Acta Sci. Math., 41 (1979), 253—257

Nilpotent torsion-free rings and triangles of types

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1. Introduction. This note modifies an idea of VINSONHALER and WICKLESS [7] concerning associative rings having torsion-free additive group. In [7] the necessary and sufficient conditions for a group to support only trivial rings given by REE and WISNER [4] are generalised in such a way that certain groups supporting only nilpotent rings are characterised. In fact more precise information can be obtained giving a bound on the nilstufe of a group. The nilstufe, a notion due to SZELE [6], $n(X)$, of a group X is the largest integer n such that there is an associative ring on X with a non-zero product of n elements. If no such largest integer exists then $n(X)=\infty$. Several authors [1], [3], [5], [8], [9] have obtained bounds for the nilstufe of a group in certain circumstances. The bound obtained here applies in quite general circumstances. In this note the basic tool of [7] is modified and then used to prove results based on two of the main theorems from [7], in one case giving a considerable generalisation.

From now on all groups are torsion-free abelian groups and all undefined concepts are standard from FUCHS [2]. In particular the product of a pair of types $\mathbf{t}_1, \mathbf{t}_2$ is written $\mathbf{t}_1\mathbf{t}_2$ not $\mathbf{t}_1+\mathbf{t}_2$ as in [7], and $T(X)=\{\mathbf{t}(x)\colon x\in X\}$ is the type-set of the group X.

2. **The results.** The following is a modification of a definition of [7].

Definition. A collection of types $\{\mathbf{t}_{ij}\colon 1\leqq j\leqq n-i+1,\ 1\leqq i\leqq n\}$ is a *triangle of base size* n if for any three integers i, j, k satisfying $1\leqq k\leqq n-i-j+1$, $1\leqq j\leqq n-i+1$, $1\leqq i\leqq n$,

$$\mathbf{t}_{i,j}\mathbf{t}_{k,(i+j)} \leqq \mathbf{t}_{(i+k),j}.$$

This is strongly related to the definition in [7] but, as will become apparent below, it is easy to handle. For example, the following based on Theorem 2.1 of [7] has a quite straightforward proof.

Received July 27, 1977, in revised form April 17, 1978.

Theorem 1. *For any torsion-free group X and any associative ring $(X, \circ)$, a non-zero product of n elements in the ring defines a triangle of base size n from the type-set of X.*

Proof. Suppose that for the elements $x_1, \ldots, x_n$ from X the product $x_1 \circ \ldots \circ x_n$ is non-zero. Then denoting by $\langle x \rangle^*$ the pure subgroup generated by an element x of X, the following pure rank 1 subgroups of X are defined: for each pair of integers i, j satisfying $1 \leqq j \leqq n-i+1$, $1 \leqq i \leqq n$,

$$X_{i,j} = \langle x_j \circ \ldots \circ x_{j+i-1} \rangle^*.$$

As a consequence, for the integers i, j, k satisfying $1 \leqq k \leqq n-i-j+1$, $1 \leqq j \leqq \leqq n-i+1$, $1 \leqq i \leqq n$,

$$\mathbf{t}(X_{(i+k),j}) = \mathbf{t}(x_j \circ \ldots \circ x_{j+i+k-1}) \geqq \mathbf{t}(x_j \circ \ldots \circ x_{j+i-1}) \mathbf{t}(x_{j+i} \circ \ldots \circ x_{j+i+k-1}).$$

However,

$$\mathbf{t}(X_{i,j}) \mathbf{t}(X_{k,(i+j)}) = \mathbf{t}(x_j \circ \ldots \circ x_{j+i-1}) \mathbf{t}(x_{j+i} \circ \ldots \circ x_{j+i+k-1})$$

implying $\mathbf{t}(X_{(i+k),j}) \geqq \mathbf{t}(X_{i,j}) \mathbf{t}(X_{k,(i+j)})$. So if $\mathbf{t}_{i,j} = \mathbf{t}(X_{i,j})$ for $1 \leqq j \leqq n-i+1$, $1 \leqq i \leqq n$ the set of types so defined form a triangle of base size n.

Recalling the definition of nilstufe there is the following corollary.

