

Kernel systems of directed graphs

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0. In graph theory there is a number of min—max theorems of quite similar type such that one is not a direct consequence of the other. For instance, a theorem of J. Edmonds states that in a directed graph there exist k edge disjoint spanning arborescences rooted at a fixed vertex r (see the exact definitions and formulation below) if and only if the indegree of every subset of vertices, not containing r , is at least k . A version of Menger's theorem resembles Edmonds' one: in a directed graph there exist k edge disjoint paths from r to another fixed vertex s if and only if the indegree of every subset of vertices, containing s but not r , is at least k .

It is a natural question whether there exists a common generalization of these theorems of similar type. The purpose of this paper is to present a tool, by means of which such a unification can be obtained on the one hand, and new min—max theorems can be concluded on the other hand. This tool is the notion of a kernel system, which is, roughly, a family of subsets of vertices of a directed graph which is closed under intersection.

Perhaps the most interesting consequences of min—max theorems concerning kernel systems are the following:

a) A conjecture of J. Edmonds and R. Giles concerning directed cuts is solved for graphs possessing an arborescence.

b) A min—max formula is given for the maximum number of edges which can be covered by K spanning arborescences rooted at a fixed vertex.

Some further corollaries of our results will be published in another paper [7] where, among others, a min—max formula is given for the maximum number of edges of a digraph which can be covered by k branchings.

At this point we refer to a recent, fundamental article of EDMONDS and GILES [2] concerning min—max relations for submodular functions.

Some of our notions are similar to those of Edmonds and Giles and in the proof of Theorem 3 a relevant idea of their work will be used. However our results

seem to be independent of the main theorem of [2]. The exact relation will be explained in the last section.

Let $G=(V, E)$ be a finite directed graph with vertex set V and edge set E . Multiple edges are allowed, loops are excluded. Let r be a distinguished vertex, called the *root* of G . An arborescence rooted at r (or briefly r -arborescence) is a directed spanning tree such that every vertex can be reached by a directed path from r (see [1]). An r - s -path is a directed path from r to the vertex s .

We say that a directed edge e enters a subset X of vertices if the head of e is in X but the tail is not. We say that a subset E' of edges enters a subset X of V if at least one element of E' enters X . The *indegree* $\varrho(X)$ and the *outdegree* $\delta(X)$ of a subset X of V is the number of edges entering X or $V \setminus X$, respectively. It is well known that the function $\varrho(X)$ is submodular, i.e. $\varrho(X) + \varrho(Y) \geq \varrho(X \cup Y) + \varrho(X \cap Y)$ for every pair X, Y of subsets of vertices.

For an arbitrary set X , $X' \subseteq X$ means that X' is a family of not necessarily distinct elements of X . $|X|$ denotes the cardinality of X . We shall use the notation $V \setminus r$ instead of $V \setminus \{r\}$. Two subsets X and Y of $V \setminus r$ are called *crossing* if $X \cap Y \neq \emptyset$, $X \setminus Y \neq \emptyset$, $Y \setminus X \neq \emptyset$. Otherwise X and Y are *non-crossing*. A family of subsets of $V \setminus r$ is called *laminar* if its members are pairwise non-crossing. (These notions occur slightly more generally in previous papers [2, 9].) A *directed cut* of G is a nonempty set of edges entering a vertex set X provided $\delta(V \setminus X) = 0$.

1. Definition. A family \mathcal{M} of distinct subsets of vertices of $V \setminus r$ is called a *kernel system* with respect to G if

- 1) $\varrho(M) > 0$ for every $M \in \mathcal{M}$;
- 2) if $M, N \in \mathcal{M}$ and $M \cap N \neq \emptyset$ then $M \cap N, M \cup N \in \mathcal{M}$. The members of \mathcal{M} are called *kernels*.

Examples. 1. $\mathcal{M}_1 = \{M : M \subseteq V \setminus r\}$. The second axiom is trivially satisfied, the first one holds if G has an r -arborescence.

2. Let s be another fixed vertex of G and $\mathcal{M}_2 = \{M : M \subseteq V \setminus r, s \in M\}$. The first axiom holds if there exists an r - s -path.

