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Remarks on finitely projected modular lattices

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1. Introduction. Let **K** be a variety of lattices. A lattice L in **K** is called finitely **K**-projected if for any surjective $f: K \twoheadrightarrow L$ in **K** there is a finite sublattice of K whose image under f is L. These lattices are important by the investigations of subvarieties of **K**, in fact, every finite **K**-projected subdirectly irreducible lattice L is splitting in **K**, i.e. there is a largest subvariety of **K** not containing L (see DAY [1]). Let $\mathbf{B}_2$ be the variety generated by all breadth 2 modular lattices. In [2] there is given a necessary condition for a lattice $L \in B_2$ to be $\mathbf{B}_2$-projected. Our goal here is to give some further necessary conditions for a lattice to be **M**-projected, where **M** denotes the variety of all modular lattices.

2. **Preliminaries.** Let M be a finite modular lattice and let Q be the chain of bounded rationals, say $Q=[0, 1]$. $M(Q)$ is the lattice of all continous monotone maps of the compact totally ordered disconnected space X of all ultrafilters of Q into the discrete space M. The constant mappings form a sublattice of $M(Q)$ which is isomorphic to M; we identify M with this sublattice. If a/b is a prime quotient of M then the corresponding quotient a/b of $M(Q)$ is isomorphic to Q, we have a natural isomorphism $\varepsilon_{ab}: Q \to a/b$. If a/b runs over all prime quotients then all a/b generate a sublattice $M[Q]$ of $M(Q)$.

Let A and B be two modular lattices with isomorphic sublattices $C \cong C'$ where C is a filter of A and C' is an ideal of B. Then $L = A \cup B$ can be made into a modular lattice by defining $x \leqq y$ if and only if one of the following conditions is satisfied: $x \leqq y$ in A or $x \leqq y$ in B or $x \leqq c$ in A and $c' \leqq y$ in B where c, c' are corresponding elements under the isomorphism $C \cong C'$. We say that L is the lattice obtained by gluing together A and B identifying the corresponding elements under the isomorphism $C \cong C'$. This useful construction is due to Hall and Dilworth. In this case A is an ideal and B is a filter of L, $L = A \cup B$ and $C = A \cap B$. Conversely if A is an ideal and B is a filter of a lattice L such that $L = A \cup B$ then L is obviously the lattice obtained by gluing together A and B.

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3. The Hall-Dilworth construction.

Theorem 1. *Let A be an ideal and let B be the filter of the finite modular lattice M such that $M=A\cup B$ and $C=A\cap B$ is a chain. Let a/b and c/d be two different prime quotients of C which are projective in A and in B. Then M is not finitely* **M**-*projected.*

Proof. $A[Q]$ is an ideal and $B[Q]$ is a filter of $M[Q]$. Consequently, $M[Q]==A[Q]\cup B[Q]$. It is easy to see that $A[Q]\cap B[Q]=C[Q]$. Let $B'[Q]$ be a disjoint copy of $B[Q]$ with the isomorphism $\varphi\colon B[Q]\to B'[Q]$ $(x\to x')$. The restriction of φ to $C[Q]$ give a sublattice $C'[Q]$ of $B'[Q]$.

Let a/b and c/d two different prime quotients of C. Then we can assume that $a>b\geqq c>d$. First we define an injection $\psi\colon C[Q]\to C'[Q]$ which is different from φ. To define this ψ we distinguish two cases:

(a) We assume that there exists a $u\in C$ covering a. The quotients u/b, a/b, u/a of $C[Q]$ are all isomorphic to Q. Let further δ be an automorphism of u/b and we set $a_0=a$, $a_1=\delta a_0, \dots, a_{i+1}=\delta a_i$ and $\bar{a}_1=\delta^{-1}a_0, \dots, a_{i+1}=\delta^{-1}\bar{a}_i$. Obviously, if r is an arbitrary irrational number between 0 and 1 then there exists an automorphism δ of u/b satisfying the following two conditions (see Fig. 1.).

$$a_1 < a, \tag{1}$$

$$\varepsilon_{ab}(\inf\{a_i\}) = \varepsilon_{ua}(\sup\{\bar{a}_i\}) = r \tag{2}$$

(ε_{ab} (resp. ε_{ua})) denotes the natural isomorphism $a/b\to Q$ (resp. $u/a\to Q$)). Defining ψ_0 to be the product $\varphi\circ\delta$, ψ_0 is an isomorphism of u/b onto u'/b'. ψ_0 can be extended to an isomorphism $\psi\colon C[Q]\to C'[Q]$ as follows:

$$\psi(x) = \begin{cases} \varphi(x) & \text{if} \quad x\notin u/b \\ \psi_0(x) & \text{if} \quad x\in u/b. \end{cases}$$

(b) In the second case a is a maximal element of C. Then we can choose an arbitrary t such that $a'>t>b'$. t/b' is isomorphic to Q, hence there exists an isomorphism $\psi_0\colon a/b\to t/b'$. The extension of ψ_0 is defined by

$$\psi(x) = \begin{cases} \varphi(x) & \text{if} \quad x\notin a/b \\ \psi_0(x) & \text{if} \quad x\in a/b. \end{cases}$$

We take in both cases the lattice L obtained by gluing together $A[Q]$ and $B'[Q]$ identifying the corresponding elements of $C[Q]$ and $\psi(C[Q])$ under the isomorphism ψ (Fig. 2).

