $K(T, G)$ be the weak closure of $K_0(T, G)$. In § 3 we shall give sufficient conditions for T in order that $K(T, G) \cap A^G$ be nonempty (Theorem 3.1.), and that $K(T, G) \cap A^G$ consist of exactly one operator.

1. Definitions and preliminaries. Let us consider a pair (A, G) of a von Neumann algebra A and a semigroup G of normal endomorphisms of A. We shall denote by A^+ the positive portion of A.

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A non-negative, finite or infinite valued function φ defined on A^+ is called a weight on A^+, if it has the following properties:

(i) $$\varphi(T+S) = \varphi(T)+\varphi(S) \quad \text{for every} \quad T, S \in A^+; \quad \text{and}$$

(ii) $$\varphi(cT) = c\varphi(T) \quad \text{for every} \quad c \geqq 0 \quad \text{and} \quad T \in A^+,$$

(with the convention that $0 \cdot \infty = 0$).

We call φ G-invariant if for every $T \in A^+$ and $g \in G$ we have $\varphi(T) = \varphi(g(T))$.

The notion of a G-invariant weight is a very natural generalization of that of a trace.

A weight φ on A^+ is said to be faithful if the conditions $T \in A^+$ and $\varphi(T)=0$ imply $T=0$; normal if, for every increasing directed set $\mathcal{F} \subset A^+$ with $\sup_{S \in \mathcal{F}} S = T \in A^+$, we have $\varphi(T) = \sup_{S \in \mathcal{F}} \varphi(S)$; semi-finite if, for every $T \in A^+$, $T \neq 0$ there exists $S \in A^+$, $S \neq 0$ such that $S \leqq T$ and $\varphi(S) < \infty$.

A weight φ on A^+ is said to be non-infinite if there exists $S \in A^+$, $S \neq 0$ such that $\varphi(S) < \infty$.

For later purposes we state an important fact concerning weights.

Proposition 1.1. ([8], Lemma 1.5) *For any weight φ on A^+ the following conditions are equivalent:*

(i) *φ is normal,*

(ii) *φ is ultraweakly lower semicontinuous,*

(iii) *there exists a family of vectors $\{x_\iota\}$ in H such that*

$$\varphi(T) = \sum_\iota (Tx_\iota, x_\iota) \quad \textit{for every} \quad T \in A^+.$$

Now we shall define special subspaces of A. Denote by Γ the set of normal faithful G-invariant non-infinite and non-zero weights defined on A^+.

Definition 1.1. A *projection* $E \in A$ is called *finite*, if there is a $\varphi \in \Gamma$ such that $\varphi(E) < \infty$. An *operator* in A is called *simple*, if it is a linear combination of finite projections. Denote the set of simple operators by M_0.

Let $\varphi \in \Gamma$ and let $M_\varphi^+ = \{T \in A^+ | \varphi(T) < \infty\}$. Denote by M the smallest norm closed subspace of A that contains M_φ^+ for every $\varphi \in \Gamma$. Since φ defines a linear form $\dot{\varphi}$ on the linear span of M_φ^+, it is not hard to see that the norm closure of M_0 is identical with M.

Let $N_\varphi = \{T \in A | \varphi(T^* T) < \infty\}$. N_φ is a left ideal in A. Denote by N the norm closed linear hull of all N_φ. It is obvious that $M_0 \subseteqq N$ and hence $M \subseteqq N$.

Definition 1.2. A pair (A, G) is said to have *property Π* if for every proper projection $P \in A$ such that $g(P) \leqq P$ for every $g \in G$, we have that $P \in A^G$.

We classify the pairs (A, G) by their weights.

Definition 1.3. A *pair* (A, G) is called *finite* (resp. *semifinite*) if for every $T \in A^+$, $T \neq 0$ we can find a normal G-invariant finite (resp. semifinite) weight φ such that $\varphi(T) \neq 0$.

To facilitate the statement of the next proposition it will be convenient to introduce the following notations.