3. $\mathcal{M}_3 = \{M : M \subseteq V \setminus r, \delta(M) = 0\}$. If G is connected (in the undirected sense) then the first axiom is fulfilled. The proof of the second one, as an easy exercise, is left to the reader.

4. If \mathcal{M} is an arbitrary kernel system with respect to G then the kernels of minimum indegree form another kernel system

$$\mathcal{M}' = \{M : M \in \mathcal{M}, \varrho(M) = \min_{X \in \mathcal{M}} \varrho(X)\}.$$

The proof of the second axiom is as follows: Let $k = \min_{X \in \mathcal{M}} \varrho(X)$ and $M, N \in \mathcal{M}'$.

Then

$$k+k = \varrho(M)+\varrho(N) \cong \varrho(M \cup N)+\varrho(M \cap N) \cong k+k$$

whence $\varrho(M \cup N)=\varrho(M \cap N)=k$, therefore $M \cup N, M \cap N \in \mathcal{M}'$.

5. Let \mathcal{M} be a kernel system and F be a subset of edges, then

$$\mathcal{M}_F = \{M: M \in \mathcal{M}, F \text{ does not enter } M\}$$

is again a kernel system. The axioms trivially hold.

2. Let k be a positive integer.

Definition. A subset E' of edges is called k -entering with respect to the kernel system \mathcal{M} , if in the subgraph formed by E' , the indegree of every kernel is at least k .

Theorem 1. *A subset E' of edges is k -entering if and only if E' can be partitioned into k 1-entering subsets.*

Proof. The necessity is trivial. For the sufficiency it can be assumed that $E'=E$. We are going to prove that E can be partitioned into a 1-entering subset E_1 and a $(k-1)$ -entering subset E_2 . This assertion proves our theorem.

The subset E_1 will be constructed sequentially and once an edge has been inserted into E_1 it is never changed. In an intermediate stage of the algorithm a kernel M is called *dangerous* with respect to the current E_1 if

$$\varrho_{G-E_1}(M) = k-1.$$

Starting from the empty set E_1 , in every step we consider a maximal kernel M such that E_1 does not enter M . Insert an edge e into E_1 which enters M but does not enter any dangerous kernel, and then we say that e was inserted into E_1 because of M . The process stops when E_1 is 1-entering.

To verify this algorithm we have to justify that the required edge e always exists.

Claim 1. *If $f \in E_1$ then the head of f is not in M .*

Proof. Suppose the contrary then the tail of f is also in M , by the algorithm. Let E_f denote the set of edges which were inserted into E_1 before f , and suppose that f was inserted into E_1 because of M_f . Now $M_f \cap M \neq \emptyset$ therefore $M_f \cap M$ is a kernel. E_f does not enter $M_f \cap M$ and $M_f \cup M \neq M_f$ which contradict the maximality of M_f . \square

Claim 2. *If M_D is dangerous with respect to E_1 then $M_D \not\subseteq M$.*

Proof. Since M_D is dangerous, there exists an edge $e_1 \in E_1$ entering M_D . The head of this edge is in M_D but not in M by Claim 1. \square

Claim 3. *If M and N are dangerous kernels and $M \cap N$ is nonempty, then $M \cap N$ is dangerous as well.*

Proof. $k-1+k-1 = \varrho_{G-E_1}(M) + \varrho_{G-E_1}(N) \cong \varrho_{G-E_1}(M \cup N) + \varrho_{G-E_1}(M \cap N) \cong \cong k-1+k-1$ whence $\varrho_{G-E_1}(M \cap N) = k-1$. \square

If every dangerous kernel is disjoint from M then an arbitrary edge entering M can be inserted into E_1 and we are done since the new set $E \setminus E_1$ remains $(k-1)$ -entering. Otherwise let M_D be a dangerous kernel such that $M_D \cap M \neq \emptyset$ and $M_D \setminus M$ is as small as possible.

By Claim 2, $M_D \setminus M \neq \emptyset$. There exists an edge e with tail in $M_D \setminus M$ and head in $M_D \cap M$ since otherwise

$$k-1 = \varrho_{G-E_1}(M_D) \cong \varrho_{G-E_1}(M_D \cap M) \cong k-1$$

whence $M_D \cap M$ is a dangerous kernel, contradicting Claim 2.