Corollary. *For any torsion-free group X, its nilstufe $n(X)$ is bounded by the maximum base size of triangles that can be formed from $T(X)$.*

Note that in the Corollary there are no restraints on the group X. Other authors [1], [5], [8] have obtained bounds for $n(X)$ but for groups whose type-sets satisfy some form of chain condition. In most cases the Corollary gives a better bound on the nilstufe and an illustrative example is given following Theorem 2. A second important result of [7] that can now be more easily proved is

Theorem 2. *If for $X = \bigoplus_{i \in I} X_i$ where each X_i has rank 1 a triangle of base size n can be formed from the set $\{\mathbf{t}(X_i) : i \in I\}$ without using the same summand twice, then $n(X) \geqq n$.*

Proof. Suppose that the triangle is $\{\mathbf{t}_{i,j} : 1 \leqq j \leqq n-i+1, 1 \leqq i \leqq n\}$ where the $\mathbf{t}_{i,j}$ are types of distinct summands X_i. Then,

$$\mathbf{t}_{(i+k),j} \geqq \mathbf{t}_{i,j} \mathbf{t}_{k,(i+j)}$$

where $1 \leqq k \leqq n-j-i+1$, $1 \leqq j \leqq n-i+1$, $1 \leqq i \leqq n$. The characteristic of an element x in X is denoted by $\chi(x)$ and so the above inequalities imply that elements $x_{i,j}$ from the corresponding rank one summands can be found such that

$$\chi(x_{(i+k),j}) \geqq \chi(x_{i,j}) \chi(x_{k,(i+j)}).$$

If Y is the direct sum of the summands of X used in defining the types then a ring $(Y, *)$ can be defined using the following products. For integers i, j, k satisfying $1 \leqq k \leqq n-l+1$, $1 \leqq j \leqq n-i+1$, $1 \leqq i \leqq n$, $1 \leqq l \leqq n$.

$$x_{i,j} * x_{k,l} = \begin{cases} x_{(i+k),j} & \text{if } l = i+j \\ 0 & \text{if not.} \end{cases}$$

These products and the distributive law define a ring which can be shown to be associative.

Take three subscripts (k_1, k_2), (l_1, l_2), (m_1, m_2); then

$$x_{k_1,k_2} * x_{l_1,l_2} = \begin{cases} 0 & \text{if } l_2 \neq k_1+k_2 \\ x_{(k_1+l_1),k_2} & \text{if } l_2 = k_1+k_2 \end{cases}$$

and

$$x_{(k_1+l_1),k_2} * x_{m_1,m_2} = \begin{cases} 0 & \text{if } m_2 \neq k_1+l_1+k_\lambda \\ x_{(k_1+l_1+m_1),k_2} & \text{if } m_2 = k_1+l_1+k_\lambda. \end{cases}$$

Also,

$$x_{l_1,l_2} * x_{m_1,m_2} = \begin{cases} 0 & \text{if } m_2 \neq l_1+l_2 \\ x_{(l_1+m_1),l_2} & \text{if } m_2 = l_1+l_2 \end{cases}$$

and

$$x_{k_1,k_2} * x_{(l_1+m_1),l_2} = \begin{cases} 0 & \text{if } l_2 \neq k_1+k_2 \\ x_{(k_1+l_1+m_1),k_2} & \text{if } l_2 = k_1+k_2. \end{cases}$$

So that the products $(x_{k_1,k_2} * x_{l_1,l_2}) * x_{m_1,m_2}$, $x_{k_1,k_2} * (x_{l_1,l_2} * x_{m_1,m_2})$ are non-zero if and only if $l_2 = k_1+k_2$ and $m_2 = l_1+l_2$ in which case they both equal $x_{(k_1+l_1+m_1),k_2}$. Furthermore, $x_{1,1} * x_{1,2} * \dots * x_{1,n} = x_{n,1}$ is non-zero.

To define an associative ring on X merely take the ring direct sum of $(Y, *)$ and the trivial ring on the complement of Y in X and so $n(X) \geqq n$.

Finally an example is given to illustrate that the bounds obtained using Theorem 1 can be lower than those obtained using other available results.

