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The singular sequence problem

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Introduction

If A and B are bounded selfadjoint operators in a Hilbert space H, and $B-A$ is compact, then A and B have the same essential spectrum.

This well-known result of H. WEYL [7] (cf. [1], section 94, Satz 1, [6], Satz 9.9) is easily proved by using Weyl's characterization of the essential spectrum by singular sequences. At the same time this proof shows that more is valid, namely: A and B have the same singular sequences. (For definitions see the end of the introduction.)

In this note we treat the question if the converse of this statement is valid, i.e.: *Let A and B be bounded selfadjoint operators with the same singular sequences. Is it possible to conclude that $B-A$ is compact?* We remark that we do not know the complete answer to this question. The purpose of this note is to present this problem and to give a positive answer in a special case.

We remark that a kind of converse of the above theorem of Weyl was proved by VON NEUMANN [4] (cf. [1], section 94, Satz 3): If A and B are bounded selfadjoint operators in a separable Hilbert space, with the same essential spectrum, then there exists a unitary operator U such that $B-UAU^{-1}$ is compact. It is easy to see by examples that $B-A$ need not be compact under this assumption; also A and B need not have the same singular sequences.

In Section 1 we review some results for unbounded operators in order to motivate the form in which we finally state the "singular sequence problem" for unbounded operators. In Section 2 we give a positive solution for the case that $\sigma(A)$ (or, equivalently, $\sigma_e(A)$) is countable. Here we need only that every singular sequence for A and s is also a singular sequence for B and s. In section 3 we show by an example that in the general case this assumption alone is not sufficient.

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We conclude the introduction by some basic facts and some notations. Let A be a selfadjoint operator in a Hilbert space H. The *essential spectrum* $\sigma_e(A)$ of A is the set consisting of the limit points of the spectrum $\sigma(A)$ of A and the eigenvalues of infinite multiplicity; this is just the set of the points of $\sigma(A)$ which are not isolated eigenvalues of finite multiplicity ([1], section 93, [3], section 1, [6], section 7.4). A real number s is in $\sigma_e(A)$ if and only if there is a *singular sequence for A and s*, i.e., a sequence (f_n) in $D(A)$ (the domain of definition of A) such that $\liminf \|f_n\| > 0$, $f_n \rightharpoonup 0$, and $(A-s)f_n \to 0$ ([7]; cf. [3], Theorem 11, [6], Satz 7.24). Let B also be a selfadjoint operator in H. If $D(A) \subset D(B)$, and for any $s \in \sigma_e(A)$ and any singular sequence (f_n) for A and s, (f_n) is also a singular sequence for B and s, then we say that *$\sigma_e(A)$ is contained in $\sigma_e(B)$ in the sense of singular sequences*, abbreviated $\sigma_e(A) \overset{S}{\subset} \sigma_e(B)$. Obviously, $\sigma_e(A) \overset{S}{\subset} \sigma_e(B)$ implies $\sigma_e(A) \subset \sigma_e(B)$. If $\sigma_e(A) \overset{S}{\subset} \sigma_e(B)$ and $\sigma_e(B) \overset{S}{\subset} \sigma_e(A)$, we say that *$A$ and B have the same singular sequences*, $\sigma_e(A) \overset{S}{=} \sigma_e(B)$.

The singular sequence problem was posed by K. JÖRGENS in connection with the work [3]. He gave this problem to W. TAFEL as the topic for his diploma thesis [5].

1. Statement of the problem

In order to give the first formulation of the problem for unbounded operators let us recall the statement of Weyl's theorem for unbounded operators (cf. [6], Satz 9.9): *Let A be selfadjoint, V symmetric and A-compact. Then $B = A + V$ is selfadjoint and $\sigma_e(A) \overset{S}{=} \sigma_e(B)$.* The following example shows that the problem for unbounded operators cannot simply be the question if the converse of the foregoing statement is true.

1.1. Example. Let $\dim H = \infty$, A selfadjoint with $\sigma_e(A) = \emptyset$, $B = 2A$. Then obviously $\sigma_e(A) \overset{S}{=} \sigma_e(B)$, but $B - A = A$ is not A-compact.

We remark that from the assumption that V is A-compact one also concludes that V is A-bounded with A-bound zero ([2], Corollary V.3.8, [6], Satz 9.7). We include this in our first formulation of the problem for unbounded operators.

1.2. Problem. Let A and B be selfadjoint operators, $\sigma_e(A) \overset{S}{=} \sigma_e(B)$, $V = B - A$ A-bounded with A-bound zero. Is it then possible to conclude that V is A-compact?

