Acta Sci. Math., 41 (1979), 215—219

A characterization of .3

GERNOT STROTH

The objective of this paper is the proof of the following theorem.

Theorem. *Let G be a finite simple group and H a 2-local subgroup of G. Assume that $H/\mathbf{O}(H)$ is an extension of $Z_4 * Q_8 * D_8$ by Σ_6. Assume further that $\mathbf{Z}(H/\mathbf{O}(H))$ is of order two. Then G is isomorphic to .3, the Conway simple group.*

Lemma 1. *Put $H_1 = H/\mathbf{O}(H)$. Then $H_1/\mathbf{Z}(H_1)$ splits over $\mathbf{O}_2(H_1/\mathbf{Z}(H_1))$.*

Proof. Put $H_2 = H_1/\mathbf{Z}(\mathbf{O}_2(H_1))$. Then $\mathbf{O}_2(H_2)$ is a symplectic space of dimension four. Thus $H_2/\mathbf{O}_2(H_2)$ is isomorphic to a subgroup of $Z_2 \times \Sigma_6$. In this group there are exactly two subgroups isomorphic to Σ_6. Since $\mathbf{Z}(H_1)$ is of order two we get that H_2 is uniquely determined. Thus we get in H_2 a subgroup isomorphic to Σ_6. Since $\mathbf{Z}(H_1)$ is of order two we get in $H_1/\mathbf{Z}(H_1)$ a subgroup isomorphic to Σ_6. This proves the lemma.

Lemma 2. *Let z be the involution in $\mathbf{Z}(H)$. Then $H \neq \mathbf{C}_G(z)$.*

Proof. By way of contradiction we assume $H = \mathbf{C}_G(z)$.

Assume first that z is conjugate to an involution x contained in $\mathbf{O}_2(H) - \langle z \rangle$. Then there is an element ϱ centralizing x such that $\varrho^3 \in \mathbf{O}(H)$. Thus $\langle \varrho, \mathbf{O}(H) \rangle$ is contained in $\mathbf{C}_G(x)$. Let π be an element of $\mathbf{O}(H)$. Then $\mathbf{C}_G(\pi)$ contains $\mathbf{O}_2(H)$. Let ν be an element in $\varrho\mathbf{O}(H)$. Then a Sylow 2-subgroup of $\mathbf{C}_H(\nu)$ is isomorphic to $(Z_4 * Q_8)\langle a \rangle$ where $a^2 \in (Z_4 * Q_8)$. Thus 64 does not divide the order of $\mathbf{C}_G(\nu)$. Let ω be an element in $H - \mathbf{O}(H)$ such that $\omega^3 \in \mathbf{O}(H)$ and $\omega\mathbf{O}(H)$ is not conjugate to $\varrho\mathbf{O}(H)$ in $H/\mathbf{O}(H)$. Let μ be an element of $\omega\mathbf{O}(H)$. Then $\mathbf{C}_H(\mu)$ possesses a Sylow 2-subgroup S such that S is of order at least 8 and $\Phi(S)$ is equal to $\langle z \rangle$. Thus 16 does not divide the order of $\mathbf{C}_G(\mu)$. Let g be an element of G such that $x^g = z$. Then ϱ^g is contained in H. Since 16 divides the order of $\mathbf{C}_G(\varrho)$ but 64 does not divide the order of $\mathbf{C}_G(\varrho)$ we may assume $\varrho^g \in \varrho\mathbf{O}(H)$. Thus we may assume that g is contained in $\mathbf{N}_G(\langle \varrho \rangle)$. Let T be a Sylow 2-subgroup of $\mathbf{C}_H(\varrho) \cap \mathbf{C}_G(x)$.

Received March 18, 1975.

Then it is easy to see that T' is equal to $\langle z\rangle$. Thus x is not conjugate to z in $\mathbf{N}_G(\langle\varrho\rangle)$. We have proved that $\langle z\rangle$ is strongly closed in $\mathbf{O}_2(H)$ with respect to G.

Assume now that z is conjugate to an involution y in $H'-\langle z\rangle$. Then $\mathbf{C}_{\mathbf{O}_2(H)}(y)$ is isomorphic to $Z_4\times Z_2$. Thus there is an involution s in $\mathbf{O}_2(H)-\langle z\rangle$ such that y is conjugate to sy in G. Let U be a Sylow 2-subgroup of H. Then every involution a of $U-\langle z\rangle$ is conjugate to za in U. Thus s is conjugate to sy in G. But then s is conjugate to z in G, which is a contradiction. Thus we have proved that $\langle z\rangle$ is strongly closed in H' with respect to G.