Definition 1.4. Let E be a projection in A^G. Let us consider the restricted von Neumann algebra A_E. Since $E \in A^G$, every element g of G induces a normal endomorphism g_E on A_E. These restricted endomorphisms form a semigroup. Let us denote this semigroup by G_E. The pair (A_E, G_E) is called a *restriction* of (A, G).

Proposition 1.2. ([5], Theorem 1) *If a pair (A, G) has property Π, then there exists a maximal projection E in A^G such that the restricted pair (A_E, G_E) is finite.*

For finite pairs the following theorem will play an important role in proving Theorem 3.3.

Theorem. (I. Kovács—J. Szűcs ([10])) *Let the pair (A, G) be finite. For every $T \in A$ the convex set $K(T, G) \cap A^G$ contains exactly one element.*

In the following paragraphs we shall deal with pairs (A, G) for which the set Γ is non-empty. This requirement is fulfilled for example in the classical case, when the group $\natural$ of inner automorphisms of A plays the role of G, and A is semifinite. We do not know if this is the case in general for semifinite pairs, but we can state the following:

Proposition 1.3. *If a semifinite pair (A, G) has property Π and $\natural \subset G$, then there exists a normal faithful G-invariant and semifinite weight on A^+.*

Property Π ensures that the support of any G-invariant weight defined on A^+ does belong to A^G. It follows from the condition $\natural \subset G$ that A^G is part of the center of A and hence Dixmier's reasoning ([7], Chap. 1, § 6, Proposition 9.) can be repeated essentially word by word.

The terms and symbols introduced here will be used in what follows without further reference.

2. Let $\mathscr{F}$ be an ultrafilter in Q. Denote by $\mathscr{F}(T)$ the image of $\mathscr{F}$ which is ultrafilter, too. Since the unit ball of A is weakly compact, $K(T, G)$ is weakly compact, too, for every $T \in A$, and so the ultrafilter $\mathscr{F}(T)$ of the means of T converges weakly to an element S of $K(T, G)$. Let this fact be expressed by the symbol $\lim_{\mathscr{F}} V(T) = S$.

Now we define two notions to express 'good behaviour' of the means of an operator.

Definition 2.1. Let the operator $T\in A$ be called *weakly quasi-ergodic* if it has the following properties:

(Li) $K(T, G)\cap A^G$ is non-empty

(Lii) for each $R\in K(T, G)$ the set $K(R, G)\cap A^G$ is non-empty.

Denote by L the subset of weakly quasi-ergodic elements of A.

Definition 2.2. Let the operator $T\in A$ be called *weakly ergodic* if it has the following properties:

(Ei) $K(T, G)\cap A^G$ consists of exactly one element,

(Eii) for each $R\in K(T, G)$ the set $K(R, G)\cap A^G$ consists of exactly one element.

Denote by E the subset of weakly ergodic elements of A. It is obvious that $A^G\subset\subset E\subset L$.

Proposition 2.1. *L is a norm closed, G-invariant subspace of A.*

Proof. The G-invariance and the homogeneity of L are rather obvious. First we prove the additivity of L. Let T_1 and T_2 be arbitrary elements of L. We shall show that the operator $T=T_1+T_2$ belongs to L. By assumption there is an operator S_1 such that $S_1\in K(T_1, G)\cap A^G$. Let $\mathcal{F}_1$ be an ultrafilter in Q such that $\lim\limits_{\mathcal{F}_1} V(T_1)=S_1$. The limits $\lim\limits_{\mathcal{F}_1} V(T)=S_0$ and $\lim\limits_{\mathcal{F}_1} V(T_2)=R_2$ exist, $S_0\in K(T, G)$ and $R_2\in K(T_2, G)$. By condition (Lii) there exists an ultrafilter $\mathcal{F}_2$ in Q such that $\lim\limits_{\mathcal{F}_2} V(R_2)=R\in K(R_2, G)\cap A^G$. It follows taking account of the facts that $S_0=S_1+R_2$ and $K(S_0, G)\subset K(T, G)$ that $S=\lim\limits_{\mathcal{F}_2} V(S_0)=S_1+R\in K(T, G)\cap A^G$.