We assert that the edge e enters no dangerous set. If e entered a dangerous set M_e then $M' = M_e \cap M_D$ would also be dangerous by Claim 3. The existence of such an M' is in contradiction with the minimum property of M_D . \square

Corollary 1. (J. EDMONDS [4]) *A digraph G has k edge-disjoint r -arborescences if and only if the indegree of every subset of $V \setminus r$ is at least k .*

Proof. Apply Theorem 1 to the first example. The corollary follows from the simple fact that a 1-entering edge set surely contains an r -arborescence. \square

Corollary 2. (Directed edge version of Menger's theorem [1]) *In a digraph there exist k edge disjoint r - s -paths if and only if the indegree of every subset of $V \setminus r$ containing s is at least k .*

Proof. Apply Theorem 1 for the second example. The corollary follows from the simple fact that a 1-entering edge set surely contains an r - s -path. \square

The next consequence settles in the affirmative a conjecture of J. EDMONDS and R. GILES [2] in a special case.

Conjecture. An edge set E' is a k -covering of directed cuts of a directed graph if and only if E' can be partitioned into k 1-coverings of directed cuts. (An edge set E' is called a k -covering of directed cuts if every directed cut contains at least k edges of E').

Corollary 3. *The conjecture of Edmonds—Giles is true for graphs possessing an arborescence.*

Proof. Applying Theorem 1 to the third example we obtain that a k -covering (that is a k -entering edge set) of those directed cuts which are directed away from

r can be partitioned into k 1-coverings. However when the graph has an r -arborescence then all of the directed cuts are of this type.

Remark. The proof of Theorem 1 can be considered as a generalization of Lovász' proof in [8] of the afore mentioned theorem of Edmonds. It is, in fact, a polynomial bounded algorithm provided that some simple operations can be carried out in polynomial time on the kernels. These operations are as follows:

a) Find a maximal kernel M such that E' does not enter M for an arbitrary edge set E' .

b) Decide whether E'' is k -entering for arbitrary edge set E'' .

The above three corollaries are of this type. In Corollary 1 we obtain Lovász' algorithm. In Corollary 2 our proof does not mean a new algorithm for Menger's theorem since the only way at hand to check b) is to use the classical augmenting path method.

In Corollary 3 operation a) is simple because the required maximal kernel M consists of those vertices which cannot be reached by a directed path from r in the graph arising from G after contracting the edges of E' . Operation b) can be carried out as follows: Let G^+ denote the graph which arises from G after inserting $k-1$ reversed copies of all the edges of E'' . It can easily be checked that E'' is k -entering if and only if there exist k edge disjoint $r-s$ -paths in G^+ for every vertex $s \in V \setminus r$. This latter problem is polynomially solvable.

3. Let c be a nonnegative integer function defined on the edge set E of G . $c(e)$ is called the *weight* of e .

Definition. A family \mathcal{M}' of not necessarily distinct kernels of \mathcal{M} (i.e. $\mathcal{M}' \subset \mathcal{M}$) is called *c-edge-independent* if each edge e enters at most $c(e)$ members of \mathcal{M}' .

Theorem 2.

$$(1) \quad \max |\mathcal{M}'| = \min \sum_{e \in E'} c(e)$$

where the maximum is taken over all the *c-edge-independent* subfamilies \mathcal{M}' of \mathcal{M} while the minimum is taken over all the 1-entering edge sets E' .

(2) The maximum can be realized by a laminar \mathcal{M}' too.

Proof. $\max \leq \min$. A simple enumeration shows that $|\mathcal{M}'| \leq \sum_{e \in E'} c(e)$ for any *c-edge-independent* \mathcal{M}' and for any 1-entering E' .

$\max = \min$. We are going to construct a *c-edge-independent* family \mathcal{M}' and a 1-entering edge set E' such that $|\mathcal{M}'| = \sum_{e \in E'} c(e)$.

The algorithm consists of two parts constructing \mathcal{M}' and E' , respectively. It has the interesting feature that both of its parts are of the greedy type, i.e. both

\mathcal{M}' and E' will be produced sequentially and once a kernel or edge has been inserted into \mathcal{M}' or E' , respectively, it is never changed.

