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C_0-Fredholm operators. I

HARI BERCOVICI

In this note we introduce the notions of C_0-Fredholm and C_0-semi-Fredholm operators, which are generalisations of the Fredholm and semi-Fredholm operators, and we study some properties of these operators. The study of index problems in connection with operators that intertwine contractions of class C_0 was suggested by [10], Theorem 2 and Conjecture.

In § 1 of this note we introduce some notions and we define and study the determinant function of an arbitrary operator of class C_0. In § 2 the notions of C_0-fredholmness, C_0-semi-fredholmness, and index are defined. Here we find (Corollary 2.8) a generalisation of [10], Theorem 2 under weaker assumptions. We also show that the index defined for C_0-semi-Fredholm operators is multiplicative. At the end of § 2 we prove a perturbation theorem. In § 3 we show that there exist C_0-Fredholm operators with given index (Proposition 3.1). We also prove that the conjecture from [10] is generally false (Proposition 3.2) but is verified in the bicommutant of a C_0 contraction of arbitrary multiplicity (Proposition 3.4). At the end of § 3 we show that the set of C_0-Fredholm operators is not generally open.

§ 1. Preliminaries. The determinant function

For any (linear and bounded) operator T acting on the Hilbert space $\mathfrak{H}$ we denote by Lat (T) the set of invariant subspaces of T and by $\mathrm{Lat}_{1/2}(T)$ the set of all semi-invariant subspaces of T (that is, subspaces of the form $\mathfrak{M}\ominus\mathfrak{N}$, where $\mathfrak{M}, \mathfrak{N}\in\mathrm{Lat}(T)$ and $\mathfrak{M}\supset\mathfrak{N}$). It is known (see [4], Lemma 0) that a subspace $\mathfrak{M}$ of $\mathfrak{H}$ is semi-invariant for T if and only if

(1.1) $$T_{\mathfrak{M}} = P_{\mathfrak{M}}T|\mathfrak{M}$$

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is a "power-compression", that is, if

(1.2) $$T^n_{\mathfrak{M}} = P_{\mathfrak{M}} T^n | \mathfrak{M}, \quad n = 1, 2, \ldots.$$

If T is a completely non-unitary contraction this is equivalent to

(1.3) $$u(T_{\mathfrak{M}}) = P_{\mathfrak{M}} u(T) | \mathfrak{M}, \quad u \in H^\infty.$$

It is obvious that $\mathrm{Lat}_{1/2}(T) = \mathrm{Lat}_{1/2}(T^*)$ (we have $\mathfrak{M} \ominus \mathfrak{N} = \mathfrak{N}^\perp \ominus \mathfrak{M}^\perp$). Let us recall that the multiplicity μ_T of the operator T is the minimum cardinality of a subset $\mathfrak{A}$ of $\mathfrak{H}$ such that $\bigvee_{n \geq 0} T^n \mathfrak{A} = \mathfrak{H}$. For each $\mathfrak{M} \in \mathrm{Lat}_{1/2}(T)$ let us put $\mu_T(\mathfrak{M}) = \mu_{T_{\mathfrak{M}}}$. If T is an operator of class C_0, we have by [7] that $\mu_T = \mu_{T^*}$. In this case we shall have

(1.4) $$\mu_T(\mathfrak{M}) = \mu_{T^*}(\mathfrak{M}), \quad M \in \mathrm{Lat}_{1/2}(T).$$

For any two operators T, T' acting on $\mathfrak{H}, \mathfrak{H}'$, respectively, we denote by $\mathscr{I}(T', T)$ the set of those operators $X \colon \mathfrak{H} \to \mathfrak{H}'$ which satisfy the relation

(1.5) $$T'X = XT.$$

Obviously, $(\mathscr{I}(T, T'))^* = \mathscr{I}(T'^*, T^*)$.

We are now going to define the determinant function of a C_0 operator acting on a separable Hilbert space.

Definition 1.1. Let T be a C_0 operator acting on a separable space and let $S(M)$, $M = \{m_j\}_{j=1}^\infty$ be the Jordan model of T [2]. We define the determinant function d_T as the limit of any convergent subsequence of $\{m_1 m_2 \ldots m_j\}$ $(j = 1, 2, \ldots)$.

The function d_T is uniquely determined up to a constant factor of modulus one because $|d_T| = \prod_{j=1}^\infty |m_j|$. If $d_T \neq 0$ then d_T is an inner function.

The C_0 operators of finite multiplicity have nonvanishing determinant function. Indeed, if $S(m_1, m_2, \ldots, m_n)$ is the Jordan model [6] of T, we have $d_T = m_1 m_2 \ldots m_n$. For any C_0 operator T the relation $d_{T^*} = d_T^{\sim}$ holds (where $f^{\sim}(z) = \overline{f(\bar{z})}$).

With this definition of the determinant function, it is obvious that d_T is invariant with respect to quasi-affine transforms. It is also obvious that $d_T = 1$ if and only if T acts on the trivial space $\{0\}$. We shall use the general notation

(1.6) $$d_T(\mathfrak{M}) = d_{T_{\mathfrak{M}}}$$

for any C_0 operator T and any $\mathfrak{M} \in \mathrm{Lat}_{1/2}(T)$.

Lemma 1.2. *A contraction T of class C_0 on a separable Hilbert space is a weak contraction if and only if $d_T \neq 0$. If T is a weak contraction of class C_0, d_T coincides with the determinant of the characteristic function of T.*

Proof. If $d_T \neq 0$ it follows that the Jordan model $S(M)$ of T is a weak contraction (cf. [3], Lemma 8.4). Thus, by Proposition 4.3.a of [3], it follows that T is a weak contraction. Conversely, if T is a weak contraction, by Lemma 8.4 and Theorem 8.5 of [3] we have $d_T \neq 0$. The coincidence of d_T with the determinant of the characteristic function of T follows from [3], Theorem 8.7.

