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Essential spectrum for a Banach space operator

RICHARD BOULDIN

§ 1. Introduction

Essential spectrum has been much studied with papers [4], [5], [6], [10], [14] taking the point of view of describing the Weyl spectrum or showing when different notions of "essential spectrum" coincide. A principal result of the significant paper [8] says that if the Weyl spectrum of T coincides with the Fredholm spectrum and T is essentially normal then T is the sum of a normal operator and a compact operator. The papers [1], [2], [3], [7] develop theories such as triangular representations for nonnormal operators by using the fine structure of index theory. The purpose of this note is to show that points identified by the fine structure of index theory are either very bad or very nice. Points in the semi-Fredholm domain which satisfy a "modest" hypothesis are very nice.

Let X be a fixed Banach space. Throughout this note "operator" will mean a linear map of X into X which is defined on a vector space dense in X and has closed graph. We adopt the notation of [15], which is our basic source for the theory of closed operators on a Banach space.

For the operator T let $\operatorname{nul}(T-\lambda)$ be the dimension of the kernel of $T-\lambda$, denoted $N(T-\lambda)$, and let $\operatorname{def}(T-\lambda)$ be the codimension of the range of $T-\lambda$, denoted $R(T-\lambda)$. The operator $T-\lambda$ is semi-Fredholm provided $R(T-\lambda)$ is closed and either $\operatorname{nul}(T-\lambda)$ or $\operatorname{def}(T-\lambda)$ is finite; for such λ the index of $T-\lambda$, denoted $\operatorname{ind}(T-\lambda)$, is $\operatorname{nul}(T-\lambda)-\operatorname{def}(T-\lambda)$. The operator $T-\lambda$ is Fredholm provided $R(T-\lambda)$ is closed and both $\operatorname{nul}(T-\lambda)$ and $\operatorname{def}(T-\lambda)$ are finite.

Lemma 1. (Index Theorem) *If the operator $T-\mu$ is semi-Fredholm then there is a neighborhood of μ, say G, such that the following are true:*

(i) $\lambda\in G$ *implies* $T-\lambda$ *is semi-Fredholm with* $\operatorname{nul}(T-\lambda)\leqq\operatorname{nul}(T-\mu)$, $\operatorname{def}(T-\lambda)\leqq\operatorname{def}(T-\mu)$ *and* $\operatorname{ind}(T-\lambda)=\operatorname{ind}(T-\mu)$;

(ii) $\operatorname{nul}(T-\lambda)$ *and* $\operatorname{def}(T-\lambda)$ *are constant on* $G\setminus\{\mu\}$ *(that is* $\{z: z\in G, z\neq\mu\}$*)*;

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(iii) *provided* nul $(T-\mu)<\infty$, nul $(T-\lambda)$ *is constant on G if and only if* $N(T-\mu)\subset\subset\bigcap\{R((T-\mu)^k)\colon k=1, 2, \ldots\}$;

(iv) *provided* $\operatorname{def}(T-\mu)<\infty$, $\operatorname{def}(T-\lambda)$ *is constant on G if and only if* $N(T'-\mu)\subset\bigcap\{R((T'-\mu)^k)\colon k=1, 2, \ldots\}$.

Proof. Parts (i) and (ii) are well known; part (iii) is Problem 5.32 of [12, p. 242] and (iv) results from applying (iii) to $(T'-\lambda)$.

The next lemma summarizes many useful facts. The spectrum of the operator T is denoted $\sigma(T)$. The dimension of the subspace X_0 in the lemma is called the algebraic multiplicity of λ.