First part: Construction of \mathcal{M}' .

First let \mathcal{M}' be empty. In the general step we decide whether there exists a kernel M which can be inserted into the current \mathcal{M}' without destroying its c -edge-independence. If the answer is "no" then the construction of \mathcal{M}' terminates.

Otherwise let M be a minimal kernel which can be inserted into \mathcal{M}' and let us insert into \mathcal{M}' as many copies of M as possible without destroying the c -edge-independence.

The family \mathcal{M}' produced by the first part is obviously c -edge-independent.

In order to describe the second part we need some notations. Let the different kernels of \mathcal{M}' be M_1, M_2, \dots, M_k (i.e. the first part terminated at the $(k+1)$ -th step), and suppose that these kernels have been inserted into \mathcal{M}' in this order. We call an edge e *saturated* with respect to \mathcal{M}' (or briefly saturated) if it enters exactly $c(e)$ members of \mathcal{M}' . Let E_i ($i=1, 2, \dots, k$) denote the set of those saturated edges which have been saturated in the i^{th} step of the first part. It is easy to see that

(3a) $E_i \neq \emptyset$ for $i=1, 2, \dots, k$;

(3b) $E_i \cap E_j = \emptyset$ for $1 \leq i < j \leq k$;

(3c) If $e \in E_i$ then e enters M_i ;

(3d) If $e \in E_i$, $i < j$ then e does not enter M_j .

Taking into consideration the construction of \mathcal{M}' , the following claim can be checked easily.

Claim 1. If $M_i \in \mathcal{M}'$, $M \subset M_i$, and $M \in \mathcal{M}$ then there exists a saturated edge e which enters M but not M_i , and then e is in E_h where $h < i$. \square

In order to verify (2) we show that \mathcal{M}' is laminar. For, otherwise, let M_i and M_j be two crossing members of \mathcal{M}' ($i < j$). Applying Claim 1 with the choice M^i and $M = M_i \cap M_j$ we obtain that there exists an edge e in E_h (for some $h < i$) which enters M but not M_i . Then e enters M_j , a contradiction to (3d).

Second part: Construction of E' .

First let E' be empty. In the general step we decide whether E' is 1-entering. If the answer is "yes" then the second part terminates.

Otherwise, let M be a maximal kernel such that the current E' does not enter M . Let i be the minimum index for which E_i enters M . Let us insert an edge e of E_i which enters M into E' . (We say that e has been inserted because of M .)

The set E' produced by the second part is obviously 1-entering.

To verify (1) and the algorithm we have to show that there exists a unique edge of E' entering M_i for each member M_i of \mathcal{M}' . This implies $|\mathcal{M}'| = \sum_{e \in E'} c(e)$, taking into consideration the fact that the edges of E' are saturated.

Claim 2. *If an edge e has been inserted into E' because of N , and e enters a member M_i of \mathcal{M}' , then $N \supseteq M_i$.*

Proof. Since e enters M_i , using (3d) we obtain that e is in E_j for some $j \geq i$. If $N \not\supseteq M_i$, then with the choice M_i and $M = N \cap M_i$, Claim 1 implies that there exists an edge e' in E_h (for some $h < i$) which enters $M_i \cap N$ but not M_i . Then e' enters N which is in contradiction with the minimality of j , since $h < j$. \square

Now suppose, indirectly, that two edges e_1, e_2 of E' enter a kernel M_i of \mathcal{M}' . Suppose that e_1 and e_2 have been inserted into E' because of N_1 and N_2 , respectively, and e_2 was inserted later than e_1 . By Claim 2, $N_1, N_2 \supseteq M_i$ and e_1 does not enter N_2 . Hence $N_1 \cup N_2 \neq N_1$ which contradicts the maximality of N_1 . \square

Remark. The proof of Theorem 2 can be considered as a generalization of that of FULKERSON [5] given for maximum packing of rooted r -cuts. Our algorithm is polynomial bounded provided that the following simple operations can be carried out in polynomial time.

a) Find a minimal kernel M such that E' does not enter M for an arbitrarily given edge set E' .

b) Decide whether E'' is 1-entering for a given edge set E'' , and if it does then find a maximal kernel M such that E'' does not enter M .

All the following corollaries and problems are of such type.