Assume now that z is conjugate to an involution u of $H-H'$. Then z is a non-square in $\mathbf{C}_H(u)$. Thus $\mathbf{C}_{\mathbf{O}_2(H)}(u)$ is elementary abelian of order eight. But then there is an involution b in $\mathbf{O}_2(H)-\langle z\rangle$ such that u is conjugate to bu in G. As above we get a contradiction.

Thus we have proved that $\langle z\rangle$ is strongly closed in a Sylow 2-subgroup of G. Hence [2; Corollary 1, p. 404] yields the assertion.

Lemma 3. *Let M be a finite simple group which possesses a 2-local subgroup L such that $L/\mathbf{O}(L)$ is isomorphic to a faithful extension of E_{16} by A_6. Then M is isomorphic to $L_4(q)$, $q\equiv 5(8)$; $U_4(q)$, $q\equiv 3(8)$; M_{22}, M_{23} or M^c.*

Proof. By [6; Theorem 3], L contains a Sylow 2-subgroup of M. Now [4] yields the assertion.

Lemma 4. *Let M be a finite group which possesses an involution z such that $\mathbf{C}_M(z)/\mathbf{O}(\mathbf{C}_M(z))$ is isomorphic to one of the following groups:*

(i) $\mathrm{SL}_4(q)$, $q\equiv 5(8)$;
(ii) $\mathrm{SU}_4(q)$, $q\equiv 3(8)$.

Then $z\in\mathbf{Z}^*(M)$.

Proof. In $\mathbf{C}_M(z)$ there are only two classes of involutions. Let v be an involution of $\mathbf{C}_M(z)$ not equal to z.

Put $C=\mathbf{C}_M(z)$. Then $\mathbf{C}_C(v)$ contains a subgroup $E=S_1\times S_2$ where S_1 and S_2 are isomorphic to $SL_2(q)$. Now we get $\mathbf{Z}(S_1)=\langle v\rangle$ and $\mathbf{Z}(S_2)=\langle zv\rangle$, implying that $\mathbf{C}_C(v)/\mathbf{O}(\mathbf{C}_C(v))$ is equal to $\mathbf{Z}(C/\mathbf{O}(C))*(E\langle a\rangle)$ where a induces the diagonal automorphism on S_1 and S_2. Let R be a Sylow 2-subgroup of $\mathbf{C}_C(v)$. Then R' is isomorphic to $Z_4\times Z_4$ and $\mathbf{C}_R(R')$ is isomorphic to $Z_2\times Z_4\times Z_8$. Since $\mho^2(\mathbf{C}_R(R'))$ is equal to $\langle z\rangle$ we get that z is not conjugate to v in G. Hence [2; Corollary 1] yields the assertion.

Lemma 5. *Let M be a finite group. Assume that z is an involution in M such that $\mathbf{C}_M(z)/\mathbf{O}(\mathbf{C}_M(z))$ is isomorphic to one of the following groups:*

(i) *$SL_4(q)\langle x\rangle$, $q\equiv 5(8)$, x induces the graph-automorphism on $SL_4(q)$ and $x^2\in\mathbf{Z}(SL_4(q))$;*

(ii) $SU_4(q)\langle x\rangle$, $q\equiv 3(8)$, *x induces the field-automorphism of order* 2 *on* $SU_4(q)$ *and* $x^2\in \mathbf{Z}(SU_4(q))$.
Then $z\in \mathbf{Z}^*(M)$.

Proof. Put $C=\mathbf{C}_M(z)$. Then $\mathbf{C}_C(x)/\mathbf{O}(\mathbf{C}_C(x))$ is isomorphic to $Sp_4(q)\langle x\rangle$. Let T be a Sylow 2-subgroup of $\mathbf{C}_C(x)$. Then $\langle z\rangle=\mathbf{Z}(T)\cap T'$. Thus $\mathbf{C}_C(x)$ contains a Sylow 2-subgroup of $\mathbf{C}_G(x)$.