Lemma 2. *Let T be an operator and let λ be an isolated point of $\sigma(T)$. Then there is a direct sum decomposition of X, say $X_0\oplus X_1$, such that X_0 and X_1 are invariant under $T-\lambda$. The restriction of $T-\lambda$ to X_0, denoted $(T-\lambda)|X_0$, is quasinilpotent and $(T-\lambda)|X_1$ is invertible. If $T-\lambda$ is semi-Fredholm then the dimension of X_0, denoted* $\dim X_0$, *is finite.*

Proof. There are many sources for the information about the decomposition corresponding to $\{\lambda\}$ and its complement (for example, see [12, pp. 178—181]). Since $T-\lambda$ is semi-Fredholm, it follows that $(T-\lambda)|X_0$ is semi-Fredholm. Since $R((T-\lambda)|X_0)$ is closed, $\mathrm{nul}'(T-\lambda)|X_0=\mathrm{nul}(T-\lambda)|X_0$ and $\mathrm{def}'(T-\lambda)|X_0==\mathrm{def}(T-\lambda)|X_0$ by [12, Theorem 5.10, p. 233]. By [12, Theorem 5.30, p. 240] we know that $\dim X_0=\infty$ implies $\mathrm{nul}'(T-\lambda)|X_0=\infty$. This proves that $\dim X_0<\infty$.

§ 2. Main result

The set of points μ such that $T-\mu$ is a Fredholm operator is denoted $\Phi(T)$ and the set of λ for which ind $T-\lambda$ is zero is denoted $\Phi_0(T)$. Provided there are nonnegative integers k such that $N(T^k)$ equals $N(T^{k+1})$, T is said to have finite ascent and the smallest such k is the ascent of T. Provided there are nonnegative integers m such that $R(T^m)$ equals $R(T^{m+1})$, T is said to have finite descent and the smallest such m is the descent of T.

To say that $N(T-\lambda)$ is not an asymptotic eigenspace for the operator T means that whenever there is a sequence of distinct eigenvalues say $\{\lambda_n\}$, converging to λ then $|\lambda_n-\lambda|=o(d(\lambda_n, \lambda))$ where

$$d(\lambda_n, \lambda) = \sup\{\operatorname{dist}(x, N(T-\lambda))\colon \|x\| = 1,\ x\in N(T-\lambda_n)\}.$$

This concept was introduced in [6].

Theorem 3. *If $T-\lambda$ is a semi-Fredholm operator with $\lambda\in\sigma(T)$ and one of the conditions* (1), (2), (3) *below holds then λ is an isolated eigenvalue with finite* *algebraic multiplicity. Furthermore, if μ is any isolated eigenvalue with finite algebraic multiplicity then μ belongs to $\Phi_0(T)$ and satisfies* (2) *and* (3).

(1) *λ is an isolated point of $\sigma(T)$.*

(2) *$N(T-\lambda)$ and $N(T'-\lambda')$ are not asymptotic eigenspaces for T and T', respectively.*

(3) *$T-\lambda$ has finite ascent and finite descent.*

Proof. First it is noted that (1) suffices for the conclusion about λ. Since $T-\lambda$ is semi-Fredholm, Lemma 2 implies that the spectral subspace X_0 corresponding to λ is finite dimensional. Consequently the quasinilpotent $(T-\lambda)|X_0$ is nilpotent, and $N((T-\lambda)|X_0)$ is non-trivial. Thus, λ is an isolated eigenvalue with finite algebraic multiplicity, and it suffices to show (1) is implied by each of the conditions (2) and (3).

Assume (2) holds and for the sake of a contradiction assume $\sigma(T)$ contains $\{\lambda_n\}$ which converges to λ with $\lambda_n \neq \lambda$. Lemma 1 shows that it may be assumed that each $T-\lambda_n$ is semi-Fredholm. Either nul $T-\lambda_n$ or def $T-\lambda_n$ is positive, and first we consider the case nul $T-\lambda_n > 0$. It will be shown that since $N(T-\lambda)$ is not an asymptotic eigenspace, $R(T-\lambda)$ is not closed, a contradiction. Since $\{|\lambda_n-\lambda|/d(\lambda_n, \lambda)\}$ converges to zero there is a sequence of unit vectors $\{x_n\}$ such that $x_n \in N(T-\lambda_n)$ and

$$\operatorname{dist}(x_n, N(T-\lambda)) > d(\lambda_n, \lambda) - |\lambda_n - \lambda|.$$

It follows that

$$\|(T-\lambda)x_n\|/\operatorname{dist}(x_n, N(T-\lambda)) = |\lambda_n-\lambda|/\operatorname{dist}(x_n, N(T-\lambda)) < |\lambda_n-\lambda|/(d(\lambda_n, \lambda)-|\lambda_n-\lambda|)$$

and clearly the last fraction converges to zero. Thus, $R(T-\lambda)$ is not closed (see Theorem 5.3, p. 72, [15]) and this contradiction proves that λ is an isolated point of $\sigma(T)$. If def $(T-\lambda_n)$ where positive then one would use that $N(T'-\lambda)$ is not an asymptotic subspace to show $R(T'-\lambda)$ to be not closed.