Let us note that V is A-compact (A-bounded with A-bound zero) if and only if V is B-compact (B-bounded with B-bound zero); therefore Problem 1.2 is symmetric with respect to A and B.

In our second formulation of the problem for unbounded operators we do not want to assume the A-boundedness with A-bound zero. Instead of the A-compactness we want to conclude a "local" compactness of V ("local" with respect to the spectral measure of A).

1.3. Definition. Let A be a selfadjoint operator, with spectral measure E. An operator V is called *A-locally compact* if $R(E(J))\subset D(V)$ and $VE(J)$ is compact for all compact intervals J.

Let us recall some known facts.

1.4. Theorem (cf. [6], Satz 9.8, Satz 9.11 b, c). *Let A be a selfadjoint operator.*

a) *An operator V is A-compact if and only if V is A-locally compact and A-bounded with A-bound zero.*

b) *An A-bounded operator V is A-locally compact if and only if V is A^p-compact for some (and then for all) $p>1$.*

1.5. Theorem (cf. [6], Satz 9.13). *Let A and B be selfadjoint operators, $D(A)=D(B)$. Let $V=B-A$ be A-locally compact. Then $\sigma_e(A) \overset{s}{=} \sigma_e(B)$.*

We conjecture that the converse of Theorem 1.5 is true.

1.6. Problem. Let A and B be selfadjoint operators, $\sigma_e(A) \overset{s}{=} \sigma_e(B)$. Is it then possible to conclude that $V=B-A$ is A-locally compact?

If the answer to Problem 1.6 is yes, then Theorem 1.4 shows that the answer to Problem 1.2 is also yes.

Also we remaık that for bounded operators A and B Problems 1.2 and 1.6 are just the problem formulated in the introduction.

Finally let us note that both Theorem 1.5 and Problem 1.6 are symmetric with respect to A and B. To see this it is sufficient to show: If A, B are selfadjoint operators, $D(A)=D(B)$, then $V=B-A$ is A-locally compact if and only if V is B-locally compact. This statement follows from [6], Satz 9.11 b, c and Satz 9.12.

2. A special case

In this section let A and B be selfadjoint operators in a Hilbert space H, with $D(A)\subset D(B)$. Let E be the spectral measure of A.

2.1. Lemma. *Let $s\in\mathbf{R}$ and $\varepsilon>0$. Assume that every singular sequence for A and s is also a singular sequence for B and s. Then there exist $\delta>0$ and a finite dimensional subspace M of H such that for all $f\in R(E((s-\delta, s+\delta)))\cap M^{\perp}$*[1]) *we have the inequality* $\|(B-A)f\|\leqq\varepsilon\|f\|$.

[1]) R denotes range.

Proof. We proceed by contradiction. So we can define inductively a sequence (f_n) in H, with the following properties:

$$f_n \in R\left[E\left(\left(s-\frac{1}{n},\ s+\frac{1}{n}\right)\right)\right] \cap \operatorname{span}\{f_1, \ldots, f_{n-1}\}^{\perp}, \quad \|f_n\|=1, \quad \|(B-A)f_n\| > \varepsilon,$$

for all $n \in \mathbf{N}$. Obviously (f_n) is a singular sequence for A and s, and therefore by assumption also a singular sequence for B and s. This implies $\|(B-A)f_n\| \leqq \leqq \|(B-s)f_n\| + \|(A-s)f_n\| \to 0$ $(n \to \infty)$, in contradiction to $\|(B-A)f_n\| > \varepsilon$ $(n \in \mathbf{N})$. □

2.2. Theorem. *Assume that for some compact interval J the set $\sigma(A) \cap J$ is countable and that every singular sequence for A and $s \in J$ is also a singular sequence for B and s. Then $(B-A)E(J)$ is compact.*

Proof. Let (f_n) be a sequence in H with $f_n \rightharpoonup 0$ and $\|f_n\| \leqq 1$ $(n \in \mathbf{N})$; we have to show $(B-A)E(J)f_n \to 0$.

Let $\varepsilon > 0$. Let $\sigma(A) \cap J = \{s_1, s_2, \ldots\}$. (We disregard the trivial case $\sigma(A) \cap \cap J = \emptyset$.) For s_j and $\varepsilon 2^{-j}$, $j \in \mathbf{N}$, we choose δ_j and M_j according to Lemma 2.1. Then $\sigma(A) \cap J \subset \bigcup_{j=1}^{\infty} J_j$, where $J_j := (s_j - \delta_j, s_j + \delta_j)$, and by the compactness of $\sigma(A) \cap J$ we find $m \in \mathbf{N}$ such that $\sigma(A) \cap J \subset \bigcup_{j=1}^{m} J_j$.