Assume that x is an element of order two. Then 2^9 does not divide the order of $\mathbf{C}_G(x)$. Thus x is not conjugate to an involution of $SL_4(q)$ or $SU_4(q)$. Thus [12; Lemma (5.38)] yields that M possesses a subgroup M_1 of index two. Consequently, $\mathbf{C}_{M_1}(z)/\mathbf{O}(\mathbf{C}_{M_1}(z))$ is isomorphic to $SL_4(q)$, $q\equiv 5(8)$ or $SU_4(q)$, $q\equiv 3(8)$ whence by Lemma 4 the assertion follows.

Put $\langle u\rangle=\mathbf{Z}(SL_4(q))$, resp. $\mathbf{Z}(SU_4(q))$. Then we may assume that $\langle u, x\rangle$ is isomorphic to Q_8.

We shall prove that $\langle z\rangle$ is strongly closed in C' with respect to M. Let v be an involution of $C'-\langle z\rangle$. Then $\mathbf{C}_C(v)/\mathbf{O}(\mathbf{C}_C(v))$ contains a subgroup $E=S_1\times S_2$ where S_1 and S_2 are isomorphic to $SL_2(q)$. We may assume $\mathbf{Z}(S_1)=\langle v\rangle$ and $\mathbf{Z}(S_2)=\langle zv\rangle$. Now $\mathbf{C}_C(v)$ contains a subgroup Q isomorphic to Q_8 such that Q' is equal to $\langle z\rangle$. Then $\mathbf{C}_C(v)/\mathbf{O}(\mathbf{C}_C(v))$ is equal to an extension of order 2 of $Q*E$. Assume that z is conjugate to v in M. Then there is a Sylow 2-subgroup B of $Q*E$ such that z is conjugate to v in $\mathbf{N}_M(B)$. Now B is isomorphic to $Q_8*(Q_8\times Q_8)$. Thus $\mathbf{N}_M(\mathbf{Z}(B))/\mathbf{C}_M(\mathbf{Z}(B))$ is isomorphic to Σ_3. However, since $\mathbf{C}_B(\mathbf{O}_3(\mathbf{C}_M(\mathbf{Z}(B))/B))$ is isomorphic to Q_8, we get a contradiction. Thus $\langle z\rangle$ is strongly closed in C' with respect to M.

Now we know that $\mathbf{C}_C(x)$ contains an element s such that sx is an involution and sx is centralized by s. Thus z is a square in $\mathbf{C}_M(xs)$. This implies that xs is not conjugate to an element of C'. Hence by [12; Lemma (5.38)] M possesses a subgroup M_1 of index two. Thus $\mathbf{C}_{M_1}(z)/\mathbf{O}(\mathbf{C}_{M_1}(z))$ is isomorphic to $SL_4(q)$, $q\equiv 5(8)$ or $SU_4(q)$, $q\equiv 3(8)$, which by Lemma 4 yields the assertion.

Lemma 6. *Let M be a finite group and z a* 2*-central involution in M such hatt* $\mathbf{C}_M(z)/\mathbf{O}(\mathbf{C}_M(z))$ *is isomorphic to a split extension of an elementary abelian group E of order* 32 *by* A_6 *where* A_6 *acts undecomposable on E. Then* $z\in \mathbf{Z}^*(M)$.

Proof. Assume first that z is conjugate in M to an involution u of $\mathbf{C}_M(z)-(E\mathbf{O}(\mathbf{C}_M(z)))$. Put $C=\mathbf{C}_M(z)$. Then there are only two classes of involutions in $C-\mathbf{O}_{2',2}(C)$. Thus $\mathbf{C}_C(u)/\mathbf{O}(\mathbf{C}_C(u))$ is isomorphic to a split extension of E_8 by D_8. Hence $C/\mathbf{O}(C)$ involves a subgroup A_5 such that EA_5 is equal to $\langle z\rangle\times(E_{16}A_5)$ where A_5 acts intransitively on E_{16}. Thus we may assume that there is an involution r in $\mathbf{Z}(\mathbf{C}_C(u)/\mathbf{O}(\mathbf{C}_C(u)))$ such that u is conjugate to ru and r is contained in $(\mathbf{C}_C(u)/\mathbf{O}(\mathbf{C}_C(u)))'$. Let S be a Sylow 2-subgroup of $\mathbf{C}_M(u)$ containing a Sylow 2-subgroup of $\mathbf{C}_C(u)$. Assume that z is conjugate neither to r nor to zr. Then $\mathbf{Z}(S)$

is equal to $\langle r, u\rangle$. But this is a contradiction. Thus we have proved that $\langle z\rangle$ is not strongly closed in E with respect to M if z is conjugate to an involution of $C-\mathbf{O}_{2',2}(C)$.