Theorem 1.3. *For any C_0 operator T acting on a separable space and any $\mathfrak{H}' \in \mathrm{Lat}(T)$ we have $d_T = d_T(\mathfrak{H}') d_T(\mathfrak{H}'')$, where $\mathfrak{H}'' = \mathfrak{H}'^{\perp}$.*

Proof. If $d_T \neq 0$, T is a weak contraction and the Theorem follows from [3], Proposition 8.2. If $d_T = 0$ we must show that either $d_T(\mathfrak{H}') = 0$ or $d_T(\mathfrak{H}'') = 0$. Equivalently, we have to show that T is a weak contraction whenever $T_{\mathfrak{H}'}$ and $T_{\mathfrak{H}''}$ are weak contractions. So, let us assume that $T_{\mathfrak{H}'}$ and $T_{\mathfrak{H}''}$ are weak contractions. Let $S(M)$, $S(M')$, $S(M'')$ be the Jordan models of T, T', T'', respectively. We consider firstly the case $\mu_T(\mathfrak{H}') < \infty$. For every natural number k we can find a subspace $\mathfrak{H}_k \in \mathrm{Lat}(T)$ such that $T|\mathfrak{H}_k$ is quasisimilar to $S(m_1, m_2, \ldots, m_k)$. The subspace $\mathfrak{H}'_k = \mathfrak{H}' \vee \mathfrak{H}_k \in \mathrm{Lat}(T)$ and $T|\mathfrak{H}'_k$ is also of finite multiplicity. From [3], Proposition 8.2 we infer

$$d_T(\mathfrak{H}'_k) = d_T(\mathfrak{H}') d_T(\mathfrak{H}''_k), \quad \mathfrak{H}''_k = \mathfrak{H}'_k \ominus \mathfrak{H}' = \mathfrak{H}'_k \cap \mathfrak{H}''. \tag{1.7}$$

Again by [3], Proposition 8.2, $m_1 m_2 \ldots m_k$ divides $d_T(\mathfrak{H}'_k)$ and $d_T(\mathfrak{H}''_k)$ divides $d_T(\mathfrak{H}'')$. Thus (1.7) implies that $m_1 m_2 \ldots m_k$ divides $d_T(\mathfrak{H}') d_T(\mathfrak{H}'')$. In particular $d_T \neq 0$ and by [3], Proposition 8.2, we have $d_T = d_T(\mathfrak{H}') d_T(\mathfrak{H}'')$ in this case.

Let us remark now that from the preceding argument it follows that the equality $d_T = d_T(\mathfrak{H}') d_T(\mathfrak{H}'')$ also holds under the assumption $\mu_T(\mathfrak{H}'') < \infty$. Indeed, we have only to replace T by T^* and to use the relation $d_{T^*} = d_T^{\sim}$.

We are now considering the general case. Let $\mathfrak{H}_k$, $\mathfrak{H}'_k$, $\mathfrak{H}''_k$ have the same meaning as before. It is clear that $\mu_T(\mathfrak{H}''_k) < \infty$ and by the preceding remark it follows that $T_{\mathfrak{H}'_k}$ is a weak contraction and (1.7) holds. Arguing as in the case $\mu_T(\mathfrak{H}') < \infty$ we obtain $d_T \neq 0$, that is T is a weak contraction. This finishes the proof.

Let T, T' be two operators and $X \in \mathscr{I}(T', T)$. For every $\mathfrak{M} \in \mathrm{Lat}(T)$, $(X\mathfrak{M})^- \in \mathrm{Lat}(T')$. We shall prove now a lemma which is not particularly concerned with operators of class C_0.

Lemma 1.4. *Let T, T' be two operators and let $X \in \mathscr{I}(T', T)$. The mapping $\mathfrak{K} \mapsto (X\mathfrak{K})^-$ is onto $\mathrm{Lat}(T')$ if and only if $\mathfrak{K}' \mapsto (X^*\mathfrak{K}')^-$ is one-to-one on $\mathrm{Lat}(T'^*)$*

Proof. Let us assume that $\mathfrak{K}' \mapsto (X^*\mathfrak{K}')^-$ is one-to-one on $\mathrm{Lat}(T'^*)$ and let us take $\mathfrak{K}' \in \mathrm{Lat}(T')$. If we put $\mathfrak{K} = X^{-1}(\mathfrak{K}')$ and $\mathfrak{K}'_1 = (X\mathfrak{K})^-$, we have $(X^*(\mathfrak{K}'^{\perp}_1))^- = (\mathrm{ran}\, X^* P_{\mathfrak{K}'^{\perp}_1})^- = (\ker P_{\mathfrak{K}'^{\perp}_1} X)^{\perp} = (X^{-1}(\mathfrak{K}'_1))^{\perp} = (X^{-1}(\mathfrak{K}'))^{\perp}$ and by the same computation $(X^*(\mathfrak{K}'^{\perp}))^- = (X^{-1}(\mathfrak{K}'))^{\perp}$. By the assumption we have $\mathfrak{K}'^{\perp}_1 = \mathfrak{K}'^{\perp}$, $\mathfrak{K}'_1 = \mathfrak{K}'$ so that $\mathfrak{K}' = (X\mathfrak{K})^-$.

Conversely, let us assume that $\mathfrak{K}\mapsto(X\mathfrak{K})^-$ is onto Lat (T') and let us take $\mathfrak{K}'\in$ Lat (T'^*). Then $\mathfrak{K}'^\perp=(X\mathfrak{K})^-$ where $\mathfrak{K}=X^{-1}(\mathfrak{K}'^\perp)$. We have $\mathfrak{K}'=(X\mathfrak{K})^\perp=$ $=(\text{ran } XP_{\mathfrak{K}})^\perp=\ker P_{\mathfrak{K}}X^*=X^{*-1}(\mathfrak{K}^\perp)=X^{*-1}((X^{-1}(\mathfrak{K}'^\perp))^\perp)=X^{*-1}(\ker P_{\mathfrak{K}'}X)^\perp=$ $=X^{*-1}(\text{ran } X^*P_{\mathfrak{K}'})^-=X^{*-1}((X^*\mathfrak{K}')^-)$ which shows that $\mathfrak{K}'$ is determined in this case by $(X^*\mathfrak{K}')^-$. The lemma follows.