Apply Theorem 2 to the first example:

Corollary 4. (EDMONDS [3], FULKERSON [5]) *In an edge-weighted digraph the minimum weight of an r -arborescence is equal to the maximum number of c -edge-independent vertex sets of $V \setminus r$.* \square

(A family of c -edge-independent vertex sets corresponds to a packing of r -directed cuts in [5]).

Apply Theorem 2 for the second example:

Corollary 5. (FORD—FULKERSON [6]) *In an edge-weighted digraph the minimum weight of an r - s -path is equal to the maximum number of c -edge-independent vertex sets containing s but not r .* \square

The following corollaries seem to be new.

Problem 1. Suppose that the maximum number of edge disjoint r -arborescences of a (weakly) connected digraph $G=(V, E)$ is k ($k \geq 0$). We want to increase this maximum by using new edges. Let the set E_1 of possible new edges be such that $G^+=(V, E \cup E_1)$ has $k+1$ arborescences. Assign to each edge e of E_1 a nonnegative integer weight $c(e)$. What is the minimum sum of weights of the required new edges?

Solution. Let us define a kernel system \mathcal{M} with respect to $G_1=(V, E_1)$ as follows:

$$\mathcal{M} = \{M: \varrho_G(M) = k, M \subseteq V \setminus r\}.$$

(Observe that the kernel system \mathcal{M} with respect to G_1 is defined by means of G .) Due to the above theorem of Edmonds (Corollary 1) we have to assure that the indegree of all the subsets of $V \setminus r$ is at least $k+1$, that is, we have to find a minimum weight 1-entering subset of kernel system \mathcal{M} . Applying Theorem 2 for this \mathcal{M} we get:

Corollary 6. *The minimum value of the weight sum of those edges of E_1 whose insertion into G increases the maximum number of edge disjoint r -arborescences by one, is equal to the maximum number of not necessarily distinct subsets of $V \setminus r$ such that (i) the indegree of the set in G is minimum ($=k$) and (ii) an arbitrary edge e of E_1 enters at most $c(e)$ subsets of them. \square*

Remark. A possible generalization arises naturally. Let $G=(V, E)$ be strongly k -edge-connected and E_1 be a set of new edges. Find a minimum subset E_2 of E_1 such that $G^+=(V, E \cup E_2)$ is strongly $(k+1)$ -edge-connected. However it is easy to check that the Hamilton circuit problem is contained in this one in the case $k=0$. Therefore this problem is NP-hard and this direction is hopeless.

Now a simple application of Corollary 6 will be presented.

Problem 2. Let us suppose that $G=(V, E)$ has an r -arborescence. Let $F=(E, \mathcal{A})$ be the hypergraph of all r -arborescence of G . Here the vertex set E of F is the edge set of G and the edge set of F is the family of r -arborescences of G . Determine the rank-function r of F . We recall the definition of the rank-function r of an arbitrary hypergraph:

$$(4) \quad r(E') = \max_{a \in \mathcal{A}} |a \cap E'| \quad (E' \subseteq E)$$

(i.e. $r(E')$ shows at most how many edges of E' can occur in an r -arborescence). Since every arborescence consists of $|V|-1$ edges, our problem is equivalent to the following:

Let us complete E' by a minimum number edges of $E \setminus E'$ so that the completed E' contains an r -arborescence. Applying Corollary 6 for the case when the original graph is $G'=(V, E')$, $E_1=E \setminus E'$, $c \equiv 1$ and $k=0$, we obtain

Corollary 7. $r(E') = \min_{V_1, V_2, \dots, V_t} (|V|-1-t)$ where the minimum is taken over all those laminar families of subsets V_1, V_2, \dots, V_t of $V \setminus r$ for which E' does not enter any V_i and an arbitrary edge of $E \setminus E'$ enters at most one V_i .

Hence one can easily obtain

Corollary 8. *A subset E' of edges of G is a subset of an r -arborescence if and only if $|V|-1 \cong |E'|+t$ for an arbitrary 1-edge-independent laminar family of subsets V_1, V_2, \dots, V_t of $V \setminus r$ such that E' enters no V_i .*

Remarks 1. One can immediately prove a slightly sharper version of this corollary when in the necessary and sufficient condition the cardinalities of all but one V_i are one.

