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The dual discriminator function in universal algebra

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1. Introduction and summary of results

For any set S the *(ternary) discriminator* t of S is the function from S^3 to S defined by $t(x, y, z)=x$ if $x \neq y$ and $=z$ if $x=y$.

The discriminator function has proved useful in the study of varieties generated by quasi-primal algebras — which includes the variety of Boolean algebras — and related areas of universal algebra. (See [14] and, for example, [15], [18], [19], [21].) Indeed, in the two element Boolean algebra $(\{0, 1\}, \vee, \wedge, ')$,

$$(x \wedge y') \vee (y' \wedge z) \vee (x \wedge z) \tag{1.1}$$

is a polynomial representing the discriminator of the set $\{0, 1\}$.

In the present paper we introduce the study of a closely related function, the *dual discriminator*, the function d from S^3 to S defined by $d(x, y, z)=x$ if $x=y$, and $=z$ if $x \neq y$.

We may think of the dual discriminator as playing a role which generalizes the "median" polynomial on the two element lattice in the same way that t generalizes (1.1); indeed, for the lattice $(\{0, 1\}, \vee, \wedge)$, the median,

$$(x \wedge y) \vee (y \wedge z) \vee (x \wedge z), \tag{1.2}$$

is a polynomial representing the dual discriminator of $\{0, 1\}$. More generally, the algebras in which the dual discriminator is a polynomial stand, roughly speaking, in the same relation to the two element lattice as quasi-primal algebras stand to the two element Boolean algebra; the purpose of the present paper is to give some grounds for this analogy. The two element lattice is, however, the only lattice in which the dual discriminator function is a polynomial. On the other hand, weakly associative lattices with the unique bound property ([5]) provide important examples of this extension of the theory of distributive lattices. Within this extended theory the special results of [7] are also of particular interest.

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As is suggested by the examples of Boolean algebras and distributive lattices, the discriminator is strictly "stronger" than the dual discriminator. In fact, from the definition we obviously have

$$d(x, y, z) = t(x, t(x, y, z), z), \tag{1.3}$$

but there is no way of expressing t in terms of d (as one sees by considering the two element lattice). There are, however, interesting relations connecting the two. Among these are the following dual functional equations:

$$t(x, y, d(x, y, z)) = x, \quad d(x, y, t(x, y, z)) = x. \tag{1.4}$$

Also, if f is the *4-ary discriminator* defined by

$$f(x, y, u, v) = u \quad \text{if} \quad x = y, \quad \text{and} \quad = v \quad \text{if} \quad x \neq y, \tag{1.5}$$

then

$$t(x, y, z) = f(x, y, z, x), \quad \text{and} \quad d(x, y, z) = f(x, y, x, z), \tag{1.6}$$

dually. (As is well known, f is equivalent to t since $f(x, y, u, v) = t(t(x, y, u), t(x, y, v), v)$.)

For terminology in the paper we shall generally follow GRÄTZER [9]. In particular, for a given type of algebras a *polynomial symbol* $p(x, y, \ldots)$ is simply a term in the first order theory of that type. If $\mathbf{A} = (A, F)$ is an algebra of this type, the *polynomial* $p^{\mathbf{A}}(x, y, \ldots)$ of $\mathbf{A}$ is the mapping induced on A by $p(x, y, \ldots)$. An *algebraic function* is a mapping of A obtained by inserting fixed elements of A in some of the argument places of a polynomial.

Summary of results. In Section 2 we shall discuss some simple relations between "discriminator" and "dual discriminator" varieties. We also show (Theorem 2.3) that a finite algebra $\mathbf{A}$ of more than two elements is functionally complete if and only if the dual discriminator is an algebraic function of $\mathbf{A}$. Finite algebras in which the dual discriminator is a polynomial are characterized (Theorem 2.4) in a way which generalizes a characterization of quasi-primal algebras. In Sections 3 and 4 we obtain an equational characterization (Theorem 3.2) of dual discriminator varieties. This result parallels an earlier result of MCKENZIE [11] for discriminator varieties. We also show (Theorem 3.11) that dual discriminator varieties have equationally definable principal congruences in the sense of [8], and examine the duality between "principal" and "co-principal" congruences in discriminator varieties. It is further shown (Theorem 4.2) that the discriminator behaves, in certain respects, like a generalized complementation operation. In Section 5 we examine weakly associative lattices and, in particular, obtain an explicit finite equational base for the variety generated by all weakly associative lattices having the unique bound property (Theorem 5.8).

2. Discriminator and dual discriminator varieties; functional completeness

A *discriminator variety* is a variety V having a ternary polynomial symbol $p(x, y, z)$ such that for each subdirectly irreducible (SI) algebra $\mathbf{A}\in V$, $p^{\mathbf{A}}(x, y, z)$ is the discriminator of A. Finite SI members of a discriminator variety are usually called *quasi-primal* algebras ([14]). Dually, let us say that a variety V is a *dual discriminator variety* if V has a ternary polynomial $q(x, y, z)$ such that $q^{\mathbf{A}}(x, y, z)$ is the dual discriminator of $\mathbf{A}$ for each SI algebra $\mathbf{A}\in V$.

Note that if V is a discriminator (respectively, dual discriminator) variety in which the polynomial symbols p and p' (respectively q and q') each induce the discriminator (respectively, dual discriminator) on each SI member of V, then $p=p'$ (respectively $q=q'$) is an equation of V. Briefly, the discriminator and dual discriminator are unique. Also note that by (1.3) each discriminator variety is a dual discriminator variety.

In addition to the variety of Boolean algebras, discriminator varieties include, as a few examples, all varieties of arithmetical rings (i.e.: varieties generated by finite sets of finite fields), varieties generated by simple relation algebras, and simple cylindric algebras. (See [21] for other examples.) Beyond the variety of distributive lattices the simplest dual discriminator variety is the variety W_3 generated by the "triangle" algebra $\mathbf{W}_3=(\{0, 1, a_1\}, \vee, \wedge)$ where $\vee, \wedge$ are the l.u.b. and g.l.b. respectively for the following reflexive and antisymmetric relation: $0\leqq 1\leqq a_1\leqq 0$. (See [6].) In this case the polynomial

$$[(z\wedge(x\wedge y))\vee(x\vee y)]\wedge[z\vee(x\wedge y)]$$

will be shown, in Section 5, to be the dual discriminator of the set $\{0, 1, a_1\}$. More generally, varieties generated by weakly associative lattices having the unique bound property ([5]) will also be shown to be dual discriminator varieties. Interesting special cases include the varieties $W_n, 2\leqq n<\omega$, generated by the algebras $\mathbf{W}_n=(\{0, 1, a_1, \dots, a_{n-2}\}, \vee, \wedge)$ where $\vee, \wedge$ are l.u.b. and g.l.b. for the reflexive and antisymmetric relation $0\leqq 1\leqq a_i\leqq 0$, $i=1, \dots, n-2$. These were introduced in [6].

The following are some simple comparative properties of the discriminator and dual discriminator.