Remark 1.5. Because the Jordan model of a C_0 operator acting on a non-separable Hilbert space contains uncountably many direct summands of the form $S(m)$ (cf. [1]) it is natural to extend the definition of the determinant function by taking $d_T=0$ for T acting on a non-separable space. With this extension Lemma 1.2 and Theorem 1.3 remain valid with the condition of separability dropped. For Lemma 1.4 it is enough to remark that a completely non-unitary weak contraction acts on a necessarily separable space and for the Theorem 1.3 we have to remark that T acts on a separable space if and only if $\mathfrak{H}'$ and $\mathfrak{H}''$ are separable spaces.

§ 2. C_0-Fredholm operators

Definition 2.1. Let T, T' be two operators and let $X\in\mathscr{I}(T', T)$. X is called a *(T', T)-lattice-isomorphism* if the mapping $\mathfrak{M}\mapsto(X\mathfrak{M})^-$ is an isomorphism of Lat (T) onto Lat (T').

For $T=0$ and $T'=0$ a (T', T)-lattice-isomorphism is simply an invertible operator. It is clear that a lattice-isomorphism is always a quasi-affinity but the converse is not true as shown by the example $T=0$, $T'=0$. By Lemma 1.4, X is a (T', T)-lattice-isomorphism if and only if X^* is a (T^*, T'^*)-lattice-isomorphism. We shall say simply lattice-isomorphism instead of (T', T)-lattice-isomorphism whenever it will be clear which are T and T'.

Definition 2.2. Let T and T' be two operators of class C_0 and $X\in\mathscr{I}(T', T)$. X is called a *(T', T)-semi-Fredholm* operator if $X|(\ker X)^\perp$ is a $(T'|(\text{ran } X)^-, T_{(\ker X)^\perp})$-lattice-isomorphism and either $d_T(\ker X)\neq 0$ or $d_{T'}(\ker X^*)\neq 0$. A (T', T)-semi-Fredholm operator X is (T', T)-*Fredholm* if both $d_T(\ker X)$ and $d_{T'}(\ker X^*)$ are different from zero. The *index* of the (T', T)-Fredholm operator X is the meromorphic function

$$j(X)=j_{(T,T')}(X)=d_T(\ker X)/d_{T'}(\ker X^*). \tag{2.1}$$

If X is (T', T)-semi-Fredholm and not (T', T)-Fredholm we define

$$j(X)=0 \quad \text{if} \quad d_T(\ker X)=0; \quad j(X)=\infty \quad \text{if} \quad d_{T'}(\ker X^*)=0. \tag{2.2}$$

We shall say simply C_0-semi-Fredholm, C_0-Fredholm instead of (T', T)-semi-Fredholm, (T', T)-Fredholm, respectively, whenever it will be clear which are the C_0 operators T and T'. We shall denote by $\mathrm{sF}(T', T)$ (respectively $\mathrm{F}(T', T)$) the set of all (T', T)-semi-Fredholm (respectively (T', T)-Fredholm) operators. If $T=T'$ we shall write $\mathrm{sF}(T)$, $\mathrm{F}(T)$ instead of $\mathrm{sF}(T, T)$, $\mathrm{F}(T, T)$, respectively.

We can easily see how the preceding definition is related to the usual definition of Fredholm operators. Let us note that the operator $T=0$ acting on the Hilbert space $\mathfrak{H}$ is a C_0 operator; it is a weak contraction if and only if $n=\dim \mathfrak{H}<\infty$ and in this case $d_T(z)=z^n$ $(|z|<1)$. If $T=T'=0$ and $X\in\mathscr{I}(T', T)=\mathscr{L}(\mathfrak{H})$ then $X|(\ker X)^\perp$ is a lattice-isomorphism if and only if X has closed range. From these remarks it follows that an operator $X\in\mathscr{I}(0, 0)$ is C_0-Fredholm if and only if it is Fredholm in the usual sense, and $j(X)(z)=z^{i(X)}$, where $i(X)=\dim\ker X-\dim\ker X^*$ is the (usual) index of the Fredholm operator X.

Proposition 2.3. *Let T, T', T'' be C_0-operators acting on $\mathfrak{H}, \mathfrak{H}', \mathfrak{H}''$, respectively, and let $A\in\mathscr{I}(T, T')$, $B\in\mathscr{I}(T, T'')$ be such that $A\mathfrak{H}'\subset(B\mathfrak{H}'')^-$. If $d_{T'}\neq 0$, we have:*

$$(A^{-1}(B\mathfrak{H}''))^- = \mathfrak{H}'; \tag{2.3}$$

$$(A\mathfrak{H}'\cap B\mathfrak{H}'')^- \supset A\mathfrak{H}'. \tag{2.4}$$

Proof. It is enough to prove (2.3) because (2.4) is a simple consequence of (2.3).

We may suppose that B is a quasi-affinity and A is one-to-one. Indeed, we have only to replace A, B respectively by $A|(\ker A)^\perp$ and $B|(\ker B)^\perp$, and $\mathfrak{H}$ by $(B\mathfrak{H}'')^-$. It follows that $d_{T''}=d_T$ and T' is quasisimilar to the restriction of T to some invariant subspace. By Theorem 1.3 we have $d_{T'}\neq 0$ and therefore

$$d_{T'\oplus T''} = d_{T'}d_{T''} = d_{T'}d_T \neq 0. \tag{2.4}$$

The operator $X\colon \mathfrak{H}'\oplus\mathfrak{H}''\to\mathfrak{H}$ defined by $X(h'\oplus h'')=Ah'-Bh''$ has dense range and satisfies $TX=X(T'\oplus T'')$.
Thus $(T'\oplus T'')_{(\ker X)^\perp}$ is a quasi-affine transform of T, in particular

$$d_{T'\oplus T''}((\ker X)^\perp) = d_T. \tag{2.5}$$

From (2.4) and (2.5) we infer

$$d_{T'\oplus T''}(\ker X) = d_{T'}. \tag{2.6}$$

The operator $Y\colon \ker X\to\mathfrak{H}'$ defined by $Y(h'\oplus h'')=h'$ is one-to-one. Indeed, $Y(h'\oplus h'')=0$ and $h'\oplus h''\in\ker X$ imply $h'=0$ and $Bh''=Ah'=0$; it follows that $h''=0$ because B is one-to-one. Moreover, we have $Y\in\mathscr{I}(T', (T'\oplus T'')|(\ker X))$. It is easy to verify that $\operatorname{ran} Y=A^{-1}(B\mathfrak{H}'')$. By the invariance of the determinant function we have

$$d_{T'}((A^{-1}(B\mathfrak{H}''))^-) = d_{T'\oplus T''}(\ker X) = d_{T'}. \tag{2.7}$$

