Acta Sci. Math., 40 (1978), 347—369

The Taylor coefficients of certain infinite products

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In memory of Paul Turán

1. Let q be any positive fundamental discriminant; that is, squarefree and $\equiv 1 \pmod 4$ or $q=4d$ where d is squarefree and $\equiv 2$ or $3 \pmod 4$. Let $\chi(j)=\left(\frac{q}{j}\right)$ be the Kronecker symbol. Let us define C_m, $m=0,1,2,\ldots$ by

$$(1.0)\qquad \sum_{m=0}^{\infty} C_m t^m = \prod_{n=0}^{\infty}\prod_{j=1}^{q-1}(1-t^{qn+j})^{-\zeta\chi(j)} = F(t),$$

where ζ is either 1 or -1. Note that when $q=5$ and $\zeta=1$, the infinite product is

$$\frac{(1-t^2)(1-t^7)\ldots(1-t^3)(1-t^8)\ldots}{(1-t)(1-t^6)\ldots(1-t^4)(1-t^9)\ldots} = 1+\frac{t}{1+}\frac{t^2}{1+\ldots};$$

that is, $F(t)$ is Ramanujan's continued fraction [6; p. 294]. If $q=8$ and $\zeta=1$, $F(t)$ has a similar continued fraction representation

$$\frac{(1-t^3)(1-t^{11})\ldots(1-t^5)(1-t^{13})\ldots}{(1-t)(1-t^9)\ldots(1-t^7)(1-t^{15})\ldots} = 1+t+\frac{t^2}{1+t^3+}\frac{t^4}{1+t^5+}\frac{t^6}{1+t^7+\ldots}.$$

This representation is due to Basil Gordon [5], but there are indications, according to Gordon, that it might have been known to Ramanujan.

In this paper we shall determine the asymptotic behaviour of the coefficients C_m, first by the saddle point method using a transformation due to Iseki [7], and then more precisely by the circle method of Hardy and Ramanujan as modified by Rademacher, to obtain a convergent series representation of the C_m. Throughout the paper we make extensive use of results of Iseki [8]. The exact formula for C_m is given by equation (4.14) in Theorem 4.1. The asymptotic formula (3.9) shows the interesting fact that if the product is turned upside down; that is, if the sign of ζ is reversed, the coefficients have the same asymptotic behaviour in the sense that they

Received September 12, 1977.

oscillate with the same amplitude and a common period of oscillation. In the classical case of Ramanujan the amplitude is $(5m)^{-3/4}\exp\left(\frac{4\pi}{5\sqrt{5}}\sqrt{m}\right)$ and the oscillation is a pure cosine wave of the form $\cos\left(\frac{2\pi}{5}\left(m-\frac{2}{5}\right)\right)$. In the case of Gordon's continued fraction the oscillating part of the asymptotic term has the form $\cos\frac{(m-1)\pi}{4}$, hence vanishes for $m\equiv 3 \pmod 4$. This suggests that $C_{4k+3}=0$, for all $k\geqq 0$ and we shall be able to verify this by means of the exact series. If the product is turned upside down $(q=8, \zeta=-1)$ then we shall find similarly that $C_{4k+2}=0$ for $k\geqq 0$.

We require the following results concerning the Kronecker symbol:

$$\chi(q-j)=\chi(j) \tag{1.1}$$

$$\sum_{j=1}^{q}\chi(j)=\sum_{j=1}^{q}j\chi(j)=0 \tag{1.2}$$

$$\sum_{j=1}^{q}\chi(j)\exp\left(2\pi i\frac{n}{q}j\right)=\sqrt{q}\chi(n). \tag{1.3}$$

These results are found for example in LANDAU [11]. Equation (1.3) is Theorem 215 of [11]. We shall often use without mention that $\chi(j)=0$ if and only if $(q, j)>1$ and that $\chi(mn)=\chi(m)\chi(n)$.