2.1 Lemma. *If V is a discriminator variety or a dual discriminator variety, each nontrivial SI member of V is simple and has only simple nontrivial subalgebras.*

Proof. For the dual discriminator, if $\theta>\omega$ is a congruence of any subalgebra of $\mathbf{A}\in V$, let $(x, y)\in\theta$, $x\neq y$. Then for any z in the subalgebra,

$$x = q^{\mathbf{A}}(x, x, z)\ \theta\ q^{\mathbf{A}}(x, y, z) = z$$

For the discriminator,

$$x = p^{\mathbf{A}}(x, y, z)\ \theta\ p^{\mathbf{A}}(x, x, z) = z.$$

2.2 Lemma. i) *A discriminator variety V is arithmetical (i.e.: congruence permutable and congruence distributive); equivalently,*

$$p(x, x, z) = z, \quad p(x, y, x) = x, \quad p(x, z, z) = x$$

are equations of V.

ii) *A dual discriminator variety V is congruence distributive, in fact 2-distributive; equivalently,*

$$q(x, x, z) = x, \quad q(x, y, x) = x, \quad q(x, z, z) = z$$

are equations of V.

iii) *A dual discriminator variety is a discriminator variety if and only if it is congruence permutable.*

Proof. i) and ii) are well known; see, e.g., [14] and [10]. For iii), if V is congruence permutable and $m(x, y, z)$ is any Mal'cev polynomial symbol for V, then for any SI member $\mathbf{A}\in V$, $m^{\mathbf{A}}(x, q^{\mathbf{A}}(x, y, z), z)$ is clearly the discriminator of A. If V is a discriminator variety, then $p(x, y, z)$ is a Mal'cev polynomial symbol for V, by i). (Note that for *any* Mal'cev polynomial symbol $m(x, y, z)$ for V, and in particular for $m=p$, $m^{\mathbf{A}}(x, p^{\mathbf{A}}(x, y, z), z)$ is the dual discriminator of any SI $\mathbf{A}\in V$. In general the discriminator and its dual are interdefinable through any Mal'cev function.)

Recall that a finite algebra $\mathbf{A}$ is *functionally complete* ([13]) if each function $f\colon A^n\to A$, $0\leqq n<\omega$, is an algebraic function of $\mathbf{A}$. A well known criterion (due to Werner [20]) for functional completeness is that the discriminator of A be an algebraic function of $\mathbf{A}$. (Hence a quasi-primal algebra is obviously functionally complete.) From the remarks above it is clear that a finite SI algebra in a dual discriminator variety is quasi-primal (respectively, functionally complete), if and only if there is a polynomial $m(x, y, z)$ (respectively, algebraic function) of $\mathbf{A}$, satisfying $m(x, x, y)=y$ and $m(x, y, y)=x$. According to the following theorem, if $|A|>2$ and the dual discriminator is an algebraic function of $\mathbf{A}$, then such an algebraic function $m(x, y, z)$ always exists.

2.3 Theorem. *Let $\mathbf{A}=(A, F)$ be a finite algebra of order greater than 2. The dual discriminator of A is an algebraic function of $\mathbf{A}$ if and only if $\mathbf{A}$ is functionally complete.*

Proof. The "if" direction is trivial. To prove the "only if" direction we establish two claims:

Claim 1. If $\mathbf{A}$ is a finite algebra having a majority polynomial (i.e.: a ternary polynomial satisfying the equations of Lemma 2.2, ii)) and if $\mathbf{A}\times\mathbf{A}$ has only two subalgebras (the diagonal $\Delta=\{(a, a): a\in A\}$ and $\mathbf{A}\times\mathbf{A}$), then $\mathbf{A}$ is primal.

Proof of Claim 1. Since $\mathbf{A}$ has a majority polynomial it follows from [3, Corollary 5.1] that the polynomials of $\mathbf{A}$ are exactly the functions $f\colon A^n\to A$ such that each subalgebra of $\mathbf{A}\times\mathbf{A}$ is closed under f; i.e.: if $\mathbf{S}$ is a subalgebra of $\mathbf{A}\times\mathbf{A}$ and $(x_i, y_i)\in S$, $i=1, \dots, n$, then

$$f((x_1, y_1), \dots, (x_n, y_n)) = (f(x_1, \dots, x_n), f(y_1, \dots, y_n))\in S.$$

But since the subalgebras Δ and $\mathbf{A}\times\mathbf{A}$ are closed under any $f\colon A^n\to A$ and since these are the only subalgebras of $\mathbf{A}\times\mathbf{A}$, the claim follows.

Claim 2. Suppose $\mathbf{A}=(A, F)$ is finite with $n>2$ elements and the dual discriminator of A is an algebraic function of $\mathbf{A}$. Let $\mathbf{S}$ be a subalgebra of $\mathbf{A}\times\mathbf{A}$ such that $\Delta\subseteqq S$. Then $S=\Delta$ or $S=A\times A$.

Proof of Claim 2. Let the distinct elements of A be $a_1, \dots, a_n$, $n>2$. Since the dual discriminator of A is an algebraic function q of $\mathbf{A}$ and since $\Delta\subseteqq S$, q extends (coordinate-wise) to S. Suppose $(a_i, a_j)\in S$ for some $i\neq j$, i.e.: suppose S contains some off-diagonal element. Then for all r,

$$q((a_i, a_i), (a_i, a_j), (a_r, a_r)) = (q(a_i, a_i, a_r), q(a_i, a_j, a_r)) = (a_i, a_r)\in S.$$

Hence for all $s, r, r\neq i$,

$$q((a_r, a_r), (a_i, a_r), (a_s, a_s)) = (q(a_r, a_i, a_s), q(a_r, a_r, a_s)) = (a_s, a_r)\in S.$$

Finally, choose m different from both i and j, which is possible since $n>2$. Then for all s,

$$q((a_s, a_j), (a_s, a_m), (a_i, a_i)) = (q(a_s, a_s, a_i), q(a_j, a_m, a_i)) = (a_s, a_i)\in S.$$

Hence $S=A\times A$ so Claim 2 is proved.

To complete the proof of Theorem 2.3 let $\mathbf{A}$ satisfy the hypotheses. Let $\mathbf{A}^+$ be the algebra obtained from $\mathbf{A}$ by adjoining as new nullary operations all elements of A. Then the dual discriminator is a majority polynomial of $\mathbf{A}^+$ and for each subalgebra $\mathbf{S}$ of $\mathbf{A}^+\times\mathbf{A}^+$, $\Delta\subseteqq S$. By Claim 2, $\mathbf{A}^+\times\mathbf{A}^+$ has only Δ and $\mathbf{A}^+\times\mathbf{A}^+$. as subalgebras. Hence by Claim 1, $\mathbf{A}^+$ is primal. Thus $\mathbf{A}$ is functionally complete.

Note. The two element lattice, for which the median is the dual discriminator, is not functionally complete — since algebraic functions of lattices are isotone. Hence the condition that $\mathbf{A}$ be of order greater than 2 is essential.