From Theorem 1.3 and relation (2.7) it follows that

$$(2.8)\qquad d_{T'} = d_{T'}((A^{-1}(B\mathfrak{H}''))^{-})d_{T'}((A^{-1}(B\mathfrak{H}''))^{\perp}) = d_{T'}d_{T'}((A^{-1}(B\mathfrak{H}''))^{\perp})$$

and therefore

$$d_{T'}((A^{-1}(B\mathfrak{H}''))^{\perp}) = 1, \quad (A^{-1}(B\mathfrak{H}''))^{\perp} = \{0\} \text{ and (2.3) follows.}$$

The Proposition is proved.

Corollary 2.4. *Let T, T' be two C_0 operators such that $d_T \neq 0$ and let $A \in \mathscr{I}(T', T)$ be a quasi-affinity. Then A is a lattice-isomorphism.*

Proof. The correspondence $\mathfrak{K} \mapsto (A\mathfrak{K})^{-}$ is onto Lat (T') by Proposition 2.3. Corollary follows by Lemma 1.4 since A^* is also a quasi-affinity.

Lemma 2.5. *Let T, T' be C_0 operators and $A \in \mathscr{I}(T', T)$. We always have $d_{T'}d_T(\ker A) = d_T d_{T'}(\ker A^*)$.*

Proof. From Theorem 1.3 and the invariance of the determinant function with respect to quasi-affine transforms we infer $d_{T'} = d_{T'}(\ker A^*)d_{T'}((\operatorname{ran} A)^{-}) = d_{T'}(\ker A^*)d_T((\ker A)^{\perp})$ and $d_T = d_T(\ker A)d_T((\ker A)^{\perp})$. The Lemma obviously follows from these relations.

Corollary 2.6. *Let T, T' be weak contractions of class C_0. Then* $\mathrm{F}(T', T) = \mathscr{I}(T', T)$ *and* $j(A) = d_T/d_{T'}$, *for* $A \in \mathscr{I}(T', T)$.

Proof. For each $A \in \mathscr{I}(T', T)$, $A|(\ker A)^{\perp}$ is a lattice-isomorphism by Corollary 2.4. Also we have $d_T(\ker A) \neq 0$ and $d_{T'}(\ker A^*) \neq 0$ by Theorem 1.3. The value of $j(A)$ follows then from Lemma 2.5.

Remark 2.7. From the preceding proof it easily follows that $\mathrm{sF}(T', T) = \mathscr{I}(T', T)$ and $\mathrm{F}(T', T) = \emptyset$ if exactly one of the contractions T and T' is weak.

The following Corollary is a generalisation of [10], Theorem 2.

Corollary 2.8. *Let T and T' be weak contractions of class C_0 such that $d_T = d_{T'}$. Then each injection $A \in \mathscr{I}(T', T)$ is a lattice-isomorphism (in particular a quasi-affinity).*

Proof. Let $A \in \mathscr{I}(T', T)$ be an injection. By Corollary 2.6 $A \in \mathrm{F}(T', T)$ and $j(A) = d_T/d_{T'} = 1$; it follows that $d_{T'}(\ker A^*) = d_T(\ker A) = 1$, thus $\ker A^* = \{0\}$ and A is a quasi-affinity. The conclusion follows by Corollary 2.4.

Corollary 2.9. *Let T be a weak contraction of class C_0 and let $A \in \{T\}'$ be an injection. Then the restriction of A to each hyper-invariant subspace of T is a quasi-affinity.*

Proof. Obviously follows from the preceding Corollary.

Lemma 2.10. *For any two C_0 operators T and T' we have* $\mathrm{sF}(T,T')^* = \mathrm{sF}(T'^*, T^*)$, $\mathrm{F}(T,T')^* = \mathrm{F}(T'^*, T^*)$, *and*

(2.9) $\quad j(A^*) = \big(j(A)^{\sim}\big)^{-1}, \quad A \in \mathrm{sF}(T',T) \quad$ (*here* $0^{-1} = \infty$ *and* $\infty^{-1} = 0$).

Proof. If $A \in \mathscr{I}(T',T)$, we have $(A|(\ker A)^{\perp})^* = A^*|(\ker A^*)^{\perp}$, $d_{T'^*}(\ker A^*) = d_{T'}(\ker A^*)^{\sim}$ and $d_{T^*}(\ker A) = d_T(\ker A)^{\sim}$. The Lemma follows.

Theorem 2.11. *Let T, T', T'' be operators of class C_0, $A \in \mathrm{sF}(T',T)$, $B \in \mathrm{sF}(T'',T')$. If the product $j(B)j(A)$ makes sense we have $BA \in \mathrm{sF}(T'',T)$ and $j(BA) = j(B)j(A)$.*

Proof. We shall show firstly that $BA|(\ker BA)^{\perp}$ is a lattice-isomorphism. To do this we will show that the range of BA is dense in each cyclic subspace of T'', contained in $(\mathrm{ran}\, BA)^-$. The whole statement will follow from Lemma 1.4 and Lemma 2.10 and the same argument applied to $(BA)^* = A^*B^*$.