For $j = 1, \ldots, m$ define $K_j := J_j \setminus \bigcup_{i=1}^{j-1} J_i$. Then $\sigma(A) \cap J \subset \bigcup_{j=1}^{m} K_j$, and $K_1, \ldots \ldots, K_m$ are mutually disjoint. For $j = 1, \ldots, m$, denote by P_j, P_j' the orthogonal projections onto $R(E(K_j))$, $R(E(K_j)) \cap M_j^{\perp}$, and define $P_j'' = P_j - P_j'$. P_j'' is finite dimensional because $R(P_j'') = \overline{P_j M_j}$, and M_j has finite dimension. Now we decompose

$$E(J) = \sum_{j=1}^{m} P_j E(J) = \sum_{j=1}^{m} P_j' E(J) + P,$$

where $P = \sum_{j=1}^{m} P_j'' E(J)$ is finite dimensional and therefore compact. Also the assumptions imply that $(B-A)E(J)$ is a bounded operator, and so $(B-A)P = (B-A)E(J)P$ is compact. This implies

$$\begin{aligned} &\limsup \|(B-A)E(J)f_n\| \\ &\leqq \sum_{j=1}^{m} \limsup \|(B-A)P_j' E(J) f_n\| + \limsup \|(B-A)Pf_n\| \\ &\leqq \sum_{j=1}^{m} \limsup (\varepsilon 2^{-j}) \|P_j' E(J) f_n\| + 0 \leqq \varepsilon \sum_{j=1}^{m} 2^{-j} < \varepsilon. \end{aligned}$$

This shows $(B-A)E(J)f_n \to 0$ $(n \to \infty)$. □

2.3. Corollary. *Let* $\sigma_e(A) \overset{s}{\subset} \sigma_e(B)$, *and assume that* $\sigma(A)$ *is countable. Then* $B-A$ *is A-locally compact.*

Proof. By Theorem 2.2 $(B-A)E(J)$ is compact for each compact interval J. □

We note that Corollary 2.3 applies especially to the case that A has purely discrete spectrum, i.e., $\sigma_e(A)=\emptyset$.

3. An example

In this section we show by an example that in the general setting of Problem 1.6 the assumption $\sigma_e(A) \overset{s}{=} \sigma_e(B)$ cannot be replaced by $\sigma_e(A) \overset{s}{\subset} \sigma_e(B)$, as was done in the special case of Corollary 2.3.

3.1. Example. We are going to construct bounded selfadjoint operators A and V with the properties:

(i) V is not compact,

(ii) $\sigma_e(A) \overset{s}{\subset} \sigma_e(A+V)$,

(iii) $[0, 1]=\sigma_e(A)\neq\sigma_e(A+V)$.

Property (iii) shows that the example is not a counterexample to Problem 1.6.

We take the Hilbert space $H=L_2(0, 1)$, and as A we take the multiplication by the independent variable, $Af(x)=xf(x)$. The spectral measure of A is then given by $E(\Sigma)f=\chi_\Sigma\cdot f$ (Σ Borel set of $\mathbf{R}$). Also $\sigma_e(A)=\sigma(A)=[0, 1]$.

To construct V, we define the function ψ: $(0, \infty)\to\mathbf{R}$ by

$$\psi(x) = (-1)^m \quad \text{for} \quad m < x \leqq m+1, \quad m\in\mathbf{N}_0$$

($\mathbf{N}_0=\{0, 1, 2, \dots\}$), and we define $v_m\in L_2(0, 1)$ by $v_m(x):=\psi(2^m x)$ for $m\in\mathbf{N}_0$; clearly (v_m) is an orthonormal sequence. We define V to be the orthogonal projection onto the subspace spanned by $\{v_m;\ m\in\mathbf{N}_0\}$, i.e. $Vf=\sum_{m=0}^{\infty}\langle v_m, f\rangle v_m$. Now we show that (i), (ii), (iii) are valid.

(i) is obvious.