Quasi-primal algebras were originally defined ([14]) as finite algebras $\mathbf{A}$ having the following property:

Let $f\colon A^n \to A$ be any function. If each subalgebra $\mathbf{S}$ of $\mathbf{A}\times\mathbf{A}$ with subuniverse of the form $S=\{(x, x\alpha): x\in\operatorname{dom}(\alpha)\}$, α an internal isomorphism of $\mathbf{A}$, is closed under the coordinate-wise extension $f\times f$ of f, then f is a polynomial of $\mathbf{A}$. It is therefore natural to ask for a similar characterization of finite algebras having the dual discriminator as a polynomial. To do this we introduce the following definition:

For an algebra $\mathbf{A}$, a subalgebra $\mathbf{S}$ of $\mathbf{A}\times\mathbf{A}$ is *projectively rectangular* (or, briefly *p-rectangular*) if S has the following two properties:

i) (x, y_1), (x, y_2), $(u, v)\in S$ and $y_1\neq y_2$ imply $(x, v)\in S$,

ii) (x_1, y), (x_2, y), $(u, v)\in S$ and $x_1\neq x_2$ imply $(u, y)\in S$.

2.4. Theorem. *For a finite algebra* $\mathbf{A}$ *the following are equivalent:*

a) *The dual discriminator of* A *is a polynomial of* $\mathbf{A}$.

b) *If* $f\colon A^n\to A$ *is any function such that each p-rectangular subalgebra of* $\mathbf{A}\times\mathbf{A}$ *is closed under the coordinate-wise extension* $f\times f$ *of* f, *then* f *is a polynomial of* $\mathbf{A}$.

Proof. a)$\Rightarrow$b). The dual discriminator $q^{\mathbf{A}}$ is a majority polynomial of $\mathbf{A}$. Hence, by [3, Corollary 5.1], the polynomials of $\mathbf{A}$ are just the functions $f\colon A^n\to A$ such that all subalgebras $\mathbf{S}$ of $\mathbf{A}\times\mathbf{A}$ are closed under f. Hence we need only show that each subalgebra $\mathbf{S}$ of $\mathbf{A}\times\mathbf{A}$ is p-rectangular. But if (x, y_1), (x, y_2), $(u, v)\in S$ and $y_1\neq y_2$, then $q^{\mathbf{A}}((x, y_1), (x, y_2), (u, v))=(x, v)\in S$, and if (x_1, y), (x_2, y), $(u, v)\in S$ and $x_1\neq x_2$, then $q^{\mathbf{A}}((x_1, y), (x_2, y), (u, v))=(u, y)\in S$.

b)$\Rightarrow$a). Let $\mathbf{S}$ be a p-rectangular subalgebra of $\mathbf{A}\times\mathbf{A}$ and let (x_1, y_1), (x_2, y_2), $(x_3, y_3)\in S$. Then

$$\begin{aligned} d((x_1, y_1), (x_2, y_2), (x_3, y_3)) &= (d(x_1, x_2, x_3),\ d(y_1, y_2, y_3)) = \\ &= (x_1, y_1) \quad \text{if} \quad x_1 = x_2 \quad \text{and} \quad y_1 = y_2 \\ &= (x_1, y_3) \quad \text{if} \quad x_1 = x_2 \quad \text{and} \quad y_1 \neq y_2 \\ &= (x_3, y_1) \quad \text{if} \quad x_1 \neq x_2 \quad \text{and} \quad y_1 = y_3 \\ &= (x_3, y_3) \quad \text{if} \quad x_1 \neq x_2 \quad \text{and} \quad y_1 \neq y_3, \end{aligned}$$

hence $\mathbf{S}$ is closed under the dual disciiminator d. Thus d is a polynomial of $\mathbf{A}$.

Note. The conditions defining a p-rectangular subalgebra are equivalent to the following (p_1, p_2 are, respectively, the first and second projections):

i)′ If S contains 2 points of $\{x\}\times Sp_2$ then S contains $\{x\}\times Sp_2$,

ii)′ If S contains 2 points of $Sp_1\times\{y\}$ then S contains $Sp_1\times\{y\}$.

To compare quasi-primal algebras with algebras having the dual discriminator as a polynomial, let $\mathbf{A}$ be quasi-primal and suppose $\mathbf{S}$ is a subalgebra of $\mathbf{A}\times\mathbf{A}$ such that S contains two points, (x, y_1), (x, y_2) of $\{x\}\times Sp_2$. Then for any (u_1, v_1), $(u_2, v_2)\in S$,

$f((x, y_1), (x, y_2), (u_1, v_1), (u_2, v_2)) = (u_1, v_2) \in S$ if f is the 4-ary discriminator (1.5). Hence $S = Sp_1 \times Sp_2$. On the other hand, Theorem 2.3 shows that for sets with at least three elements, the constant functions together with either the discriminator or the dual discriminator, generate all functions.

3. Characterization and properties of dual discriminator varieties

The following theorem, which gives an equational characterization of discriminator varieties, appears in McKenzie [11].

3.1 Theorem. *For a variety V and ternary polynomial symbol $p(x, y, z)$, the following are equivalent:*

1) *V is a discriminator variety with $p(x, y, z)$ the discriminator on each SI member of V.*

2) *The following are equations of V:*

a) $p(x, z, z) = x, \quad p(x, y, x) = x, \quad p(x, x, z) = z,$

b) $p(x, p(x, y, z), y) = y,$

c) *for each operation symbol f of V,*

$$p(x, y, f(z_1, \dots, z_k)) = p(x, y, f(p(x, y, z_1), \dots, p(x, y, z_k)))$$

(where f is k-ary).

The proof depends essentially on observing that for any $\mathbf{A} \in V$ and $a, b \in A$, the principal congruence $\theta(a, b)$ is given by

$$\theta(a, b) = \{(x, y) \in A \times A \colon p^{\mathbf{A}}(a, b, x) = p^{\mathbf{A}}(a, b, y)\}.$$

For dual discriminator varieties we have the following corresponding result:

3.2 Theorem. *For a variety V and ternary polynomial symbol $q(x, y, z)$, the following are equivalent:*

1) *V is a dual discriminator variety with $q(x, y, z)$ the dual discriminator on each SI member of V.*

2) *The following are equations of V:*

a) $q(x, z, z) = z, \quad q(x, y, x) = x, \quad q(x, x, z) = x,$

b) $q(x, y, q(x, y, z)) = q(x, y, z),$

c) $q(z, q(x, y, z), q(x, y, w)) = q(x, y, z),$

d) *for each operation symbol f of V,*

$$q(x, y, f(z_1, \dots, z_k)) = q(x, y, f(q(x, y, z_1), \dots, q(x, y, z_k)))$$

(where f is k-ary).

If V is an idempotent variety the equations d) *may be replaced by*

d′) $q(x, y, f(z_1, \dots, z_k)) = f(q(x, y, z_1), \dots, q(x, y, z_k))$ [1].

Proof. If 1) holds then it is easy to check that a)—d) are equations of each SI member of V and hence of V. If V is idempotent then 1) clearly implies d′).