Let us remark that from the C_0-semi-fredholmness of B it follows that

$$B^{-1}\big((\mathrm{ran}\, BA)^-\big) \subset \big((\mathrm{ran}\, A)^- + \ker B\big)^-.$$

Therefore, for each $f \in (\mathrm{ran}\, BA)^-$ and $\varepsilon > 0$ we can find $g \in \big((\mathrm{ran}\, A)^- + \ker B\big)^-$ such that

(2.10) $$Bg \in \mathfrak{H}_f = \bigvee_{n \geq 0} T''^n f \quad \text{and} \quad \|Bg - f\| < \varepsilon.$$

Now, let us denote by $\mathfrak{R}$ the subspace $\big((\mathrm{ran}\, A)^- + \ker B\big)^- \ominus (\mathrm{ran}\, A)^-$ and by P the orthogonal projection of $\big((\mathrm{ran}\, A)^- + \ker B\big)^-$ onto $\mathfrak{R}$. We claim that

(2.11) $$d_{T'}(\mathfrak{R}) \neq 0.$$

Indeed, if $j(A) \neq \infty$, we have $d_{T'}(\ker A^*) \neq 0$ and $\mathfrak{R} \subset \ker A^*$. If $j(A) = \infty$ it follows from the hypothesis that $j(B) \neq 0$ and therefore $d_{T'}(\ker B) \neq 0$. But

(2.12) $$\big((\mathrm{ran}\,(P|\ker B)\big)^- = \mathfrak{R}$$

and

(2.13) $$T'_{\mathfrak{R}} = PT'|\big((\mathrm{ran}\, A)^- + \ker B\big)^-.$$

From Theorem 1.3 and the invariance of the determinant function with respect to quasi-affine transforms we infer that $d_{T'}(\mathfrak{R})$ divides $d_{T'}(\ker B)$; thus (2.11) is proved.

From the relations (2.11—13) it follows, via Proposition 2.3, that $\{k \in \mathfrak{H}_g;\ Pk \in P(\ker B)\}$ is dense in $\mathfrak{H}_g$, that is $\mathfrak{H}_g \cap \big((\mathrm{ran}\, A)^- + \ker B\big)$ is dense in $\mathfrak{H}_g$. Thus there exist $u \in (\mathrm{ran}\, A)^-$ and $v \in \ker B$ such that

(2.14) $$u + v \in \mathfrak{H}_g, \quad \|u + v - g\| < \varepsilon.$$

Now, by the C_0-semi-fredholmness of A, there exists $k\in\mathfrak{H}$ such that

$$(2.15)\qquad Ak\in\mathfrak{H}_u,\quad \|Ak-u\|<\varepsilon.$$

We have $Bu=B(u+v)\in B\mathfrak{H}_g\subset\mathfrak{H}_f$ and it follows that $B\mathfrak{H}_u\subset\mathfrak{H}_f$. Therefore $BAk\in B\mathfrak{H}_u\subset\mathfrak{H}_f$. From (2.10), (2.14) and (2.15) we infer $\|BAk-f\|\leqq\|BAk-Bu\|+$ $+\|B(u+v)-Bg\|+\|Bg-f\|<(2\|B\|+1)\varepsilon$. Because ε is arbitrarily small, the first part of the proof is done.

We obviously have

$$(2.16)\qquad \ker BA=A^{-1}(\ker B),\quad \ker(BA)^*=B^{*-1}(\ker A^*).$$

Let us consider the triangularisation $T|\ker BA=\begin{bmatrix}T_1 & X\\ 0 & T_2\end{bmatrix}$ determined by the decomposition $\ker BA=\ker A\oplus(\ker BA\ominus\ker A)$. By the C_0-semi-fredholmness of A, T_2 is a quasi-affine transform of $T'|\mathfrak{H}_1$, where

$$(2.17)\qquad \mathfrak{H}_1=(\operatorname{ran}A)^-\cap\ker B.$$

If $d_{T'}(\ker B)\neq 0$ and $d_T(\ker A)\neq 0$ it follows that

$$(2.18)\qquad d_T(\ker BA)=d_T(\ker A)\,d_{T'}(\mathfrak{H}_1)\neq 0,$$

thus $BA\in\mathrm{sF}(T'',T)$. Analogously, if $d_{T''}(\ker B^*)\neq 0$ and $d_{T'}(\ker A^*)\neq 0$ it follows that $BA\in\mathrm{sF}(T'',T)$. From the hypothesis it follows that at least one of the situations considered must occur. Thus we always have $BA\in\mathrm{sF}(T'',T)$.

It is obvious that $d_{T''}(\ker(BA)^*)=0$ whenever $d_{T''}(\ker B^*)=0$ since $\ker(BA)^*\supset\ker B^*$. Thus the relation $j(BA)=\infty=j(B)j(A)$ is proved in this case. Let us suppose now that $j(B)=0$. Then, by Theorem 1.3 we have

$$0=d_{T'}(\ker B)=d_{T'}(\mathfrak{H}_1)\,d_{T'}(\ker B\ominus\mathfrak{H}_1).$$

The projection onto $\ker A^*$ is one-to-one on $\ker B\ominus\mathfrak{H}_1$, thus $T'_{\ker B\ominus\mathfrak{H}_1}$ is a quasi-affine transform of some restriction of $T'_{\ker A^*}$. It follows that $d_{T'}(\ker B\ominus\mathfrak{H}_1)\neq 0$ and the preceding relation implies $d_{T'}(\mathfrak{H}_1)=0$. By (2.18), the relation $j(BA)=$ $=j(B)j(A)$ $(=0)$ is proved in this case also. If $j(A)\in\{0,\infty\}$ we have $j(BA)=$ $=\big(j((BA)^*)^{\check{}}\big)^{-1}=\big(j(A^*)^{\check{}}\,j(B^*)^{\check{}}\big)^{-1}=j(B)j(A)$ by Lemma 2.10.