We also note that

$$\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2}\cos 2n\pi\lambda=((\lambda))^2-\frac{1}{12} \tag{1.4}$$

where, for any real λ, $((\lambda))=\lambda-[\lambda]-\frac{1}{2}$ (see [9], formula 573)[1]). Hence by (1.2) and (1.3)

$$\begin{aligned}\frac{\sqrt{q}}{\pi^2}\sum_{n=0}^{\infty}\frac{\chi(n)}{n^2}&=\frac{1}{\pi^2}\sum_{n=1}^{\infty}\sum_{j=1}^{q-1}\frac{1}{n^2}\chi(j)\exp\left(2\pi i\frac{n}{q}j\right)=\\&=\sum_{j=1}^{q-1}\chi(j)\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2}\cos\left(2\pi\frac{n}{q}j\right)=Q/q^2\end{aligned} \tag{1.5}$$

where

$$Q=\sum_{j=1}^{q-1}j^2\chi(j). \tag{1.6}$$

[1]) Note that when λ is an integer then $((\lambda))=-1/2$ which is not the usual convention.

From (1.3) and (1.5) we obtain for any integer m

$$\sum_{j=1}^{q-1} \chi(j) \sum_{n=1}^{\infty} \frac{1}{\pi^2 n^2} \cos\left(2\pi \frac{n}{q} mj\right) = \chi(m) Q/q^2. \tag{1.7}$$

Formula (1.5) shows incidentally that $Q>0$ since $\sum_{n=1}^{\infty} \frac{\chi(n)}{n^2} > 1 - \sum_{n=2}^{\infty} \frac{1}{n^2} > 0.$

2. In this section we derive a transformation equation for the generating function $F(t)$. The transformation is obtained from a formula of ISEKI and we largely follow his notation in [7] and [8]. Let z be a complex number with $\mathscr{R}z>0$ and h, k be co-prime positive integers with $k>1$. Let D and $K=kq/D$ denote the g.c.d. and l.c.m. of k and q respectively. Put $k=k_1 D$, $q=q_1 D$ so that $(k_1, q_1)=1$ and $K=k_1 q=kq_1$. Choose integers γ, δ satisfying

$$\gamma k_1 - \delta q_1 = 1. \tag{2.0}$$

Let H be any solution of

$$hH \equiv \delta \pmod{k}. \tag{2.1}$$

Set

$$x = \exp\left(2\pi i \frac{h}{k} - 2\pi \frac{z}{k}\right), \quad \tilde{x} = \exp\left(2\pi i \frac{H}{k} - \frac{2\pi}{Kz}\right),$$

and for $1 \leqq a < q$,

$$F_a(\tilde{x}; b, D, \varrho) = \prod_{m=0}^{\infty} (1-\varrho \tilde{x}^{Dm+b})^{-\zeta\chi(a)} (1-\bar{\varrho}\tilde{x}^{Dm+d-b})^{-\zeta\chi(a)} \tag{2.2}$$

where

$$b = ha - D\left[\frac{ha}{D}\right] = \{ha\}_D = D\left(\left(\left(\frac{ha}{D}\right)\right) + \frac{1}{2}\right), \quad \varrho = \varrho_a = \exp\left(-2\pi i a \frac{\gamma}{q}\right). \tag{2.3}$$

The notation $\{x\}_r$ will be used to denote the reduced residue of the integer x modulo r, that is $0 \leqq \{x\}_r < r$, $x \equiv \{x\}_r \pmod{r}$.