Now let us consider an arbitrary element Y of $K(T, G)$. Then we can find an ultrafilter $\mathcal{F}$ in Q such that $Y=\lim\limits_{\mathcal{F}} V(T)$. The limits $\lim\limits_{\mathcal{F}} V(T_1)=Y_1$ and $\lim\limits_{\mathcal{F}} V(T_2)=Y_2$ exist, and both belong to L. Since $Y=Y_1+Y_2$, then using the previous result it is obvious that $K(Y, G)\cap A^G$ is non-empty, so we have finished proving that $T\in L$.

Now we are going to show that L is norm closed. Let the sequence $\{T_n\}$ of operators converge to the operator T uniformly. Let us suppose that for each n, $T_n\in L$. Passing, if necessary, to a subsequence, we can assume without loss of generality that $\|T_{n+1}-T_n\| < 1/2^{n+1}$ for each n.

Using the technique of the previous part of the present proof we can construct a sequence $\{S_n\}$ recursively in the following way:

$$S_n\in K(T_n, G)\cap A^G \quad\text{and}\quad S_{n+1}-S_n\in K(T_{n+1}-T_n, G)$$

for each n. It is an obvious consequence of these facts that the sequence $\{S_n\}$ converges in norm, and the limit S of it belongs to A^G.

Now we prove that for any $\varepsilon > 0$ and for any finite system of vectors $x_1, x_2, \ldots, x_k; y_1, y_2, \ldots, y_k$ of H we can find an operator $R \in K_0(T, G)$ such that

$$(*) \qquad |((S-R)x_i, y_i)| < \varepsilon \quad \text{for each} \quad i = 1, 2, \ldots, k.$$

Let us choose a sufficiently large index p, for which $\|S-S_p\|$ and $\|T-T_p\|$ are both sufficiently small. Since $S_p \in K(T_p, G)$, there exists a $V_0 \in Q$ such that $|((S_p - V_0(T_p))x_i, y_i)|$ is sufficiently small for each $i = 1, 2, \ldots, k$. Let $R = V_0(T)$. This operator satisfies $(*)$, and this means that $S \in K(T, G) \cap A^G$.

Now let us consider an arbitrary element Y of $K(T, G)$. We can find an ultrafilter $\mathscr{F}$ in Q such that $Y = \lim_{\mathscr{F}} V(T)$. Let us set $Y_n = \lim_{\mathscr{F}} V(T_n)$. It is clear that $Y_n \in L$ for every n, and that the sequence $\{Y_n\}$ converges in norm to Y. Applying the preceding part to the sequence $\{Y_n\}$, we get that $K(Y, G) \cap A^G$ is non-empty.

The next proposition might bear the name 'The Theorem of Linear Choice'.

Proposition 2.2. *For every $T_0 \in L$ and $S_0 \in K(T_0, G) \cap A^G$ we can find a positive linear mapping τ of L onto A^G which possesses the following properties:*

(i) $\tau(T) \in K(T, G)$ *for each* $T \in L$,

(ii) $\tau(TS) = \tau(T)S$ *and* $\tau(ST) = S\tau(T)$ *for every* $T \in L$ *and* $S \in A^G$,

(iii) $\tau(T_0) = S_0$.

We omit the proof. It can be done by J. T. Schwartz's method developed in ([11], Lemma 5).

Proposition 2.3. *The weakly ergodic elements of A form a norm closed, G-invariant subspace E of A. Denote by $\tau_0(T)$ the single element of $K(T, G) \cap A^G$ for every $T \in E$. The mapping τ_0 is positive linear and has the property that*

$$\tau_0(TS) = \tau_0(T)S \quad \textit{and} \quad \tau_0(ST) = S\tau_0(T) \quad \textit{for every } T \in E \textit{ and } S \in A^G.$$

Proof. The G-invariance of E is based upon the fact that for every $T \in A$ the elements of G map $K(T, G)$ into itself.