It remains now to prove the relation $j(BA)=j(B)j(A)$ for $A\in\mathrm{F}(T',T)$ and $B\in\mathrm{F}(T'',T')$. From the second relation (2.14) we infer, as before,

$$(2.18)^*\qquad d_{T''}(\ker(BA)^*)=d_{T''}(\ker B^*)\,d_{T'}(\mathfrak{H}_1^*)$$

where

$$(2.17)^*\qquad \mathfrak{H}_1^*=(\operatorname{ran}B^*)^-\cap\ker A^*=(\ker B)^\perp\cap(\operatorname{ran}A)^\perp.$$

Let us denote by Q the orthogonal projection of $\mathfrak{H}'$ onto $(\operatorname{ran}A)^\perp=\ker A^*$. If we consider the decompositions

$$(2.19)\qquad \ker B=\mathfrak{H}_1\oplus\mathfrak{H}_2,\quad \ker A^*=\mathfrak{H}_1^*\oplus\mathfrak{H}_2^*,$$

we claim that $Q|\mathfrak{H}_2$ is a quasi-affinity from $\mathfrak{H}_2$ into $\mathfrak{H}_2^*$. Indeed, if $h\in\mathfrak{H}_2$ and $g\in\mathfrak{H}_1^*$, we have $(g, Qh)=(g, h)=0$ as $g\in(\ker B)^\perp$, thus $Q\mathfrak{H}_2\subset\mathfrak{H}_2^*$. Because $\mathfrak{H}_1=\ker B\cap\cap(\operatorname{ran} A)^-=\ker(Q|\ker B)$, Q is one-to-one on $\mathfrak{H}_2$. We have only to show that $\ker A^*\ominus(Q\mathfrak{H}_2)^-=\mathfrak{H}_1^*$. If $h\in\ker A^*\ominus(Q\mathfrak{H}_2)^-$ and $g\in\ker B$ we have $(h, g)=(h, Qg)=0$ because $(Q\mathfrak{H}_2)^-=(Q(\ker B))^-$ (as $Q|\mathfrak{H}_1=0$); the inclusion $\ker A^*\ominus(Q\mathfrak{H}_2)^-\subset\mathfrak{H}_1^*$ follows and the assertion concerning $Q|\mathfrak{H}_2$ is proved.

Now, because $\mathfrak{H}_1=\ker(Q|\ker B)$, we have the intertwining relation $T'_{\mathfrak{H}_2^*}(Q|\mathfrak{H}_2)=$ $=(Q|\mathfrak{H}_2)T'_{\mathfrak{H}_2}$; in particular

$$d_{T'}(\mathfrak{H}_2) = d_{T'}(\mathfrak{H}_2^*). \tag{2.20}$$

By (2.18—20) and Theorem 1.3 we have

$$\begin{aligned} j(BA) &= d_T(\ker BA)/d_{T''}(\ker(BA)^*) = \\ &= (d_T(\ker A)/d_{T''}(\ker B^*))(d_{T'}(\mathfrak{H}_1)/d_{T'}(\mathfrak{H}_1^*)) = \\ &= (d_T(\ker A)/d_{T''}(\ker B^*))(d_{T'}(\mathfrak{H}_1)d_{T'}(\mathfrak{H}_2)/d_{T'}(\mathfrak{H}_1^*)d_{T'}(\mathfrak{H}_2^*)) = \\ &= (d_T(\ker A)/d_{T'}(\ker A^*))(d_{T'}(\ker B)/d_{T''}(\ker B^*)) = j(B)j(A). \end{aligned}$$

Theorem 2.11 is proved.

Theorem 2.12. *Let T be an operator of class C_0 acting on $\mathfrak{H}$ and let $X\in\{T\}'$ be such that $d_T((X\mathfrak{H})^-)\neq 0$. Then $I+X\in F(T)$ and $j(I+X)=1$.*

Proof. We firstly show that the mapping $\operatorname{Lat}(T)\ni\mathfrak{M}\mapsto((I+X)\mathfrak{M})^-$ is onto $\operatorname{Lat}(T|((I+X)\mathfrak{H})^-)$. To do this let us take $\mathfrak{N}\in\operatorname{Lat}(T)$, $\mathfrak{N}\subset((I+X)\mathfrak{H})^-$ and let P denote the orthogonal projection of $\mathfrak{H}$ onto $(\ker X)^\perp$. Because $P\mathfrak{N}\subset(P(I+X)\mathfrak{H})^-$, $T_{(\ker X)^\perp}P=PT$ and $d_T((\ker X)^\perp)\neq 0$, it follows by Proposition 2.3 that $\mathfrak{N}'=\{h\in\mathfrak{N};\ Ph\in P(I+X)\mathfrak{H}\}$ is dense in $\mathfrak{N}$. Now we can show that $\mathfrak{N}'\subset(I+X)\mathfrak{H}$; indeed $\mathfrak{N}'\subset(I+X)\mathfrak{H}+\ker X$ and $\ker X\subset(I+X)\mathfrak{H}$ $(h=(I+X)h$ for $h\in\ker X)$. Therefore we have $N=((I+X)\mathfrak{M})^-$, where $\mathfrak{M}=(I+X)^{-1}\mathfrak{N}$.

From the preceding argument applied to $I+X^*$ and from Lemma 1.4 it follows that $(I+X)|(\ker(I+X))^\perp$ is a lattice-isomorphism. Because $\ker(I+X)\subset X\mathfrak{H}$ $(h=-Xh$ whenever $(I+X)h=0)$ and $\ker(I+X)^*\subset X^*\mathfrak{H}$, by Theorem 1.3 it follows that $I+X\in F(T)$.

It remains only to compute $j(I+X)$. To do this let us consider the decomposition $\mathfrak{H}=\mathfrak{U}\oplus\mathfrak{V}$, $\mathfrak{U}=(X\mathfrak{H})^-$. With respect to this decomposition we have $I=\begin{bmatrix} I_{\mathfrak{U}} & 0 \\ 0 & I_{\mathfrak{V}}\end{bmatrix}$, $X=\begin{bmatrix} X' & X'' \\ 0 & 0\end{bmatrix}$, where $X'\in\{T|\mathfrak{U}\}'$. Since by the hypothesis $T|\mathfrak{U}$ is a weak contraction, we infer by Corollary 2.6

$$d_T(\ker(I+X')) = d_T(\ker(I+X')^*). \tag{2.21}$$

Now, we can easily verify that $\ker(I+X)=\ker(I+X')$. The inclusion $\ker(I+X')\subset$ $\subset\ker(I+X)$ is obvious. If $h\in\ker(I+X)$ we have $h=-Xh\in\mathfrak{U}$ so that $h=-X'y$