Finally let

$$\sigma_a(h, k) = \sum_{\mu \equiv a \pmod{q}}^{(K)} \left(\left(\frac{\mu}{K}\right)\right) \left(\left(\frac{h\mu}{k}\right)\right) \tag{2.4}$$

where $\sum^{(K)}$ signifies that μ runs through a complete set of residues modulo $K=k_1 q$, subject to the condition $\mu \equiv a \pmod{q}$. In particular

$$\sigma_a(k, q) = \left(\left(\frac{a}{q}\right)\right) \left(\left(\frac{ah}{q}\right)\right). \tag{2.5}$$

We also note that for $1 \leqq a < q$, $(a, q)=1$, $D>1$

$$\sigma_{q-a}(h, k) = \sum_{\mu \equiv -a \pmod{q}}^{(K)} \left(\left(\frac{\mu}{K}\right)\right) \left(\left(\frac{h\mu}{k}\right)\right) = \sigma_a(h, k) \tag{2.6}$$

since under the conditions μ is not divisible by D, hence neither $\frac{\mu}{K}$ nor $\frac{h\mu}{k}$ are integers, and $((-x))=-((x))$ for non-integer x. If $D=1$, $(k,q)=1$ then there is a unique $\mu_a\equiv a \pmod q$ such that $\mu_a\equiv 0 \pmod k$ and we obtain, noting that for integer x, $((x))=-1/2$,

$$\sigma_{q-a}(h,k)=\sigma_a(h,k)+\frac{\mu_a}{K}. \tag{2.6'}$$

In the following $\sum_a'$, $\prod_a'$ denote sums and products over $a=1,2,\dots,\left[\frac{q}{2}\right]$.

Theorem 2.1. *Let* $\omega^*(h,k)=\exp\{2\pi i\zeta\sigma^*(h,k)\}$ *where*

$$\sigma^*(h,k)=\sum_a{}'\chi(a)\sigma_a(h,k). \tag{2.7}$$

Then

$$F(x)=\omega^*(h,k)\exp\left\{\frac{\zeta\pi Q}{2kq}\left(\frac{D^2}{q^2}\chi\left(\frac{qh}{D}\right)z^{-1}-z\right)\right\}\times\prod_a{}'F_a(\tilde{x};b,D,\varrho).$$

Proof. From Theorem 1 of [8] we obtain that

$$\prod_{m=0}^{\infty}(1-x^{qm+a})^{-\zeta\chi(a)}(1-x^{qm+q-a})^{-\zeta\chi(a)}= \tag{2.8}$$

$$=\omega_a(h,k)\exp\left\{\frac{\pi\zeta\chi(a)}{6qk}(Bz^{-1}-Az)\right\}\times F_a(\tilde{x};b,D,\varrho)$$

where $\omega_a(h,k)=\exp\{2\pi i\zeta\chi(a)\sigma_a(h,k)\}$ and

$$A=6a^2-6qa+q^2,\quad B=6b^2-6Db+D^2=6D^2\left(\left(\left(\frac{ha}{D}\right)\right)^2-\frac{1}{12}\right) \tag{2.9}$$

by (2.3). It follows at once that

$$\prod_a{}'\omega_a(h,k)=\omega^*(h,k) \tag{2.10}$$

Next we show that

$$\sum_a{}'\chi(a)A=3\sum_{j=1}^{q-1}j^2\chi(j)=3Q. \tag{2.11}$$

Using equation (1.1)

$$\sum_a{}'\chi(a)A=\sum_a{}'\chi(a)\{6a^2-6qa+q^2\}=\sum_a{}'\chi(a)\{6(q-a)^2-6q(q-a)+q^2\}=$$

$$=\frac{1}{2}\sum_{j=1}^{q-1}\chi(j)\{6j^2-6jq+q^2\}=3\sum_{j=1}^{q-1}j^2\chi(j)$$

by equation (1.2) which proves (2.11):

Finally we prove

$$\sum_a{}' \chi(a) B = 3\,\frac{QD^2}{q^2}\,\chi\left(\frac{hq}{D}\right). \tag{2.12}$$