Conversely, suppose a)—d) are equations of V. For each $\mathbf{A} \in V$ and $a, b \in A$, define the *co-principal* congruence $\gamma(a, b)$ by

$$\gamma(a, b) = \{(x, y) \in A \times A \colon q^{\mathbf{A}}(a, b, x) = q^{\mathbf{A}}(a, b, y)\}.$$

By d) (or d′)) if V is idempotent) $\gamma(a, b)$ is easily seen to be a congruence of $\mathbf{A}$. Next observe that, by b),

$$(3.3) \qquad (q^{\mathbf{A}}(x, y, z), z) \in \gamma(x, y) \quad \text{for all} \quad x, y, z \in A.$$

Now $q^{\mathbf{A}}(x, x, z) = x$ by a). Hence to complete the proof it will suffice to show that if $\mathbf{A}$ is SI and $x, y \in A$, then

$$(3.4) \qquad x \neq y \quad \text{implies} \quad \gamma(x, y) = \omega,$$

for, by (3.3), this will mean $q^{\mathbf{A}}(x, y, z) = z$ if $x \neq y$.

As a preliminary we first establish the following implication:

$$(3.5) \quad x \neq y \quad \text{implies} \quad \gamma(q^{\mathbf{A}}(x, y, z), z) \neq \omega, \quad \text{for any} \quad \mathbf{A} \in V \quad \text{and} \quad x, y, z \in A.$$

To prove (3.5) we observe that, by c), we have

$$q^{\mathbf{A}}(z, q^{\mathbf{A}}(x, y, z), q^{\mathbf{A}}(x, y, w)) = q^{\mathbf{A}}(z, q^{\mathbf{A}}(x, y, z), q^{\mathbf{A}}(x, y, z))$$

and hence $(q^{\mathbf{A}}(x, y, z), q^{\mathbf{A}}(x, y, w)) \in \gamma(z, q^{\mathbf{A}}(x, y, z))$ for all $x, y, z, w \in A$. If $\gamma(z, q^{\mathbf{A}}(x, y, z)) = \omega$ for some x, y, z, then $q^{\mathbf{A}}(x, y, z) = q^{\mathbf{A}}(x, y, w)$ for all $w \in A$. In particular, using a), we would then have

$$x = q^{\mathbf{A}}(x, y, x) = q^{\mathbf{A}}(x, y, z) = q^{\mathbf{A}}(x, y, y) = y.$$

This establishes (3.5).

Now let $\mathbf{A}$ be SI in V. Choose $a, b \in A$, $a \neq b$, such that $(a, b) \in \theta$ for all congruences $\theta \neq \omega$. Let us suppose we have a pair $x, y \in A$ contradicting (3.4), i.e.: such that $x \neq y$ and $\gamma(x, y) \neq \omega$. Then $(a, b) \in \gamma(x, y)$ so $q^{\mathbf{A}}(x, y, a) = q^{\mathbf{A}}(x, y, b)$. Denote the common value of these expressions by $c \in A$. By (3.5) we have

$$\gamma(a, q^{\mathbf{A}}(x, y, a)) = \gamma(a, c) \neq \omega \quad \text{and} \quad \gamma(b, q^{\mathbf{A}}(x, y, b)) = \gamma(b, c) \neq \omega$$

Hence $(a, b) \in \gamma(a, c) \cap \gamma(b, c)$ so that, by a),

$$a = q^{\mathbf{A}}(a, c, a) = q^{\mathbf{A}}(a, c, b) \quad \text{and} \quad b = q^{\mathbf{A}}(b, c, b) = q^{\mathbf{A}}(b, c, a).$$

[1]) In Theorem 3.1, if V is idempotent, the equations c) may be analogously simplified.

Since $a \neq b$, we have $c \neq a$ or $c \neq b$. If $c \neq a$ then, taking $x=a$, $y=c$, $z=b$ in (3.5), we obtain $\gamma(q^A(a, c, b), b) = \gamma(a, b) \neq \omega$. Hence $(a, b) \in \gamma(a, b)$ which implies $a = q^A(a, b, a) = q^A(a, b, b) = b$, a contradiction. If $c \neq b$ then, taking $x=b$, $y=c$, $z=a$ in (3.5), we obtain $\gamma(q^A(b, c, a), a) = \gamma(b, a) \neq \omega$ which again leads to the contradiction $a=b$. Hence (3.4) is established, completing the proof.

Notice that on SI members of a dual discriminator variety we have:

$$\theta(a, b) = \omega \quad \text{if} \quad a = b, \quad \text{and} \quad = \iota \quad \text{if} \quad a \neq b,$$

$$\gamma(a, b) = \iota \quad \text{if} \quad a = b, \quad \text{and} \quad = \omega \quad \text{if} \quad a \neq b.$$

We compare these congruences more closely. First observe that in a dual discriminator variety V, $q(x, y, u)$ and $q(x, y, v)$ are *principal intersection polynomials* in the sense of BAKER [1], i.e.: the polynomial symbols $D_1(x, y, u, v) = q(x, y, u)$ and $D_2(x, y, u, v) = q(x, y, v)$ have the property that on any SI member $\mathbf{A}$ of V,

$$D_1^{\mathbf{A}}(x, y, u, v) = D_2^{\mathbf{A}}(x, y, u, v) \quad \text{iff} \quad x = y \quad \text{or} \quad u = v. \tag{3.6}$$

From [1] it then follows that for any $\mathbf{A} \in V$, the meet of principal congruences $\theta(a, b)$ and $\theta(c, d)$ is principal and is given by

$$\theta(a, b) \wedge \theta(c, d) = \theta(q^{\mathbf{A}}(a, b, c), q^{\mathbf{A}}(a, b, d)). \tag{3.7}$$

Using this observation we have

3.8 Theorem. *Let $\mathbf{A}$ be any algebra in a dual discriminator variety. For any $a, b \in A$, the principal and co-principal congruences $\theta(a, b)$ and $\gamma(a, b)$ are complements and, in particular,*

$$\gamma(a, b) \circ \theta(a, b) \circ \gamma(a, b) = \iota \quad (\circ \textit{ denotes relation product}).$$

Proof. For all $x, y \in A$, using equations a), b) of Theorem 3.2, we have

$$x \; \gamma(a, b) \; q^{\mathbf{A}}(a, b, x) \; \theta(a, b) \; q^{\mathbf{A}}(a, a, x) = a = q^{\mathbf{A}}(a, a, y) \; \theta(a, b) \; q^{\mathbf{A}}(a, b, y) \; \gamma(a, b) \; y$$

Hence $\gamma(a, b) \circ \theta(a, b) \circ \gamma(a, b) = \iota$. For the meet, if $(x, y) \in \theta(a, b) \wedge \gamma(a, b)$ then $(x, y) \in \theta(a, b) \wedge \theta(x, y) = \theta(q^{\mathbf{A}}(a, b, x), q^{\mathbf{A}}(a, b, y))$ by (3.7). But $q^{\mathbf{A}}(a, b, x) = q^{\mathbf{A}}(a, b, y)$ since $(x, y) \in \gamma(a, b)$. Hence $x = y$, whence $\theta(a, b) \wedge \gamma(a, b) = \omega$.