2. Some further special cases of the above corollaries are interesting for their own sake. Let us apply Corollary 6 in the case if $k=0$ and E_1 consists of the reversed copies of all edges of E . We obtain a theorem of Lucchesi—Younger type (but not the Lucchesi—Younger theorem itself), which simply follows from the theorem of EDMONDS—GILES [2], too (although our proof provides a polynomial algorithm as well). The reader may find it interesting to specialize for the case $k \cong 1, E_1=E$ and $c \cong 1$. In this way a min—max theorem can be obtained for the minimum number of edges of G whose duplication increases the maximum number of edge disjoint r -arborescences.

4. In this section a generalization of Theorem 2 will be given. Unlike the proof of Theorem 2, this does not provide a polynomial algorithm. This is the reason why Theorem 2 was discussed in the previous paragraph.

Let \mathcal{M} be a kernel system with respect to $G=(V, E)$ and let f be a nonnegative integer function defined on the kernels.

Definition. The function f is called *weakly supermodular* on \mathcal{M} if $M, N \in \mathcal{M}, f(M) > 0, f(N) > 0, M \cap N \neq \emptyset$ imply that

$$(5) \quad f(M) + f(N) \cong f(M \cup N) + f(M \cap N).$$

If already $M, N \in \mathcal{M}$ and $M \cap N \neq \emptyset$ imply this inequality then f is called *supermodular*.

Definition. A family E' of not-necessarily distinct edges of E (i.e. $E' \subseteq E$) is called *f -entering*, if in the subgraph $G'=(V, E')$ the indegree of every kernel M is at least $f(M)$.

Let c be a nonnegative integer function defined on the edges of G .

Theorem 3. *Let f be a weakly supermodular function on \mathcal{M} . Then*

$$(6) \quad \max_{\substack{\mathcal{M}' \subseteq \mathcal{M} \\ \mathcal{M}' \text{ c-edge-independent}}} \sum_{M \in \mathcal{M}'} f(M) = \min_{\substack{E' \subseteq E \\ E' \text{ f-entering}}} \sum_{e \in E'} c(e)$$

where \mathcal{M}' is c -edge-independent, E' is f -entering.

(7) *The maximum can be realized by a laminar \mathcal{M}' .*

Proof. First we will prove (7) which will be used in the proof of (6), too. We note that this technique is due to N. Robertson for $\mathbf{f} \equiv 1$ and to Edmonds and Giles for an arbitrary supermodular function \mathbf{f} . It can be assumed that the optimum \mathcal{M}' consists of kernels with positive weights only. If M, N are crossing members of \mathcal{M}' then replace them by $M \cup N$ and $M \cap N$ i.e. $\mathcal{M}'' = \mathcal{M}' \setminus \{M, N\} \cup \{M \cup N, M \cap N\}$. It is easy to check that \mathcal{M}'' is \mathbf{c} -edge-independent again and, since \mathbf{f} is weakly supermodular,

$$\sum_{M \in \mathcal{M}''} \mathbf{f}(M) \cong \sum_{M \in \mathcal{M}'} \mathbf{f}(M).$$

Hence \mathcal{M}'' is another optimum \mathbf{c} -edge-independent family. Apply this method as long as there exist crossing members in the optimum family. The process terminates since $\sum_{M \in \mathcal{M}'} |M|^2$ increases at each step.

We need two simple claims.

Claim 1. Let e be an edge of G and let \mathbf{f} be a weakly supermodular function on \mathcal{M} . Let

$$\mathbf{f}_e(M) = \begin{cases} \mathbf{f}(M), & \text{if } e \text{ does not enter } M \\ 0, & \text{if } \mathbf{f}(M) = 0 \\ \mathbf{f}(M) - 1, & \text{otherwise,} \end{cases}$$

then \mathbf{f}_e is weakly supermodular.

The proof of the claim is trivial.

We note that the analogous property for supermodular functions is not necessarily true.

Claim 2. Let $\mathbf{c}_1(e) = k \cdot \mathbf{c}(e)$ for a natural number k . If $\mathcal{M}'' \subset \mathcal{M}$ is a laminar \mathbf{c}_1 -edge-independent family, then it can be partitioned into k \mathbf{c} -edge-independent families.