Example. Begin by partitioning the set of all primes into two disjoint infinite subsets P_1 and P_2 where $P_2 = \{p_1, p_2, p_3, \dots\}$. Then define the following subgroups of the rationals.

$$A_1 = A_2 = gp\left\{\frac{1}{p}: p \in P_1 \cup P_2\right\}, \quad B = gp\left\{\frac{1}{p^n}, \frac{1}{q^2}: p \in P_1,\ n \in \mathbf{Z}^+,\ q \in P_2\right\},$$

and for each integer $i \geqq 1$,

$$C_i = gp\left\{\frac{1}{p}, \frac{1}{p_1^n}, \frac{1}{p_2^n}, \dots, \frac{1}{p_i^n}: p \in P_1,\ n \in \mathbf{Z}^+\right\}.$$

For each positive integer m let

$$X_m = A_1 \oplus A_2 \oplus B \oplus C_1 \oplus \dots \oplus C_m.$$

Then $T(X_m)=\{\mathbf{t}(A_1), \mathbf{t}(B), \mathbf{t}(C_1), \dots, \mathbf{t}(C_m)\}$ contains a chain of length $m+1$, but no chain of greater length.

A type $\mathbf{t}_1$ in $T(X)$ absorbs a type $\mathbf{t}_2$ in $T(X)$ if $\mathbf{t}_1\mathbf{t}_2=\mathbf{t}_1$. It is clear that $\mathbf{t}(X_m)$ contains no absorbing types so that Proposition 1.2 of [5] implies $n(X_m) \leqq m+1$. However, a triangle of base length two can be formed from $T(X_m)$, namely $\mathbf{t}_{21}= =\mathbf{t}(B)$, $\mathbf{t}_{1i}=t(A_i)$ i$=1, 2$. So, by Theorem 2, $n(X_m) \geqq 2$. The following lemma will show that no larger triangles can be formed.

Lemma. *If X is a group such that $T(X)$ contains no absorbing types the apex of a triangle of base size n formed from $T(X)$ has a chain of length n descending from it in $T(X)$.*

Proof. The proof is by induction on n, the result being trivial for $n=1$.

Suppose $n>1$. Let $\mathbf{t}_{k,1}$ be the apex of a triangle of base size k, then the apex of any triangle of base size $(n-1)$ from $T(X)$ has a chain of length $(n-1)$ descending from it in $T(X)$. If the triangle is $\{\mathbf{t}_{i,j}\colon 1 \leqq j \leqq n-i+1,\ 1 \leqq i \leqq n\}$ then

$$\mathbf{t}_{n,1} \geqq \mathbf{t}_{(n-1),1}\mathbf{t}_{1,n}.$$

Now $\mathbf{t}_{(n-1),1}$ is the apex of a triangle of base size $(n-1)$ from $T(X)$ so is the maximal type of a chain of length $(n-1)$ in $T(X)$. Suppose $\mathbf{t}_{n,1}=\mathbf{t}_{(n-1),1}$ then $\mathbf{t}_{(n-1),1}\mathbf{t}_{1,n}= =\mathbf{t}_{(n-1),1}$ and $\mathbf{t}_{(n-1),1}$ is an absorbing type in $T(X)$. Thus $\mathbf{t}_{n,1}>\mathbf{t}_{(n-1),1}$ and so has a chain of length n descending from it in $T(X)$.

Returning to the example it should be noted that $\mathbf{t}(B)$ has only chains of at most length 2 descending from it in $T(X_n)$. Furthermore by considering the summands of X_m it is clear that in any ring on X_m, $X_m^2 \subseteqq B$ since $P_1 \cap P_2$ is empty. Since any product of more than two elements in any ring on X is in B, $\mathbf{t}(B)$ is the apex of all triangles formed by applying Theorem 1. Hence $n(X_m) \leqq 2$ which with the earlier work gives $n(X_m)=2$ for all $m \geqq 1$.

In conclusion it is noted that this technique fails to work for non-associative rings as the bracketing of a product may be quite arbitrary. Essential to the above is that for a product in an associative ring $\mathbf{t}((ab)c)=\mathbf{t}(a(bc))$.

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