3.9 Corollary. *In a dual discriminator variety the join of co-principal congruent ces is co-principal and is given by*

$$\gamma(a, b) \vee \gamma(c, d) = \gamma(q^{\mathbf{A}}(a, b, c), q^{\mathbf{A}}(a, b, d)). \tag{3.10}$$

Proof. Apply Theorem 3.8 and congruence distributivity, taking complements of both sides of (3.7).

Theorem 3.8 has several important consequences. Recall from [12] that a variety V has *definable principal congruences* if there is a formula $\beta(u, v, x, y)$ in the firs-order language of V such that for all $\mathbf{A} \in V$ and $a, b, c, d \in A$,

$$(c, d) \in \theta(a, b) \quad \text{iff} \quad \mathbf{A} \models \beta(a, b, c, d).$$

A stronger concept, *equationally definable principal congruences,* was introduced in [8].

3.11 Theorem. *If V is a dual discriminator variety, then V has equationally definable principal congruences. In particular, for $\mathbf{A} \in V$, $a, b, c, d \in A$,*

(3.12) $$(c, d) \in \theta(a, b) \quad \text{iff} \quad A \models (\forall u)[q(c, d, u) = q(c, d, q(a, b, u))].$$

Proof. First notice that

$$(c, d) \in \theta(a, b) \quad \text{iff} \quad \theta(c, d) \leqq \theta(a, b) \quad \text{iff} \quad \gamma(a, b) \leqq \gamma(c, d)$$

by Theorem 3.8. But $\gamma(a, b) \leqq \gamma(c, d)$ is equivalent to the condition .

(3.13) $$(\forall u_1, u_2 \in A)[q^{\mathbf{A}}(a, b, u_1) = q^{\mathbf{A}}(a, b, u_2) \Rightarrow q^{\mathbf{A}}(c, d, u_1) = q^{\mathbf{A}}(c, d, u_2)].$$

Clearly the right side of (3.12) implies (3.13) and, taking $u_2 = q^{\mathbf{A}}(a, b, u_1)$, the right side of (3.12) follows from (3.13) and equation b) of Theorem 3.2.

3.14 Corollary. *If V is a dual discriminator variety and $\mathbf{A} \in V$ is a subdirect product of algebras $\mathbf{A}_i$, $i \in I$, then for $a, b, c, d \in A$,*

(3.15) $$(c, d) \in \theta(a, b) \quad \text{iff} \quad (\forall i \in I)[(c_i, d_i) \in \theta(a_i, b_i)].$$

Corollary 3.14 is immediate from the universal form of the formula appearing in (3.12); it asserts that V has *factor determined principal congruences* in the sense of [8].

From [8, Theorem 4.5] we also obtain:

3.16 Corollary. *A dual discriminator variety has the congruence extension property.*

McKenzie [12] has shown that if a variety V of finite type has only finitely many SI members, all finite, and has definable principal congruences, then V has a finite equational base. On the other hand Baker's finite basis theorem [2] asserts that a congruence distributive variety of finite type generated by a finite algebra always has a finite equational base. Hence if $\mathbf{A}$ is a finite algebra in a dual discriminator variety (of finite type), then $\mathbf{A}$ has a finite equational base either as a result of Baker's theorem and Lemma 2.2, ii), or more briefly, from McKenzie's result and Theorem 3.11. Despite the fact that we have these two proofs it is still instructive to establish the result directly, illustrating McKenzie's method. We do this as follows:

If $d(x_0, x_1, x_2)$ is the dual discriminator of a set, define inductively

$$\begin{aligned} d_1(x_0, x_1, x_2) &= d(x_0, x_1, x_2), \\ d_2(x_0, x_1, x_2, x_3) &= d(x_0, d_1(x_0, x_1, x_2), x_3), \\ &\vdots \\ d_n(x_0, \dots, x_{n+1}) &= d(x_0, d_{n-1}(x_0, \dots, x_n), x_{n+1}), \end{aligned}$$

and observe that

$$\begin{aligned} d_n(x_0, \dots, x_{n+1}) &= x_0 \quad \text{if} \quad x_0 \text{ equals any of } x_1, \dots, x_n, \\ &= x_{n+1} \quad \text{otherwise.} \end{aligned}$$

It follows that on any set S the sentence

$$(\forall x_0, \dots, x_n) \bigvee_{0 \leqq i < j \leqq n} (x_i = x_j), \quad (\text{meaning } |S| \leqq n),$$

is true in S if and only if the following equation N_n holds in S:

$$\begin{aligned} & d_n(x_0, \dots, x_n, d_{n-1}(x_1, \dots, x_n, d_{n-2}(x_2, \dots, d_2(x_{n-2}, x_{n-1}, x_n, x_n) \dots) = \\ &= d_n(x_0, \dots, x_n, d_{n-1}(x_1, \dots, x_n, d_{n-2}(x_2, \dots, d_2(x_{n-2}, x_{n-1}, x_n, x_{n-1}) \dots). \end{aligned}$$

For example, N_3 is

$$d_3(x_0, x_1, x_2, x_3, d_2(x_1, x_2, x_3, x_3)) = d_3(x_0, x_1, x_2, x_3, d_2(x_1, x_2, x_3, x_2)).$$

Now let V be a dual discriminator variety of finite type and let $\mathbf{A}$ be a finite algebra in V. By congruence distributivity and either [13, Theorem 2.5] or [10], the variety generated by $\mathbf{A}$ is

$$V(\mathbf{A}) = IP_S HS(\mathbf{A})$$

and thus we can effectively determine from the finitely many isomorphism types of $HS(\mathbf{A})$, all of the isomorphism types of the SI members of $V(\mathbf{A})$. Let these be $K_0 = \{\mathbf{A}_1, \dots, \mathbf{A}_k\}$ and let $n = \max\{|A_i| : i = 1, \dots, k\}$. Then the equations a), b), c), d) of Theorem 3.2, together with N_n (with q_i replacing $d_i, i = 1, \dots, n$) are an equational base for the variety generated by

$K_1 = \{\mathbf{B} : \mathbf{B}$ is SI of the given type, $|B| \leqq n$, and $q^{\mathbf{B}}$ is the dual discriminator of $\mathbf{B}\}$.

To this extent the basis is canonical. Next we can obviously effectively list the members of K_1 and, by congruence distributivity, for each $\mathbf{B}_i$ in K_1, either $\mathbf{B}_i$ is isomorphic with some algebra of K_0 or there is an equation e_i which is an identity of each member of K_0 but not of $\mathbf{B}_i$. Let $e_{i(1)}, \dots, e_{i(m)}$, $m \geqq 0$, be such "exclusion" equations, which can clearly be effectively determined. Then the equations a), b), c), d), $N_n, e_{i(1)}, \dots, e_{i(m)}$ are a finite equational base for $\mathbf{A}$.