Proof. The members of \mathcal{M}'' will be colored one by one with colors $0, 1, \dots, k-1$. In the general step let M be a maximal non-colored member of \mathcal{M}'' . If there exist no previously colored member M' of \mathcal{M}'' containing M then let M be colored by 0 . Otherwise let M' be a previously colored kernel with $M' \supseteq M$, which received its color last. If the color of M' is i then we color M by $i+1 \pmod k$.

It is an easy exercise to verify that each subfamily of kernels with the same color is \mathbf{c} -edge-independent. \square

For the proof of (6) a simple enumeration shows that $\max \leq \min$. Let v_i denote the left-hand side in (6). We use induction on v_i . If $v_i = 0$ then the statement is trivial.

Let M be an arbitrary kernel such that $\mathbf{f}(M) > 0$ and not all the edges entering M are of zero weight. There are two cases.

(a) There is an edge e with positive weight, entering M such that all the optimum (of weight v_f) c -edge-independent families saturate e (i.e. e enters just $c(e)$ kernels of the family with positive weight).

In this case $v_{f_e} = v_f - c(e)$. By the induction hypothesis there exists an $E'_e \subset E$ for which $v_{f_e} = \sum_{e' \in E'_e} c(e')$ and E'_e is f_e -entering. Let $E' = E'_e \cup \{e\}$. Since $v_f = \sum_{e' \in E'} c(e')$ and E' is f -entering we are finished with the proof.

(b) For each edge e_i with positive weight and entering M there exists an optimum c -edge-independent family \mathcal{M}_i which does not saturate e_i . Let $\mathcal{M}'' = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \dots \cup \mathcal{M}_k \cup \{M\}$. Then \mathcal{M}'' is c_1 -edge-independent where $c_1 = k \cdot c$ and

$$\sum_{N \in \mathcal{M}''} f(N) = k \cdot v_f + f(M).$$

By the proof of (7) there exists a laminar family \mathcal{M}''' such that

$$\sum_{N \in \mathcal{M}'''} f(N) \cong \sum_{N \in \mathcal{M}''} f(N).$$

Now by Claim 2, \mathcal{M}''' can be partitioned into k c -edge-independent subfamilies. However, the weight of one of these subfamilies is greater than v_f which is impossible. Hence case (b) cannot occur. \square

Theorem 3 reduces to Theorem 2 in the case $f \equiv 1$, therefore the corollaries of Theorem 2 can be generalized. However, we emphasize only one consequence of Theorem 3.

Problem 3. Let $G=(V, E)$ be a digraph in which the maximum number of edge-disjoint r -arborescences is k ($k > 0$). We want to increase this maximum to K ($K > k$) by multiplying edges. What is the minimum number of the required new edges?

Solution. Due to the theorem of Edmonds (Corollary 1) we have to assure just that in the extended graph the indegree of every subset of $V \setminus r$ is at least K .

Let \mathcal{M} be the kernel system defined in the first example. Let the function f be defined as follows:

$$(8) \quad f(M) = \max \{K - \varrho(M), 0\}$$

that shows the number of edges still required to reach K as the indegree of M . In this way our question is translated into the problem of a minimum f -entering edge set.

Claim. The above defined f is weakly supermodular.

Proof. Trivial. \square

We note that f is not supermodular in general.

Applying Theorem 3 for this f in the case $c \equiv 1$ we obtain a min—max formula for the minimum number of new edges. Instead of the exact formulation of this theorem we mention another problem which is equivalent to this one but is more illustrative.

Problem 4. What is the maximum number of edges which can be covered by K r -arborescences?