If (3) holds then (1) follows immediately from [13, Theorem 2.1, p. 200].

It only remains to establish the properties of the isolated eigenvalue μ. If Y_0 is the algebraic eigenspace associated with μ and Y_1 is the complementary subspace in X then $(T-\mu)|Y_1$ is one-to-one and onto. Since dim Y_0 is finite, it is straightforward to see that $(T-\mu)|Y_0$ is Fredholm with index zero, and conditions (2) and (3) must hold.

If λ belongs to $\sigma(T)\cap\Phi_0(T)$ then clearly λ is an eigenvalue for the operator T. Thus, the hypothesis of the next corollary would be stronger if one of the conditions (1), (2), (3), (4) was required for each eigenvalue λ. Hence, the hypothesis of the corollary is weaker than the hypotheses for similar results in [4], [5], [11].

Corollary 4. *Let T be an operator on X. If every λ in $\sigma(T)\cap\Phi_0(T)$ satisfies one of the conditions of Theorem* 3 *or* (4) *below then each λ is an isolated eigenvalue with finite algebraic multiplicity.*

(4) *$N(T-\lambda)$ and $N(T'-\lambda)$ are not subspaces of* $\bigcap_{k=1}^{\infty} \{R((T-\lambda)^k)\}$ *and* $\bigcap_{k=1}^{\infty} \{R((T'-\lambda)^k)\}$, *respectively.*

Proof. Let (4) hold and take $\varepsilon>0$ such that $0<|\lambda-\mu|<\varepsilon$ implies that all of the conclusions of Lemma 1 hold. Lemma 1 and condition (4) imply $\text{nul}\,(T-\mu)<\text{nul}\,(T-\lambda)$. If $\text{nul}\,(T-\mu)$ were positive then one of the conditions (1), (2), (3), (4) would apply and the resulting conclusion would contradict that $\text{nul}\,(T-\mu)$ is a positive constant for $0<|\lambda-\mu|<\varepsilon$; hence, $\text{nul}\,(T-\lambda)=0$ for such μ. Similarly, $\text{def}\,(T-\lambda)$ is zero for $0<|\lambda-\mu|<\varepsilon$ and so $T-\mu$ is invertible, which proves that λ is an isolated point of $\sigma(T)$.

The conditions (1), (2), (3) of Theorem 3 can be weakened provided the hypothesis for $T-\lambda$ is strengthened.

Corollary 5. *Let T be an operator on X. Every λ in $\sigma(T)\cap\Phi_0(T)$ which satisfies one of the conditions* (1′), (2′), (3′) *below is an isolated eigenvalue with finite algebraic multiplicity.*

(1′) *λ is an isolated point of $\sigma(T)$.*

(2′) *$N(T-\lambda)$ is not an asymptotic eigenspace for T.*

(3′) *$T-\lambda$ has finite ascent.*

Proof. If λ is an isolated point then Theorem 3 proves the desired conclusion.

The argument given in the second paragraph of the proof of Theorem 3 shows that (2′) above suffices.

That (3′) suffices follows from [14, Theorem 1.1].

In the final corollary the previous results are applied to get a simple alternative proof for a recent result on Riesz operators. An operator T is a *Riesz operator* provided the following hold for every nonzero λ:

(i) $T-\lambda$ has finite ascent and finite descent;

(ii) $N((T-\lambda)^k)$ is finite dimensional for $k=1, 2, \ldots$;

(iii) $R((T-\lambda)^k)$ is closed with finite codimension for $k=1, 2, \ldots$;

(iv) nonzero points of $\sigma(T)$ are eigenvalues and the only possible accumulation point of $\sigma(T)$ is zero.