$(X'=X|\mathfrak{U})$ and $h\in\ker(I+X')$. In particular

(2.22) $$d_T(\ker(I+X)) = d_T(\ker(I+X')).$$

It is easy to see, using the matrix representation of X, that $u\oplus v\in\ker(I+X)^*$ if and only if

(2.23) $$u\in\ker(I+X')^* \quad\text{and}\quad v=-X''^*u.$$

If we denote by Q the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{U}$, it follows from (2.23) that $Q|\ker(I+X)^*$ is an invertible operator from $\ker(I+X)^*$ onto $\ker(I+X')^*$, the inverse being given by $\ker(I+X')^*\ni u\mapsto u\oplus(-X''^*u)$. Because we have also $T_{\mathfrak{U}}^*Q=QT^*$ it follows that $T_{\mathfrak{U}}^*|\ker(I+X')^*$ and $T^*|\ker(I+X)^*$ are similar, in particular

(2.24) $$d_T(\ker(I+X)^*) = d_T(\ker(I+X')^*).$$

From (2.21), (2.22), and (2.24) it obviously follows that $j(I+X)=1$. The Theorem is proved.

§ 3. Some examples

Proposition 3.1. *For any two inner functions m and n there exist a C_0 operator T and $X\in\mathrm{F}(T)$ such that $j(X)=m/n$.*

Proof. The operator $T=(S(m)\otimes I)\oplus(S(n)\otimes I)$, where I denotes the identity operator on l^2, is of class C_0. If we denote by U_+ the unilateral shift on l^2, obviously

$$X=(I_{\mathfrak{H}(m)}\otimes U_+^*)\oplus(I_{\mathfrak{H}(n)}\otimes U_+)\in\{T\}'.$$

Moreover, X has closed range so that $X|(\ker X)^\perp$ is invertible. Because $T|\ker X$ is unitarily equivalent to $S(m)$ and $T_{\ker X^*}$ is unitarily equivalent to $S(n)$, it follows that X is C_0-Fredholm and $j(X)=m/n$.

The following proposition infirms the Conjecture from [10]. Proposition 3.4 shows however that this Conjecture is true under the assumption $X\in\{T\}''$ and with the condition $\mu_T<\infty$ dropped.

Proposition 3.2. *Let K and K_* be C_0 operators of finite multiplicities such that $d_K=d_{K_*}$. Then there exist a C_0 operator T of finite multiplicity and an $X\in\{T\}'$ such that $T|\ker X$ and $T_{\ker X^*}$ are quasisimilar to K and K_*, respectively.*

Proof. Let $S=S(m_1, m_2, \ldots, m_n)$ and $S_*=S(m_1', m_2', \ldots, m_n')$ be the Jordan models of K, K_*, respectively (it may happen that some of the m_j or m_j' be equal to 1). By the hypothesis we have

(3.1) $$m_1m_2\ldots m_n = m_1'm_2'\ldots m_n'.$$

Let us consider the operator

(3.2) $$T = S(\varphi_1, \varphi_2, \dots, \varphi_n), \text{ where}$$

(3.3) $$\varphi_1 = m_1 m_2 \dots m_n, \quad \varphi_2 = m_2' m_2 \dots m_n, \quad \varphi_3 = m_2' m_3' m_3 \dots m_n, \quad \dots,$$
$$\varphi_n = m_2' m_3' \dots m_n' m_n.$$

(T is generally not a Jordan operator). The matrix over H^∞ given by

(3.4) $$A = \begin{bmatrix} 0 & 0 & \dots & 0 & m_1' \\ m_2' & 0 & \dots & 0 & 0 \\ 0 & m_3' & \dots & 0 & 0 \\ \dots & & & & \\ 0 & 0 & \dots & m_n' & 0 \end{bmatrix} = [a_{ij}]_{1 \le i, j \le n}$$

satisfies the conditions

(3.5) $$a_{ij} \varphi_j \in \varphi_i H^2$$

and therefore (cf. [2], relations (6.5—7)) the operator X defined by

(3.6) $$X = [X_{ij}]_{1 \le i, j \le n}, \quad X_{ij} h = P_{\mathfrak{H}(\varphi_i)} a_{ij} h \quad (h \in \mathfrak{H}(\varphi_j))$$

commutes with T. Now it is easy to see that

(3.7) $$T|\ker X = \bigoplus_{i=1}^{n} T|(\ker X \cap \mathfrak{H}(\varphi_i)), \quad T_{\ker X^*} = \bigoplus_{i=1}^{n} T_{(\ker X^* \cap \mathfrak{H}(\varphi_i))}.$$

Using [8], p. 315, we see that $T|(\ker X \cap \mathfrak{H}(\varphi_i))$ is unitarily equivalent to $S(m_i)$ and $T_{(\ker X^* \cap \mathfrak{H}(\varphi_i))}$ is unitarily equivalent to $S(m_i')$ so that $T|\ker X$ is unitarily equivalent to S and $T_{\ker X^*}$ is unitarily equivalent to S_*. Proposition 3.2 follows.

Lemma 3.3. *If T and T' are two quasisimilar operators of class C_0 and $\varphi \in H^\infty$ then $T|\ker \varphi(T)$ and $T'|\ker \varphi(T')$ are quasisimilar.*

Proof. Let X, Y be two quasi-affinities such that $T'X = XT$ and $TY = YT'$. Then we have also $\varphi(T')X = X\varphi(T)$ and $\varphi(T)Y = Y\varphi(T')$ which shows that

(3.8) $$X \ker \varphi(T) \subset \ker \varphi(T'), \quad Y \ker \varphi(T') \subset \ker \varphi(T).$$

From (3.8) it follows that $T|\ker \varphi(T)$ can be injected into $T'|\ker \varphi(T')$ and $T'|\ker \varphi(T')$ can be injected into $T|\ker \varphi(T)$. The Lemma follows by [10], Theorem 1.