Solution. If there exist K edge disjoint r -arborescences then this number is obviously $K \cdot (|V| - 1)$. Otherwise let a_1, a_2, \dots, a_K be r -arborescences whose union is as large as possible. Suppose that this union consists of m edges. Let us multiply every edge of G by the number of r -arborescences from a_1, a_2, \dots, a_K containing it. Of course this graph has already K r -arborescences. This means that $s = K \cdot (|V| - 1) - m$ new copies of original edges assure the existence of K edge disjoint r -arborescences. Conversely, if the insertion of s new copies of edges yields the existence of K edge disjoint r -arborescences, then $m = K \cdot (|V| - 1) - s$ edges can be covered by K r -arborescences in G . In this way Problem 4 is equivalent to Problem 3. Hence, as a consequence of Theorem 3, we obtain

Corollary 9. *The maximum number of edges which can be covered by K r -arborescences is equal to the minimum value of*

$$K(|V| - 1) - \sum_{i=1}^t f(V_i)$$

where the minimum is taken over all the 1-edge-independent laminar families of subsets V_1, V_2, \dots, V_t of $V \setminus r$ where t is arbitrary and function f is defined in (8).

There is an interesting special case of this corollary.

Corollary 10. *The edges of G can be covered by K r -arborescences if and only if for an arbitrary laminar 1-edge-independent family of subsets V_1, V_2, \dots, V_t of $V \setminus r$, the number e_t of edges entering no V_i satisfies*

$$(9) \quad e_t \leq K(|V| - 1 - t).$$

Remark. K. VIDYASANKAR [11] has proved a similar but simpler necessary and sufficient condition for the problem in Corollary 10. He requires (9) only in the case if the cardinality of all but one of the V_i 's is one, with the two side-conditions that the indegree of each vertex is at most K and every edge is in an r -arborescence. The necessity of these two latter conditions is trivial (and obviously our conditions imply them).

Now we formulate Corollary 9 in another way. Suppose again that G has an r -arborescence. Let E' be a subset of edges of G and let $r(E')$ denote the maximum

number of edges E' can have in common with an r -arborescence, i.e. r is the rank-function of the hypergraph of r -arborescences. We recall that function r was determined by a min—max formula in Corollary 7.

Corollary 9a. *The maximum number of edges which can be covered by K r -arborescences is equal to the*

$$\min_{E' \subseteq E} (K \cdot r(E') + |E \setminus E'|).$$

Proof. $\max \subseteq \min$ is true for any hypergraph. For the equality we show that

$$(10) \quad \min_{E' \subseteq E} (K \cdot r(E') + |E \setminus E'|) \subseteq K(|V| - 1) - \sum_{i=1}^t f(V_i)$$

where V_1, V_2, \dots, V_t form a 1-edge-independent family. It can be assumed that $f(V_i) > 0$ whence $f(V_i) = K - \rho(V_i)$. Let E'' be the set of edges which do not enter any V_i . We have $\sum_{i=1}^t \rho(V_i) = |E \setminus E''|$. Obviously, an arbitrary r -arborescence contains at least t edges entering one of the V_i 's. Thus $r(E'') \subseteq |V| - 1 - t$. Hence (10) follows, as required. \square

A similar version of Corollary 10 easily follows.

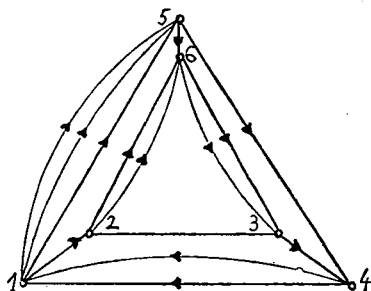
Corollary 10a. *The edges of G can be covered by K r -arborescences if and only if $K \cdot r(E') \geq |E'|$ for every $E' \subseteq E$.*

The reader can easily observe the similarity between Corollary 10a and a Theorem of C. ST. J. A. NASH-WILLIAMS [10] on the covering of a matroid by K bases.

5. In this last section we discuss the relationship between our results and those of J. Edmonds and R. Giles. Roughly speaking the main difference is that we consider entering edges only while they deal with entering and outgoing edges together.

Edmonds and Giles have defined the notion of crossing family. Our theorems concern a special type of crossing family (when the members of the family do not

contain a fixed vertex), but they cannot, however, be generalized for arbitrary crossing family. The remark after Corollary 6 justifies this statement for Theorem 2. The example in the Figure shows that Theorem 1 also fails for general crossing families.



Let $\mathcal{M} = \{M: \varrho(M) = 2\} = \{(1, 2, 3, 4, 6), (2, 3, 6), (2), (1, 2, 4, 5, 6), (4)\}$. The edges cannot be colored with two colors so that both of the color classes enter every kernel.

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