By definition

$$\sum_a{}' \chi(a) B = \sum_a{}' \chi(a)\{6b^2-6Db+D^2\} =$$

$$= 6D^2 \sum_a{}' \chi(a) \left\{\left(\left(\frac{ha}{D}\right)\right)^2 - \frac{1}{12}\right\} = 3D^2 \sum_{j=1}^{q-1} \chi(j) \left\{\left(\left(\frac{hj}{D}\right)\right)^2 - \frac{1}{12}\right\}$$

since if $D=1$, the last two expressions are 0 by (1.1) and (1.2), and if $D>1$ then

$$\left(\left(\frac{h(q-a)}{D}\right)\right) = \left(\left(-\frac{ha}{D}\right)\right) = -\left(\left(\frac{ha}{D}\right)\right) \tag{2.13}$$

when $D\nmid a$ and $\chi(a)=0$ otherwise. Hence the value of B is unchanged by substituting $q-a$ for a. It follows by equations (1.4) and (1.7) that

$$\sum_a{}' \chi(a) B = 3D^2 \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} \sum_{j=1}^{q-1} \chi(j) \cos\left(2\pi\frac{nh}{D} j\right) = \frac{3QD^2}{q^2}\,\chi\left(\frac{hq}{D}\right).$$

Since the product of the left side of equation (2.8) taken over $a=1, 2, \ldots, \left[\frac{q}{2}\right]$ is $F(x)$, the theorem follows at once from (2.8), (2.10), (2.11), and (2.12).

Lemma 2.1. *If $D=q$ and $\chi(h)=\zeta$ then*

$$\prod_a{}' F_a(\tilde{x}; b, q, \varrho) = F(\tilde{x})^{\zeta}.$$

Proof. Because $D=q$, we can take $\gamma=0$, $\delta=-1$ in (2.0), hence $\varrho=1$ and

$$\prod_a{}' F_a(\tilde{x}; b, q, \varrho) = \prod_a{}' \prod_{m=0}^{\infty} [(1-\tilde{x}^{qm+b})(1-\tilde{x}^{qm+q-b})]^{-\zeta\chi(a)}. \tag{2.14}$$

Now $b=ha-q\left[\frac{ha}{q}\right]$, $\chi(b)=\chi(ha)=\zeta\chi(a)$, and $0\leqq b<q$. Hence the exponent in (2.12) is $-\zeta\chi(a)=-\chi(b)$. Since $1=(h,k)=(h,qk_1)$, we have $(h,q)=1$. Thus as a runs through a reduced residue system mod q, so does ah, hence b. The lemma now follows from (2.14) and the definition (1.0) of $F(t)$.

Next we derive an alternative expression for the generalized Dedekind sum $\sigma^*(h,k)$ and hence for $\omega^*(h,k)$. Following Iseki we define integers f, g as follows:

$$\begin{array}{llll} f=12, & g=1 \text{ for } (k,6)=1; & f=3, & g=4 \text{ for } (k,6)=2; \\ f=4, & g=3 \text{ for } (k,6)=3; & f=1, & g=12 \text{ for } (k,6)=6. \end{array} \tag{2.15}$$

In all cases $fg=12$ and $(f,k)=1$. Thus

$$(h,k)=1 \Leftrightarrow (h,gDk)=1, \tag{2.16}$$

$$(f,gDk)=1. \tag{2.17}$$

We define integers φ and ψ to be any solution of

$$f\varphi+gDk\psi=1 \tag{2.18}$$

and choose the solution H of (2.1) so that

$$hH\equiv\delta \pmod{gDk}. \tag{2.19}$$

Set $K_1=\frac{1}{2}k_1(k_1-1)$, $K_2=\frac{1}{6}k_1(k_1-1)(2k_1-1)$,

$$U_a=gD\gamma(aK_1+qK_2)-\varphi\delta(2K^2+3K(2a-q)+A),$$

$$V_a=\varphi(k^2-B),$$

$$W_a=-\frac{\gamma k_1}{4D}(k+2b-2D)+\left(\frac{a\gamma}{2Dq}+\frac{3\varphi\delta q_1}{gD}\right)(2b-D)-g\psi\delta\left(\frac{1}{2}a+\frac{1}{2}q+\frac{1}{2}\left(b-\frac{1}{2}\right)q_1\right),$$