In Section 5 we shall apply Theorem 3.2 even more directly to obtain explicit equational bases for the varieties a) generated by all weakly associative lattices having the unique bound property, and b) the variety generated by the triangle algebra $\mathbf{W}_3$.

4. Dual discriminator varieties and the discriminator

Recall that a congruence permutable distributive lattice is necessarily relatively complemented. (Of course the converse is always true whether the lattice is distributive or not.) The following theorem generalizes this fact to dual discriminator varieties. (Cf. Lemma 2.2.)

4.1 Lemma. *Let* $\mathbf{A}$ *be any algebra in a dual discriminator variety and let* $f\colon A^3 \to A$ *be any function which is compatible with all of the congruences of* $\mathbf{A}$ *and which satisfies the following equations for all* $x, y, z \in A$:

$$q^{\mathbf{A}}(x, y, f(x, y, z)) = x, \quad f(x, x, z) = z.$$

Then f *induces the discriminator on each SI homomorphic image of* $\mathbf{A}$.

Proof. $q^{\mathbf{A}}(x, y, f(x, y, z)) = x = q^{\mathbf{A}}(x, y, x)$ so that $(f(x, y, z), x) \in \gamma(x, y)$ in any SI homomorphic image. But if $x \neq y$, $\gamma(x, y) = \omega$, so that $f(x, y, z) = x$.

4.2 Theorem. *If* $\mathbf{A}$ *is any algebra in a dual discriminator variety the following are equivalent:*

a) $\mathbf{A}$ *is congruence permutable.*

b) *There is a ternary function* $f\colon A^3 \to A$ *which is compatible with all congruences of* $\mathbf{A}$ *and which induces the discriminator on each SI homomorphic image of* $\mathbf{A}$.

Example. If $\mathbf{A}$ is a congruence permutable distributive lattice, then, by Theorem 4.2, such a function f exists and induces the discriminator on the two element lattice. Hence on $\{0, 1\}$

$$f(1, y, 0) = f(0, y, 1) = y' \quad \text{(complement).}$$

From this it follows that for y in the interval $[x, z]$ of $\mathbf{A}$, $f(x, y, z)$ is the relative complement of $y = q^{\mathbf{A}}(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (x \wedge z)$. (Cf. (1.1) and (1.2).)

Proof of Theorem 4.2. Suppose $\mathbf{A}$ is congruence permutable and $x, y, z \in A$. Then (as noted in the proof of Theorem 3.8),

$$x = q^{\mathbf{A}}(x, x, z)\ \theta(x, y)\ q^{\mathbf{A}}(x, y, z)\ \gamma(x, y)\ z.$$

Hence, by permutability, there is a $c \in A$ such that

$$x\ \gamma(x, y)\ c\ \theta(x, y)\ z,$$

which means $[x]\gamma(x, y) \cap [z]\theta(x, y) \neq \emptyset$. Thus, by the Axiom of Choice, there is a function $f\colon A^3 \to A$ such that for each $x, y, z \in A$,

$$x\ \gamma(x, y)\ f(x, y, z)\ \theta(x, y)\ z.$$

Hence,

$$\text{i) } q^{\mathbf{A}}(x, y, f(x, y, z)) = q^{\mathbf{A}}(x, y, x) = x, \quad \text{ii) } f(x, y, z)\ \theta(x, y)\ z.$$

To show that f is compatible with the congruences of **A** first let φ be a completely meet irreducible congruence (i.e.: such that $\mathbf{A}/\varphi$ is SI). Let (x, x_1), (y, y_1), $(z, z_1)\in\varphi$. If $(x, y)\in\varphi$ then $(x_1, y_1)\in\varphi$, so $\theta(x, y)\leqq\varphi$ and $\theta(x_1, y_1)\leqq\varphi$ and, by ii), $f(x, y, z)\ \varphi\ z\ \varphi\ z_1\ \varphi\ f(x_1, y_1, z_1)$. If $(x, y)\notin\varphi$ then $(x_1, y_1)\notin\varphi$ so that

$$f(x, y, z)\ \varphi\ q^{\mathbf{A}}(x, y, f(x, y, z)) = x\ \varphi\ x_1 = q^{\mathbf{A}}(x_1, y_1, f(x_1, y_1, z_1))\ \varphi\ f(x_1, y_1, z_1)$$

by i). Hence f is compatible with φ. Since any congruence is the meet of completely meet irreducible congruences, it easily follows that f is compatible with all congruences of A. Hence f meets the conditions of Lemma 4.1 so that it induces the discriminator on each SI homomorphic image of A.

Conversely, if f satisfying b) exists and θ_1, θ_2 are congruences of **A** with $x\ \theta_1\ y\ \theta_2\ z$ then, by the compatibility of f,

$$x = f(x, z, z)\ \theta_2\ f(x, y, z)\ \theta_1\ f(x, x, z) = z$$

so that **A** is congruence permutable.

Finally we observe that in any discriminator variety V (which by Lemma 2.2 is necessarily a dual discriminator variety) we have, in addition to formulas (3.6), (3.7), and (3.10), their duals. Indeed, we may call the polynomial symbols $p(x, y, u)$ and $p(x, y, v)$ *principal join polynomials,* since for SI algebras $\mathbf{A}\in V$,

(3.6)′ $\qquad p^{\mathbf{A}}(x, y, u) = p^{\mathbf{A}}(y, x, v)$ iff $x = y$ and $u = v$.

(Note the reversal of x and y.) Using (3.6)′ we can deduce

(3.7)′ $\qquad \theta(a, b)\vee\theta(c, d) = \theta(a, b)\circ\theta(c, d) = \theta(p^{\mathbf{A}}(a, b, c),\ p^{\mathbf{A}}(b, a, d)).$

($\vee=\circ$ since V is congruence permutable by Lemma 2.2.)

To prove (3.7)′ we observe that by the remark following Theorem 3.1,

$(x, z)\in\theta(a, b)\circ\theta(c, d)$ iff $(\exists y)[p^{\mathbf{A}}(a, b, x) = p^{\mathbf{A}}(a, b, y)$ and $p^{\mathbf{A}}(c, d, y) = p^{\mathbf{A}}(c, d, z)]$.

Hence on each SI factor $\mathbf{A}_i$ of **A**,

$$p^{\mathbf{A}_i}(a_i, b_i, x_i) = p^{\mathbf{A}_i}(a_i, b_i, y_i) \quad\text{and}\quad p^{\mathbf{A}_i}(c_i, d_i, y_i) = p^{\mathbf{A}_i}(c_i, d_i, z_i),$$

from which we directly infer

(4.3) $\quad p^{\mathbf{A}_i}(p^{\mathbf{A}_i}(a_i, b_i, c_i),\ p^{\mathbf{A}_i}(b_i, a_i, d_i),\ x_i) = p^{\mathbf{A}_i}(p^{\mathbf{A}_i}(a_i, b_i, c_i),\ p^{\mathbf{A}_i}(b_i, a_i, d_i),\ z_i),$

using (3.6)′. From (4.3) we have $(x, z)\in\theta(p^{\mathbf{A}}(a, b, c), p^{\mathbf{A}}(b, a, d))$ by Corollary 3.14.