(ii) Let $s\in[0, 1]=\sigma_e(A)$, and let (f_n) be a singular sequence for A and s. We are done if we show $Vf_n\to 0$. Without restriction we may assume $\|f_n\|\leqq 1$. Let $\varepsilon>0$. There exist $m'\in\mathbf{N}_0, p, q\in\mathbf{Z}, p<q$, such that $s\in J:=(p/2^{m'}, q/2^{m'})$, $(q-p)/2^{m'}\leqq\varepsilon^2$. From $(A-s)f_n\to 0$ we obtain $(I-E(J))f_n\to 0$, and therefore $V(I-E(J))f_n\to 0$. Next, we define $v'_m:=E(J)v_m=\chi_J\cdot v_m$ $(m\in\mathbf{N}_0)$. It is easy to see from the definition of the v_m that $(v'_m)_{m\geqq m'}$ is an orthogonal sequence in $L_2(0, 1)$ with $0<\|v'_m\|^2\leqq\varepsilon^2$. In

$$VE(J)f = \sum_{m=0}^{\infty}\langle v_m, E(J)f\rangle v_m = \sum_{m=0}^{m'-1}\langle v'_m, f\rangle v_m + \sum_{m=m'}^{\infty}\langle v'_m, f\rangle v_m$$

we estimate

$$\left\|\sum_{m=m'}^{\infty}\langle v'_m, f\rangle v_m\right\|^2 = \sum_{m=m'}^{\infty}|\langle v'_m, f\rangle|^2 = \sum_{m=m'}^{\infty}\|v'_m\|^2|\langle v'_m/\|v'_m\|, f\rangle|^2 \leqq \varepsilon^2\|f\|^2.$$

This estimate together with $\langle v'_m, f_n\rangle \to 0$ $(n\to\infty)$ for all $m\in\mathbf{N}_0$ implies

$$\limsup\|VE(J)f_n\| \leqq \varepsilon,$$
$$\limsup\|Vf_n\| \leqq \limsup\|VE(J)f_n\| + \limsup\left\|V\big(I-E(J)\big)f_n\right\| \leqq \varepsilon+0.$$

This shows $Vf_n\to 0$.

(iii) Consider the sequence $(v_m)_{m\in\mathbf{N}_0}$. It is orthonormal, and therefore $v_m \rightharpoonup 0$. Also,

$$\langle v_m, (A+V)v_m\rangle = \langle v_m, Av_m\rangle + \langle v_m, Vv_m\rangle = \int_0^1 x\,dx + \|v_m\|^2 = 3/2.$$

Now the following lemma shows that there exists $s\in\sigma_e(A+V)$ with $s\geqq 3/2$.

3.2. Lemma. *Let A be a bounded selfadjoint operator, E its spectral measure. Let $s\in\mathbf{R}$. If there exists a sequence (f_n) in H with $f_n\rightharpoonup 0$, $\|f_n\|=1$ $(n\in\mathbf{N})$, such that* $\limsup\langle f_n, Af_n\rangle\geqq s$, *then* $\sigma_e(A)\cap[s,\infty)\neq\emptyset$.

Proof. If we assume $\sigma_e(A)\cap[s,\infty)=\emptyset$, then there exists $\varepsilon>0$ such that $E\big((s-\varepsilon,\infty)\big)$ is a finite dimensional projection. This would imply

$$\begin{aligned}&\limsup\langle f_n, Af_n\rangle\\ &\leqq \limsup\langle E\big((-\infty, s-\varepsilon]\big)f_n, Af_n\rangle + \limsup\langle E\big((s-\varepsilon,\infty)\big)f_n, Af_n\rangle\\ &\leqq (s-\varepsilon)\limsup\left\|E\big((-\infty, s-\varepsilon]\big)f_n\right\|^2 + 0 = s-\varepsilon,\end{aligned}$$

in contradiction with the assumption $\limsup\langle f_n, Af_n\rangle\geqq s$. □

References

[1] N. I. Achieser and I. M. Glasmann, *Theorie der linearen Operatoren im Hilbert-Raum*, 5. Aufl., Akademie-Verlag (Berlin, 1968).

[2] S. Goldberg, *Unbounded Linear Operators*, McGraw-Hill (New York, 1966).

[3] K. Jörgens and J. Weidmann, *Spectral Properties of Hamiltonian Operators*, Lecture Notes in Mathematics, vol. 313, Springer-Verlag (Berlin/Heidelgerg/New York, 1973).

[4] J. von Neumann, *Charakterisierung des Spektrums eines Integraloperators*, Actualités Sci. Ind., **229** (1953), 1—20.

[5] W. Tafel, *Kriterien für die (relative) Kompaktheit der Differenz zweier selbstadjungierter Operatoren*, Diplomarbeit (München, 1974) (unpublished).

[6] J. Weidmann, *Lineare Operatoren in Hilberträumen*, B. G. Teubner (Stuttgart, 1976).

[7] H. Weyl, Über beschränkte quadratische Formen, deren Differenz vollstetig ist, *Rend. Circ. Mat. Palermo* **27** (1909), 373—392.

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