Denote by Λ the family of those linear mappings τ of L onto A^G which have properties (i) and (ii) of Proposition 2.2. Let τ and ψ be two arbitrary elements of Λ. Let us define the following subset

$$L_{\tau,\psi} = \{T \in L \mid \tau(T) = \psi(T)\}.$$

Taking into account the fact that every element of Λ is norm-continuous and linear it follows that $L_{\tau,\psi}$ is a norm closed subspace of A. Denote by L_0 the intersection of all such $L_{\tau,\psi}$ subspaces. It is obvious that L_0 is a norm closed subspace of A and by Proposition 2.2 it is identical with E.

If we restrict any τ occuring in Proposition 2.2 to E, then we get the mapping τ_0 with the desired properties.

3. In this section we shall investigate pairs (A, G) for which Γ is non-empty and hence the subspaces M and N defined in Definition 1.1. are different from the trivial subspace $\{0\}$.

Theorem 3.1. *If for a pair (A, G) the set Γ is non-empty then all elements of the subspace N are weakly quasi-ergodic.*

Proof. By virtue of Proposition 2.1 it is enough to prove that for every $\varphi\in\Gamma$ $N_\varphi\subset L$. Proving this we follow S. M. Abdalla ([1], Chap. 3, Theorem 3.4). For our purposes it is sufficient to show that for every $T\in N_\varphi$

(i) $K(T, G)\subset N_\varphi$ and (ii) $K(T, G)\cap A^G$ is non-empty.

Let $T\in N_\varphi$ and $R\in K(T, G)$. We can find a filter $\mathcal{F}$ in Q such that $\lim_{\mathcal{F}} V(T)=R$ in the strong operator topology. As $K(T, G)$ is bounded, we have $\lim_{\mathcal{F}}(V(T)^* V(T))= =R^*R$ in the weak operator topology. On the other hand, if $V\in Q$ and $V=\sum_{i=1}^n \alpha_i g_i$ $(\alpha_i>0, \sum_{i=1}^n \alpha_i=1, g_i\in G)$, then we have by Schwarz's inequality

$$\varphi(V(T)^*V(T))=\varphi\left[\left(\sum_{i=1}^n \alpha_i g_i(T)^*\right)\left(\sum_{j=1}^n \alpha_j g_j(T)\right)\right]=\sum_{i,j}\alpha_i\alpha_j\varphi\big(g_i(T)^*g_j(T)\big)\leqq$$

$$\leqq\sum_{i,j}\alpha_i\alpha_j\varphi\big(g_i(T^*)g_i(T)\big)^{\frac{1}{2}}\cdot\varphi\big(g_j(T^*)g_j(T)\big)^{\frac{1}{2}}=\sum_{i,j}\alpha_i\alpha_j\varphi(T^*T)=\varphi(T^*T).$$

Since φ is normal, it is ultraweakly lower semicontinuous and so it is weakly lower semicontinuous on any bounded part of A^+, thus $\varphi(R^*R)\leqq\varphi(T^*T)$. This proves (i).

Since φ is normal it can be represented in the following form: $\varphi(T)=\sum_\iota (Tx_\iota, x_\iota)$ for every $T\in A^+$, where the x_ι's are suitable vectors from H. It follows that the function $S\to\varphi(S^*S)$ is weakly lower semicontinuous on any bounded part of A and thus it attains its minimum on the weakly compact bounded set $K(T, G)$. Taking into account the fact that φ is faithful it follows that the function $S\to(\varphi(S^*S))^{1/2}= =\|S\|_2$ is a pre-Hilbert norm on N_φ, therefore the minimum is attained only at one point. Denote by T_0 this element. It is not hard to see that for every element g of G $g(T_0)\in K(T, G)$. On the other hand, it is evident that $\varphi(T_0^*T_0)=\varphi(g(T_0)^*g(T_0))$ and this implies that $g(T_0)=T_0$. This means that $T_0\in A^G$ and proves (ii).

The next theorem is a generalisation of J. B. Conway's result ([4], Lemma 6).