Conversely, if $(x, z)\in\theta(p^{\mathbf{A}}(a, b, c), p^{\mathbf{A}}(b, a, d))$ then (4.3) holds on each SI factor $\mathbf{A}_i$ of **A**. Thus if $a_i=b_i$ and $c_i=d_i$, then $x_i=z_i$ while $\theta(a_i, b_i)=\iota$ if $a_i\neq b_i$ and likewise for $c_i\neq d_i$. From this it follows that

$$x_i\ \theta(a_i, b_i)\ p^{\mathbf{A}_i}(p^{\mathbf{A}_i}(a_i, b_i, x_i),\ p^{\mathbf{A}_i}(a_i, b_i, z_i),\ z_i)\ \theta(c_i, d_i)\ z_i$$

since the middle term is x_i if $a_i=b_i$ and z_i if $a_i\neq b_i$. Hence, by Corollary 3.14,

$$x\,\theta(a,b)\,p^{\mathbf{A}}\big(p^{\mathbf{A}}(a,b,x),\,p^{\mathbf{A}}(a,b,z),\,z\big)\,\theta(c,d)\,z,$$

establishing (3.7)′. Complementing both sides of (3.7)′ we obtain

$$(3.10)' \qquad \gamma(a,b)\wedge\gamma(c,d)=\gamma\big(p^{\mathbf{A}}(a,b,c),\,p^{\mathbf{A}}(b,a,d)\big).$$

Formulas (3.6)′ and (3.7)′ together with some additional properties of principal congruences in discriminator varieties can also be found in [4].

5. Weakly associative lattices

Recall from [7] that an algebra $\mathbf{A}=(A,\vee,\wedge)$ is a *weakly associative lattice* (WAL) if the operations $\vee$ and $\wedge$ are binary and satisfy the following identities in A:

$$(5.1)\qquad \begin{array}{lll} x\vee x=x, & x\wedge x=x, & \text{(idempotence)}\\ x\vee y=y\vee x, & x\wedge y=y\wedge x, & \text{(commutativity)}\\ x\wedge(x\vee y)=x, & x\vee(x\wedge y)=x, & \text{(absorption)}\\ \multicolumn{2}{c}{\left.\begin{array}{l}((x\wedge z)\vee(y\wedge z))\vee z=z\\ ((x\vee z)\wedge(y\vee z))\wedge z=z\end{array}\right\}} & \text{(weak associativity)}\end{array}$$

WALs have also been called *trellises* by SKALA [17]. *Tournaments* ([7]) and the algebras $\mathbf{W}_n$ of Section 2 are special cases of WALs. A WAL has the *unique bound property* (UBP) ([5]) if for distinct $a,b\in A$, $a\leqq c$ and $b\leqq c$ imply $c=a\vee b$ and, dually, $d\leqq a$, $d\leqq b$ imply $d=a\wedge b$. For brevity we shall call a WAL with the UBP simply a UBP. In [6] it was shown that for a WAL $\mathbf{A}$ the following are equivalent:

a) $\mathbf{A}$ is a UBP,

b) $\mathbf{A}$ is SI and satisfies the congruence extension property.

c) Each subalgebra of $\mathbf{A}$ is simple.

The following theorem adds a new equivalence to this list. Combined with Theorem 2.3 it also provides a new proof that a finite UBP of more than two elements is functionally complete. (See [5] for the original proof.)

5.2 Theorem. *A WAL $\mathbf{A}$ is a UBP if and only if the dual discriminator is a polynomial of $\mathbf{A}$. In particular the WAL polynomial symbol $q_u(x,y,z)$, explicitly constructed in the proof below, has the property: For any UBP $\mathbf{A}$, $q_u^{\mathbf{A}}(x,y,z)$ is the dual discriminator of A.*

Proof. If the dual discriminator is a polynomial q of $\mathbf{A}$ and if $\mathbf{A}$ is not a UBP, then for some $a,b\in A$, $a\neq b$, $a\leqq c$, $b\leqq c$, and $a\vee b<c$; while $q(c,a\vee b,a)=a$. But for the three element chain with elements $\{a,a\vee b,c\}$ the mapping α onto $\mathbf{2}$

given by: $a\alpha=0$, $(a\vee b)\alpha=c\alpha=1$, is a homomorphism. But since $q(1,1,0)=1$, this is a contradiction. Hence, $\mathbf{A}$ must have unique upper bounds, and dually, unique lower bounds. (Essentially the same proof is used in [5, Theorem 4].)

Conversely, define the polynomial symbols h, h', g by

$$h(x,y,z)=(z\wedge y)\vee(((z\wedge x)\vee y)\wedge x),$$
$$h'(x,y,z)=(z\vee x)\wedge(((z\vee y)\wedge x)\vee y),$$
$$g(x,y,z)=h\big(h(x,y,z),\ h'(x,y,z),\ z\big).$$

Let $\mathbf{A}$ be any UBP. Then h and h' are easily seen to induce majority functions on A and hence so does g.

Consider a pair $a, b\in A$ such that $a<b$. Then

$$a<b\leqq(c\wedge a)\vee b \quad \text{for all} \quad c\in A. \tag{5.3}$$

Since $c\wedge a$ is a lower bound for the elements a and $(c\wedge a)\vee b$, which by (5.3) are distinct, we have

$$((c\wedge a)\vee b)\wedge a=c\wedge a,$$

because $\mathbf{A}$ is a UBP. Therefore $h^{\mathbf{A}}(a,b,c)=(c\wedge b)\vee(c\wedge a)$ for $a<b$. Thus, since c is an upper bound for both $c\wedge b$ and $c\wedge a$,

$$h^{\mathbf{A}}(a,b,c)=c \quad \text{unless} \quad c\wedge b=c\wedge a<c.$$

In the latter case both $c\wedge a$ and a are lower bounds for the distinct elements a and b. Hence $c\wedge a=a$ (which means $a\leqq c$). Also $c\wedge b=c\wedge a$ implies $b\neq c$. Hence we have

$$\left.\begin{aligned} h^{\mathbf{A}}(a,b,c)&=a \quad \text{if} \quad a=b \quad \text{or} \quad b\neq c>a\\ &=c \quad \text{otherwise}\end{aligned}\right\} \quad \text{for} \quad a\leqq b \quad \text{and arbitrary } c. \tag{5.4}$$

Dually, from the definition of h', we obtain

$$\left.\begin{aligned} h'^{\mathbf{A}}(a,b,c)&=b \quad \text{if} \quad a=b \quad \text{or} \quad a\neq c<b\\ &=c \quad \text{otherwise}\end{aligned}\right\} \quad \text{for} \quad a\leqq b \quad \text{and arbitrary } c. \tag{5.4$'$}$$