Assume now that $\langle z\rangle$ is not strongly closed in E with respect to M. Let T be a Sylow 2-subgroup of C. Since all involutions of E are conjugate to involutions of $\mathbf{Z}(T)$ in C we get that all involutions of E are conjugate in M. If z is not conjugate to an involution of $C-\mathbf{O}_{2',2}(C)$ in M we get that E is strongly closed in T with respect to M. Then it follows from [3] that $E\mathbf{O}(M)$ is normal in M. Thus $|M/\mathbf{O}(M):C\mathbf{O}(M)/\mathbf{O}(M)|$ is equal to 31, which is impossible.

Thus we have proved there are only two possibilities for the fusion of involutions in M. The first is that $\langle z\rangle$ is strongly closed in T with respect to M. Then [2] yields the assertion. The second is that all involutions of M are conjugate in M. Thus all 2-local subgroups of $M/\mathbf{O}(M)$ are 2-constrained, so that applying [1] we get a contradiction. Thus the lemma is proved.

Lemma 7. *Put* $\langle u\rangle=\mathbf{Z}(\mathbf{O}_2(H))$. *Then* $\mathbf{N}_G(\langle u\rangle)/\mathbf{O}(\mathbf{N}_G(\langle u\rangle))$ *is isomorphic to one of the following groups:*

(i) $H/\mathbf{O}(H)$;

(ii) $SL_4(q)\langle x\rangle$, $q\equiv 5(8)$, $x^2\in\mathbf{Z}(SL_4(q))$ *and* x *induces the graph-automorphism on* $SL_4(q)$;

(iii) $SU_4(q)\langle x\rangle$, $q\equiv 3(8)$, $x^2\in\mathbf{Z}(SU_4(q))$ *and* x *induces the field-automorphism on* $SU_4(q)$.

Proof. Put $N=\mathbf{N}_G(\langle u\rangle)$. Assume that N is not equal to H. Let M be a minimal normal subgroup of $N/(\mathbf{O}(N)\langle u\rangle)$. Then M is simple. Further, M possesses a 2-local subgroup isomorphic to a split extension of E_{16} by A_6. Then, by Lemma 3, M is isomorphic to $L_4(q)$; $q\equiv 5(8)$, $U_4(q)$; $q\equiv 3(8)$, M_{22}, M_{23} or M^c. Applying [5] we get that M is isomorphic to $L_4(q)$; $q\equiv 5(8)$ or $U_4(q)$; $q\equiv 3(8)$. Thus $N/\mathbf{O}(N)$ contains a subgroup of index 2 isomorphic to $SL_4(q)$ or $SU_4(q)$. Now the structure of $\mathrm{Aut}(SL_4(q))$ and $\mathrm{Aut}(SU_4(q))$ yields the assertion.

Lemma 8. *The group* $\mathbf{C}_G(z)/\mathbf{O}(\mathbf{C}_G(z))$ *is isomorphic to* $\widehat{Sp_6(2)}$.

Proof. Put $C=\mathbf{C}_G(z)/(\mathbf{O}(\mathbf{C}_G(z))\langle z\rangle)$. Assume first that $N=\mathbf{N}_G(\langle u\rangle)$ is not equal to H. Let F be a minimal normal subgroup of C. Assume that F is not simple. Then F is contained in $N/(\mathbf{O}(\mathbf{C}_G(z))\langle z\rangle)$. Then $\mathbf{C}_G(z)$ is equal to N, which by Lemmas 7 and 4 leads to a contradiction. Thus F is simple. Let T be a Sylow 2-subgroup of N. Since $u\langle z\rangle$ is not a square in $T/\langle z\rangle$ but all other involutions in $\mathbf{Z}(T/\langle z\rangle)$ are squares in $T/\langle z\rangle$ we get that T is a Sylow 2-subgroup of G. Thus C (possesses a Sylow 2-subgroup of type A_{12}. Since all involutions of $(N/(\mathbf{O}(\mathbf{C}_G(z))\langle z\rangle)$ are conjugate to involutions of $\mathbf{Z}(T/\langle z\rangle)$ we get that F possesses a Sylow 2-subgroup of type A_{12}. Then by [9], F is isomorphic to A_{12}, A_{13}, $PSp_6(2)$ or has the involution-fusion-pattern of $\Omega_7(3)$.