Theorem 3.2. *If for a pair (A, G) the set Γ is non-empty and A^G does not contain any finite projection except 0, then for every $T\in M$, $K(T, G)\cap A^G=\{0\}$.*

Proof. Let P be a finite projection in A. Then we can find a $\varphi\in\Gamma$ such that $\varphi(P)<\infty$. By Theorem 3.1 it follows that $K(P, G)\cap A^G$ is non-empty. Denote by S an arbitrary element of this set. Since φ is weakly lower semicontinuous on $K(P, G)$ and finite constant on $K_0(P, G)$, the values of φ are finite on $K(P, G)$, thus $\varphi(S)<\infty$. On the other hand, $P\in A^+$, hence $S\in A^+$.

Let $S=\int\lambda dE_\lambda$ be the spectral decomposition of S, where E_λ is right-continuous. Let $\mu>\lambda$ be arbitrary positive reals. It is clear that $E_\mu-E_\lambda$ belongs to A^G and that $\lambda(E_\mu-E_\lambda)\leqq S$. It follows that $\lambda\cdot\varphi(E_\mu-E_\lambda)\leqq\varphi(S)$ so the projection $E_\mu-E_\lambda$ can't be infinite, and therefore $E_\mu=E_\lambda$. This proves that $S=0$.

Now let $T\in M$ be arbitrary. For any $\varepsilon>0$ we can find finite projections $P_1, P_2, \ldots, P_n$ and complex numbers $c_1, c_2, \ldots, c_n$ such that $\left\|T-\sum_{i=1}^{n} c_iP_i\right\|<\varepsilon$. By Theorem 3.1 it follows that $K(T, G)\cap A^G$ is non-empty. Denote by S an arbitrary element of this set. By Proposition 2.2 there exists a positive linear mapping τ of L onto A^G such that for every $R\in L$, $\tau(R)\in K(R, G)\cap A^G$ and $\tau(T)=S$. Since $\|\tau\|\leqq 1$, we have $\left\|\tau(T)-\sum_{i=1}^{n} c_i\tau(P_i)\right\|<\varepsilon$. By the preceding part of the present proof we have $\tau(P_i)=0$ for all indices i, hence $\|\tau(T)\|<\varepsilon$. This proves that $\tau(T)=S=0$.

Theorem 3.3. *Let the pair (A, G) possess property Π. Let us suppose that Γ is non-empty and that $\natural\subset G$. In this case for every $T\in M$, $K(T, G)\cap A^G$ consists of a single element. In other words, $M\subset E$.*

Proof. Denote the largest projection of A^G by P. If $P=0$ then the statement of the theorem is trivial. If $P\neq 0$, then necessarily $P=I$. Indeed, if we set $R=I-P$ then we have $g(R)g(P)=0$ and $g(P)=P$ for every $g\in G$ and thus $g(R)\leqq R$ for every $g\in G$. It follows from property Π that $R\in A^G$, and, consequently, $I=P+$ $+R\in A^G$.

In virtue of Proposition 2.3 and Theorem 3.1 it is sufficient to show that for every $\varphi\in\Gamma$ and $T\in M_\varphi^+$ the set $K(T, G)\cap A^G$ contains exactly one element.

Denote by Y the maximal projection of A^G for which the restriction (A_Y, G_Y) of (A, G) is finite (Proposition 1.2.). Let $Z=I-Y$. Taking into account that $\natural\subset G$ the projections Y and Z belong to the center of A. It follows immediately from this that for every $S\in A$ the operator S is uniquely determined by its 'parts' S_Y and S_Z.

By Theorem 3.1 $K(T, G)\cap A^G$ is non-empty. Denote by R and S two elements of it. Using the facts that

(1) $$(K(T, G)\cap A^G)_Y\subseteqq K(T_Y, G_Y)\cap A_Y^{G_Y} \text{ and}$$

(2) $$(K(T, G)\cap A^G)_Z\subseteqq K(T_Z, G_Z)\cap A_Z^{G_Z}$$

the restricted operators R_Y and S_Y belong to the set (1) and the restricted operators R_Z and S_Z belong to the set (2). By the theorem of I. KovÁcs—J. Szűcs the set (1) consists of a single element, so $R_Y = S_Y$. By Theorem 3.2 it follows that $R_Z = S_Z = 0$. This means that $R = S$, and thus the set $K(T, G) \cap A^G$ has only one element.

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