Now consider $g^{\mathbf{A}}(a,b,c)$ for $a\leqq b$. By (5.4) and (5.4)$'$ and the fact that $h^{\mathbf{A}}$ is a majority function, we have four cases: $g^{\mathbf{A}}(a,b,c)$ equals one of $h^{\mathbf{A}}(a,b,c)$, $h^{\mathbf{A}}(a,c,c)=c$, $h^{\mathbf{A}}(c,b,c)=c$, or $h^{\mathbf{A}}(c,c,c)=c$, i.e.:

$$g^{\mathbf{A}}(a,b,c)=h^{\mathbf{A}}(a,b,c) \quad \text{or} \quad c.$$

The case $g^{\mathbf{A}}(a,b,c)=h^{\mathbf{A}}(a,b,c)$ occurs when $a=b$ (yielding $g^{\mathbf{A}}(a,b,c)=b$) or when $b\neq c>a$ and $a\neq c<b$. Since $\mathbf{A}$ is a UBP these two inequalities together with $a<b$ yield the contradiction $a=b$. Hence we have

$$g^{\mathbf{A}}(a,b,c)=b \quad \text{if} \quad a=b, \quad \text{and} \quad =c \quad \text{if} \quad a<b. \tag{5.5}$$

Now let $u(x, y, z)$ be any WAL ternary majority polynomial symbol. The simplest is apparently

$$u(x, y, z) = [(x\wedge z)\vee(y\wedge z)]\vee(x\wedge y).$$

Put

$$f(x, y, z, w) = u\big(g(x, y, z),\ g(y, w, z), z\big).$$

Since $u^{\mathbf{A}}$ is a majority function, if $a\leqq b\leqq d$ and c is arbitrary, from (5.5) we obtain:

$$\begin{aligned} f^{\mathbf{A}}(a, b, c, d) &= u^{\mathbf{A}}(b, b, c) = b \quad \text{if} \quad a = b = d,\\ &= u^{\mathbf{A}}(b, c, c) = c \quad \text{if} \quad a = b < d,\\ &= u^{\mathbf{A}}(c, b, c) = c \quad \text{if} \quad a < b = d,\\ &= u^{\mathbf{A}}(c, c, c) = c \quad \text{if} \quad a < b < d.\end{aligned}$$

Thus for $a\leqq b\leqq d$ and c arbitrary we have

(5.6) $\qquad f^{\mathbf{A}}(a, b, c, d) = b \quad \text{if} \quad a = b = d, \quad \text{and} \quad = c \quad \text{otherwise.}$

Finally, put

$$q_u(x, y, z) = f(x\wedge y, x, z, x\vee y).$$

By (5.6) we have

$$q_u^{\mathbf{A}}(a, b, c) = a \quad \text{if} \quad a\wedge b = a = a\vee b, \quad \text{and} \quad = c \quad \text{otherwise.}$$

Since $a\wedge b=a=a\vee b$ is equivalent to $a=b$, $A_u^{\mathbf{A}}$ is the dual discriminator of A.

Next we shall prove that the explicit polynomial symbol given in Section 2 is the dual discriminator for the triangle algebra $\mathbf{W}_3$. In fact we prove more.

5.7 Theorem. *The polynomial symbol*

$$q_t(x, y, z) = \big[(z\wedge(x\wedge y))\vee(x\vee y)\big]\wedge[z\vee(x\wedge y)]$$

induces the dual discriminator on the triangle algebra $\mathbf{W}_3$ *and on no other* WAL *of more than two elements.*

Proof. It is routine to check that $q_t^{\mathbf{A}}$ is a majority polynomial on any WAL $\mathbf{A}$. Hence on $\mathbf{W}_3$ $q_t^{\mathbf{W}_3}$ agrees with the dual discriminator if any two of its arguments are equal. Otherwise, using symmetry, we may suppose $z=a_1$ and either $x=0$, $y=1$, or $x=1$, $y=0$. In both cases $x\wedge y=0$ and $x\vee y=1$. Thus

$$q_t^{\mathbf{W}_3}(x, y, z) = [(a_1\wedge 0)\vee 1]\wedge[a_1\vee 0] = a_1 = z$$

so that q_t induces the dual discriminator on $\mathbf{W}_3$.

To complete the proof observe that if $\mathbf{A}$ is a WAL which is not $\mathbf{W}_3$ and has more than two elements, then it must contain an incomparable pair b and c. Put $a=b\wedge c$. Then $a\wedge b=a$ and $a\vee b=b$, so

$$q_t^{\mathbf{A}}(a, b, c) = [(c\wedge a)\vee b]\wedge[c\vee a] = b\wedge c = a \neq c.$$

But $a\neq b$ since b and c are incomparable. Hence $q_t^{\mathbf{A}}$ fails to be the dual discriminator on A.

In contrast to the unique property of the polynomial symbol q_t expressed by Theorem 5.7, any polynomial symbol which induces the dual discriminator on $\mathbf{W}_5$ also induces the dual discriminator on every $\mathbf{W}_n$, $n>5$, This is so since each 3-generated subalgebra of any such $\mathbf{W}_n$ is evidently isomorphic to a subalgebra of $\mathbf{W}_5$. (See Problem 2, Section 6.)

Theorems 5.6 and 5.7 together with Theorem 3.2 yield the following result immediately. (Since it is routine to check that q_u and q_t induce majority polynomials in any WAL, equations a) of Theorem 3.2 are omitted from ii) below. Also we may use d′) of Theorem 3.2 since WALs are idempotent.)

5.8. Theorem. *Let B denote the set of identities whose members are the following:*

i) *the identities* (5.1) *defining WALs,*

ii) *the identities (from Theorem* 3.2*):*

$$q(x, y, q(x, y, z)) = q(x, y, z), \quad q(z, q(x, y, z), q(x, y, w) = q(x, y, z)),$$

$$q(x, y, z\vee w) = q(x, y, z)\vee q(x, y, w), \quad q(x, y, z\wedge w) = q(x, y, z)\wedge q(x, y, w).$$

Then the set B_u, obtained from B by taking for q the polynomial symbol q_u defined in Theorem 5.3, *is an equational base for the variety U generated by all* UBPs. *The set B_t, obtained from B by taking for q the polynomial symbol q_t of Theorem* 5.7, *is an equational base for the variety T generated by the triangle algebra* $\mathbf{W}_3$.

Since the identities B of Theorem 5.8 contain only four variable symbols, we have the following corollary.

5.9 Corollary. *Let* $\mathbf{A}$ *be a WAL. If each subalgebra of* $\mathbf{A}$ *which is generated by four or fewer elements is contained in the variety U (respectively T), then* $\mathbf{A}$ *is contained in U (respectively T).*

In [7] a weaker version of Corollary 5.9 was established, namely for the variety T only and with "five" instead of "four". Hence Corollary 5.9 solves the problem raised in [7] (following Corollary 1 of Theorem 2).

6. Problems

1. Find a simple property P of varieties such that: A variety V is a dual discriminator variety if and only if V has a) a majority polynomial, b) the congruence extension property, and c) property P.

2. Does Theorem 5.7 have an analog for $\mathbf{W}_4$, i.e.: is there a WAL polynomial symbol $q(x, y, z)$ which induces the dual discriminator on the UBP $\mathbf{W}_4$ and on no other WAL of more than three elements?

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