Assume now that N is equal to H. Let F be a minimal normal subgroup of C. Lemma 2 implies that F is simple and Lemma 6 yields that $N/(\mathbf{O}(\mathbf{C}_G(z))\langle z\rangle)$ is contained in F since a Sylow 2-subgroup of N is a Sylow 2-subgroup of $\mathbf{C}_G(z)$. Hence, by [9], F is isomorphic to $A_{12}, A_{13}, PSp_6(2)$ or has the involution-fusion-pattern of $\Omega_7(3)$.

Thus in both cases we have proved that a minimal normal subgroup of C is isomorphic to $A_{12}, A_{13}, PSp_6(2)$ or has the involution-fusion-pattern of $\Omega_7(3)$.

Assume first that a minimal normal subgroup of C has the involution-fusion-pattern of $\Omega_7(3)$. Applying [10] and [7] we get that $\mathbf{C}_G(z)/\mathbf{O}(\mathbf{C}_G(z))$ is an odd extension of $\mathrm{Spin}_7(q)$, $q\equiv 3, 5(8)$. Now [11; Theorem (3.4)] yields a contradiction.

Assume now that a minimal normal subgroup of C is isomorphic to A_{12} or A_{13}. Then $\mathbf{C}_G(z)/\mathbf{O}(\mathbf{C}_G(z))$ is isomorphic to $\hat{A}_{12}$ or $\hat{A}_{13}$, so that G possesses only one class of involutions. Now [8; Corollary] yields a contradiction.

Thus we have proved that a minimal normal subgroup of C is isomorphic to $PSp_6(2)$. The structure of $\mathrm{Aut}\,(PSp_6(2))$ shows now that C is isomorphic to $PSp_6(2)$. Thus the lemma is proved.

Lemma 9. *The group G is isomorphic to .3, the Conway simple group.*

Proof. By Lemma 8, a Sylow 2-subgroup of G is of type .3, which by [11] implies the assertion.

References

[1] B. Beisiegel, Über endliche einfache Gruppen mit Sylow 2-Untergruppe der Ordnung höchstens 2^{10}, *Comm. Algebra,* **5** (1977), 113—170.

[2] G. Glauberman, Central elements in core-free groups, *J. Algebra,* **4** (1966), 403—421.

[3] D. M. Goldschmidt, 2-fusion in finite groups, *Ann. of Math.,* **99** (1974), 70—117.

[4] D. Gorenstein and K. Harada, Finite groups whose 2-subgroups are generated by at most 4 elements, *Mem. Amer. Math. Soc.,* **147** (1974).

[5] R. Griess, *Schur multipliers of the known finite simple groups,* Thesis, University of Chicago (1971).

[6] K. Harada, On finite groups having self-centralizing 2-subgroups of small order, *J. Algebra,* **33** (1975), 144—160.

[7] J. B. Olsson, Odd-order extensions of some orthogonal groups, *J. Algebra,* **28** (1974), 573—596.

[8] R. Solomon, Finite groups with components of alternating type, *Notices Amer. Math. Soc.,* **21** (1974), A—103.

[9] R. Solomon, Finite groups with Sylow 2-subgroups of type A_{12}, *J. Algebra,* **24** (1973), 346—378.

[10] R. Solomon, Finite groups with Sylow 2-subgroups of type $\Omega(7, q)$, $q\equiv \pm 3$ (mod 8), *J. Algebra.* **28** (1974), 174—181.

[11] R. Solomon, Finite groups with Sylow 2-subgroups of type .3, *J. Algebra,* **28** (1974), 182—198.

[12] J. G. Thompson, Nonsolvable groups all of whose local subgroups are solvable, *Bull. Amer. Math. Soc.,* **74** (1968), 383—437.

UNIVERSITÄT HEIDELBERG
MATHEMATISCHES INSTITUT
6900 HEIDELBERG, BRD