Note that the sum of any quasinilpotent operator and a compact operator is a Riesz operator. For bounded T the next result was proved by CARADUS [9, p. 42].

Corollary 6. *Let T be an operator with nonempty resolvent set. If $\Phi(T)$ contains $\{z\colon z\neq 0\}$ then T is a Riesz operator.*

Proof. The index, being locally constant, is continuous and integer valued; thus, it is constant on connected components, and $\Phi_0(T)$ contains $\{z\colon z\neq 0\}$. If $\sigma(T)\cap\{z\colon z\neq 0\}$ contains accumulation points of $\sigma(T)$ then the intersection of

$\{z: z\neq 0\}$ with the boundary of $\sigma(T)$ contains λ, an accumulation point of $\sigma(T)$. Since $\operatorname{nul}(T-z)$ is constant on $N=\{z: 0<|\lambda-z|<\varepsilon\}$ for some $\varepsilon>0$ and N intersects the resolvent set of T, it must be that $\operatorname{nul}(T-z)=0$ for $z\in N$. Since $\operatorname{ind}(T-z)=0$ for $z\in N$, λ is an isolated point and the only possible accumulation point of $\sigma(T)$ is zero. Now Corollary 5 and Theorem 3 complete the proof.

Because of the astonishing lack of examples of (unbounded) operators in the literature, we mention the following. If C is the complex plane endowed with Lebesgue measure and M_z is multiplication by the independent variable defined on $\{f(z)\in L^2(C): zf(z)\in L^2(C)\}$ then M_z is an operator with no λ such that $M_z-\lambda$ is semi-Fredholm. So the resolvent set of an operator might be empty.

Bibliography

[1] C. Apostol, The correction by compact perturbation of the singular behavior of operators, *preprint.*

[2] C. Apostol, Matrix models for operators, *Duke Math. J.*, **42** (1975), 779—785.

[3] C. Apostol and K. Clancey, Generalized inverses and spectral theory, *Trans. Amer. Math. Soc.*, **215** (1976), 293—300.

[4] S. K. Berberian, An extension of Weyl's theorem to a class of not necessarily normal operators, *Michigan Math. J.*, **16** (1969), 273—279.

[5] R. Bouldin, Essential spectrum for a Hilbert space operator, *Trans. Amer. Math. Soc.*, **163** (1972), 437—445.

[6] R. Bouldin, The Weyl essential spectrum, *Proc. Amer. Math. Soc.*, **28** (1971), 531—536.

[7] R. Bouldin, The triangular representation of C. Apostol, *Proc. Amer. Math. Soc.*, **57** (1976), 256—260.

[8] L. Brown, R. G. Douglas and P. A. Fillmore, Unitary equivalence modulo the compact operators and extensions of C* algebras, *Proceedings of a Conference in Operator Theory*, Springer-Verlag, (New York, 1973), 58—128.

[9] S. R. Caradus, W. E. Pfaffenberger and B. Yood, *Calkin Algebras and Algebras of Operators on Banach Spaces,* Marcel Dekker (New York, 1974).

[10] K. Gustafson, Necessary and sufficient conditions for Weyl's theorem, *Michigan Math. J.*, **19** (1972), 71—81.

[11] V. Istrăţescu, Weyl's theorem for a class of operators, *Rév. Roumaine Math. Pures. Appl.*, **13** (1968), 1103—1105.

[12] T. Kato, *Perturbation Theory for Linear Operators,* Springer-Verlag (New York, 1966).

[13] D. C. Lay, Spectral analysis using ascent, descent, nullity and defect, *Math. Ann.*, **184** (1970), 197—214.

[14] M. Schechter, On the essential spectrum of an arbitrary operator I, *J. Math. Anal. Appl.*, **13** (1966), 205—215.

[15] M. Schechter, *Principals of Functional Analysis,* Academic Press (New York, 1971).

DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF GEORGIA
ATHENS, GEORGIA 30 602
USA