Proposition 3.4. *Let T be an operator of class C_0 and $X \in \{T\}''$. Then $T|\ker X$ and $T_{\ker X^*}$ are quasisimilar. In particular we have*

$$\mathrm{sF}(T) \cap \{T\}'' = \mathrm{F}(T) \cap \{T\}'' \quad \textit{and} \quad j(X) = 1 \quad \textit{for} \quad X \in \mathrm{F}(T) \cap \{T\}''.$$

Proof. From [2] and [1] it follows that $X=(u/v)(T)$, where $u, v \in H^\infty$ and $v \wedge m_T = 1$. It is easy to see that $\ker X = \ker u(T)$ and $\ker X^* = \ker u^\sim(T^*)$. By Lemma 3.3 it suffices to prove our Proposition for T a Jordan operator and $X=u(T)$. Now, a Jordan operator is a direct sum of operators of the form $S(m)$ and it is easy to see that $S(m)|\ker u(S(m))$ and $(S(m)^*|\ker (u(S(m)))^*)^*$ are both unitarily equivalent to $S(m \wedge u)$. Thus for T a Jordan operator $T|\ker u(T)$ and $T_{\ker(u(T))^*}$ are unitarily equivalent. Thus Proposition follows.

Proposition 3.5. *Let T be an operator of class C_0 and let $X \in \{T\}''$ be an injection. Then X is a lattice-isomorphism.*

Proof. Let $\mathfrak{M} \in \operatorname{Lat}(T)$; by [9] we have $X\mathfrak{M} \subset \mathfrak{M}$. Moreover we have $X|\mathfrak{M} \in \operatorname{Alg}\operatorname{Lat}(T|\mathfrak{M})$ and obviously $X|\mathfrak{M} \in \{T|\mathfrak{M}\}'$. Again by [9] we infer $X|\mathfrak{M} \in \{T|\mathfrak{M}\}''$. From Proposition 3.4 applied to the injection $X|\mathfrak{M}$ we infer $\ker (X|\mathfrak{M})^* = \{0\}$ so that

$$(X\mathfrak{M})^- = \mathfrak{M}. \tag{3.9}$$

This shows that the mapping $\mathfrak{M} \mapsto (X\mathfrak{M})^-$ is the identity on $\operatorname{Lat}(T)$. The Proposition is proved.

Proposition 3.6. *There exist an operator T of class C_0 and operators X_n, $X \in \{T\}''$ such that $\lim_{n\to\infty} \|X_n - X\| = 0$, $X \in \mathrm{F}(T)$ but $X_n \notin \mathrm{F}(T)$, $n=1, 2, \ldots$. Thus the set $\mathrm{F}(T)$ is not generally an open subset of $\{T\}'$.*

Proof. We shall construct Blaschke products m, b and b_n $(n=1, 2, \ldots)$ such that

$$b \wedge m = 1, \quad b_n \wedge m \neq 1; \tag{3.10}$$

$$\lim_{n\to\infty} \|b_n - b\|_\infty = 0. \tag{3.11}$$

Then the required example is given by

$$T = S(m) \otimes I, \tag{3.12}$$

where I denotes the identity operator on an infinite dimensional Hilbert space, and

$$X = b(T), \quad X_n = b_n(T) \quad (n = 1, 2, \ldots). \tag{3.13}$$

It is clear that $T|\ker X_n$ is unitarily equivalent to $S(m \wedge b_n) \otimes I$ which is not a weak contraction and therefore $X_n \notin \mathrm{F}(T)$ (by Proposition 3.4, $X_n \notin \mathrm{sF}(T)$). Because $b \wedge m = 1$, $b(T)$ is a lattice-isomorphism by Proposition 3.5, in particular $X \in \mathrm{F}(T)$. The convergence $X_n \to X$ follows from (3.11).

It remains only to construct the functions m, b and b_n ($n=1, 2, \ldots$). Let us put

$$b = \prod_{k=1}^{\infty} B^k, \quad b_n = \prod_{k=1}^{\infty} B_n^k \,(n = 1, 2, \ldots), \quad m = \prod_{k=1}^{\infty} B_k^k \tag{3.14}$$

where B^k (respectively B_n^k) is the Blaschke factor with the zero k^{-2} (respectively $k^{-2}\exp(it_n^k)$, $t_n^k>0$). Because $|b-b_n|\leqq \sum_{k=1}^{\infty}|B^k-B_n^k|$, one can verify that (3.11) holds whenever $\lim_{n\to\infty}\sum_{k=1}^{\infty}k^4 t_n^k=0$. Conditions (3.10) are also verified and $b_n \wedge m = B_n^n$.

References

[1] H. Bercovici, On the Jordan model of C_0 operators, *Studia Math.*, **60** (1977), 267—284.

[2] H. Bercovici, C. Foiaş, B. Sz.-Nagy, Compléments à l'étude des opérateurs de classe C_0. III, *Acta Sci. Math.*, **37** (1975), 313—322.

[3] H. Bercovici, D. Voiculescu, Tensor operations on characteristic functions of C_0 contractions, *Acta Sci. Math.*, **39** (1977), 205—233.

[4] D. Sarason, On spectral sets having connected complement, *Acta Sci. Math.*, **26** (1966), 289—299.

[5] B. Sz.-Nagy, C. Foiaş, *Harmonic Analysis of Operators on Hilbert Space*, North-Holland/Akadémiai Kiadó (Amsterdam/Budapest, 1970).

[6] B. Sz.-Nagy, C. Foiaş, Modèle de Jordan pour une classe d'opérateurs de l'espace de Hilbert, *Acta Sci. Math.*, **31** (1970), 91—115.

[7] B. Sz.-Nagy, C. Foiaş, Compléments à l'étude des opérateurs de classe C_0, *Acta Sci. Math.*, **31** (1970), 287—296.

[8] B. Sz.-Nagy, C. Foiaş, Jordan model for contractions of class $C_{\cdot 0}$, *Acta Sci. Math.*, **36** (1974), 305—322.

[9] B. Sz.-Nagy, C. Foiaş, Commutants and bicommutants of operators of class C_0, *Acta Sci. Math.*, **38** (1976), 311—315.

[10] B. Sz.-Nagy, C. Foiaş, On injections, intertwining operators of class C_0, *Acta Sci. Math.*, **40** (1978), 163—167.

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