We prove that there exists a surjection $f\colon L\twoheadrightarrow M$. Let Θ be the congruence relation of Q defined as follows: $x\equiv y(\Theta)$ if and only if either $x, y>r$ or $x, y<r$. Then $A[Q]$ has a congruence relation Θ_A such that the restriction of Θ_A to a quotient

a/b — where a/b is a prime quotient of A — is the image of Θ by the isomorphism ε_{ab}: $Q \to a/b$. The corresponding factor lattice $A[Q]/\Theta_A$ is isomorphic to A. Similarly $B'[Q]$ has a congruence relation Θ_B corresponding to Θ, and the factor lattice is isomorphic to B. By the definition of δ, $x \equiv y(\Theta_A)$ $(x, y \in C[Q])$ if and only if $\delta x \equiv \delta y(\Theta_A)$. That means that the restriction of Θ_A to $\delta C[Q]$ corresponds by φ to the restriction of Θ_B to $C'[Q]$. If follows that the join $\Theta_A \cup \Theta_B$ has an extension $\bar{\Theta}$ to L such that the restriction of $\bar{\Theta}$ to $A[Q]$ is Θ_A and the restriction $B'[Q]$ is Θ_B. Hence $M/\bar{\Theta}$ is isomorphic to M.

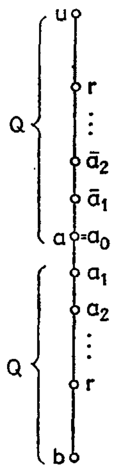

Fig. 1

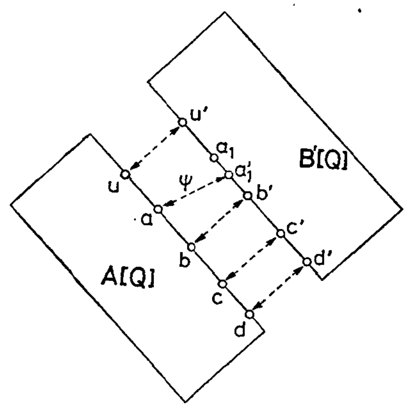

Fig. 2

Let π be the projectivity $a/b \approx c/d$ in A. ϱ denotes the projectivity $c/d \approx a/b$ in B. Thus we get the projectivity $\varrho \circ \pi : a/b \approx a/b$. It is easy to show that this projectivity has no inverse in L, i.e. by Lemma 1 of [2] we get that M is not finitely projected.

Theorem 2. *Let A be an ideal and let B be a filter of the finite modular lattice M such that $M = A \cup B$ and $C = A \cap B$ is a Boolean lattice. Let a/b and c/d be two prime quotients of C which are projective in A and in B. If M is finitely* **M**-*projected then a/b and c/d are projective in C.*

Proof. The proof is similar to the previous one. We define an injective endomorphism δ of $C[Q]$. a/b is isomorphic to Q, hence we can choose an arbitrary t such that $b < t < a$. Let u be the relative complement of b in the quotient a/o where o denotes the least element of $C[Q]$. Finally u' denotes the complement of u in $C[Q]$. Then the ideal $(t \vee u']$ of $C[Q]$ is isomorphic to $C[Q]$. We have therefore an injective endomorphism δ for which $\delta a = t$ and $\delta x = x$ for every $x \leqq u'$. We assume that $c, d \leqq u'$.

If φ denotes the isomorphism $C[Q] \to C'[Q]$ then $\psi = \varphi \circ \delta$ is an injection of $C[Q]$ into $C'[Q]$, such that the image of $C[Q]$ is an ideal of $C'[Q]$, $\psi a > a$, $\psi b = b$, $\psi c = c$, $\psi d = d$. Let L be the lattice obtained by gluing together $A[Q]$ and $B[Q]$ identifying the corresponding elements under ψ. We can finish the proof as in Theorem 1.

It is easy to generalize the previous theorems if we introduce the following notion.

Definition. Let M be a finite lattice. An injective endomorphism δ of $M[Q]$ is called a compression if the following properties are satisfied.

(i) $\delta(x) \leqq x$ for every $x \in M[Q]$ and $\delta M[Q]$ is an ideal of $M[Q]$;

(ii) there exists a $\Theta \in \mathrm{Con}\,(Q)$ with exactly two Θ-classes such that $\delta^{-1}(x) \equiv \equiv x \equiv \delta x(\bar{\Theta})$ for every x where $\bar{\Theta}$ denotes the extension of Θ to $M[Q]$.

Theorem 3. *Let A be an ideal and let B be a filter of the finite modular lattice M, such that $M = A \cup B$. Let further a/b and c/d be two prime quotients of $C = A \cap B$ which are projective in A and in B. If C has a compression δ such that $a > \delta a > \delta b = b$ and $\delta c = c$, $\delta d = d$ then M is not finitely* M-*projected.*

4. **Stable quotients.** Let a/b be a prime quotient of a finite modular lattice M. We define a new element t to M for which $a > t > b$. Then $M \cup \{t\}$ is a partial lattice with the sublattice M. $t \vee m$, $t \wedge m$ $(m \in M)$ are not defined. It is easy to show that there exists a lattice $\bar{M}$ freely generated by this partial lattice. We say that a/b is stable if $\bar{M}$ is finite. A. Mitschke and R. Wille have proved that every prime quotient of M_3 is stable. The prime-quotients of M_4 are not stable.

Conjecture. A finite modular lattice is finitely M-projected if and only if every prime quotient is stable.

It is easy to show — applying [2] — that a finite planar modular lattice is finitely M-projected if and only if every prime quotient is stable.

References

[1] A. Day, Splitting algebras and weak notion of projectivity, *Algebra Universalis*, **5** (1975), 153—162.

[2] A. Mitschke, E. T. Schmidt, R. Wille, On finitely projected modular lattices of breadth two, *in preparation.*

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