where A, B are defined in (2.9). It is shown in [8, p. 947] that

$$\sigma_a(h,k)=\frac{1}{gDk}(U_ah+V_aH)+W_a \pmod 1. \tag{2.20}$$

Now it follows from (2.6), (2.6′) and (2.7) that

$$\begin{aligned}\sigma^*(h,k)=\sum_a{}'\chi(a)\sigma_a(h,k)&=\frac{1}{2}\sum_{j=1}^{q-1}\chi(j)\sigma_j(h,k) \quad \text{if } D>1\\ &=\frac{1}{2}\sum_{j=1}^{q-1}\chi(j)\sigma_j(h,k)-\frac{1}{2}\sum_a{}'\chi(a)\frac{\mu_a}{kq} \quad \text{if } D=1\end{aligned} \tag{2.21}$$

where $\mu_a=a+\nu_aq\equiv 0 \pmod k$, $0\leqq\nu_a<k$. Writing $\mu_a=a+\nu_aq=r_ak$, $1\leqq r_a<q$ and noting that $r_a\equiv ak^{-1} \pmod q$, we can rewrite the expression $\sum_a{}'\chi(a)\frac{\mu_a}{kq}$ in the form $\frac{\chi(k)}{q}\sum_{\substack{r=1\\ 1\leqq\{kr\}\leqq q/2}}^{q-1} r\chi(r)$, and we obtain for $D=(k,q)=1$

$$2\sigma^*(h,k)=\sum_{j=1}^{q-1}\chi(j)\sigma_j(h,k)-\frac{1}{q}\chi(k)\sum_{\substack{r=1\\ 1\leqq\{kr\}_q<q/2}}^{q-1} r\chi(r). \tag{2.21′}$$

Now substitute for $\sigma_j(h,k)$ from (2.20) into (2.21), (2.21′). Making use of (2.11), (2.12), (1.2) and

$$\sum_{j=1}^{q-1}\chi(j)b=\chi(h)\sum_{j=1}^{q-1}\chi(hj)\{hj\}_q=\chi(h)\sum_{b=1}^{q-1}b\chi(b)=0,$$

we obtain

$$\sum_{j=1}^{q-1}\chi(j)U_j=-\varphi\delta\sum_{j=1}^{q-1}A\chi(j)=-6\varphi\delta Q,$$

$$\sum_{j=1}^{q-1}\chi(j)V_j=-6\varphi Q\frac{D^2}{q^2}\chi\left(\frac{hq}{D}\right),$$

$$\sum_{j=1}^{q-1}\chi(j)W_j=\frac{\gamma}{Dq}\sum_{j=1}^{q-1}\chi(j)j\{hj\}_q,$$

hence

$$\begin{aligned}(2.22)\qquad 2\sigma^*(h,k)\equiv -\frac{6\varphi\delta Q}{gDk}h-\frac{6\varphi Q}{gDk}\frac{D^2}{q^2}\chi\left(\frac{hq}{D}\right)H+\frac{\gamma}{Dq}\sum_{j=1}^{q-1}\chi(j)j\{hj\}_q-\\ -\frac{1}{q}\chi(k)\delta_{D,1}\sum_{\substack{r=1\\ 1\le\{kr\}_q<q/2}}^{q-1}r\chi(r)\pmod 1\end{aligned}$$

where $\delta_{D,1}=0$ if $D>1$, 1 if $D=1$.

We apply formula (2.22) to the case when $D=q$, $\chi(h)=\zeta$.

Lemma 2.2. *Let $\omega^*(h,k)$ be as in Theorem 2.1 and suppose that $D=q, k=qk_1$, $\chi(h)=\zeta$. Let h^* be any solution of*

$$(2.23)\qquad hh^*\equiv 1\pmod{qk}.$$

Then

$$(2.24)\qquad \omega^*(h,k)=\mu(h,k)\exp\left\{\pi i(\zeta h+h^*)\frac{Q}{2qk}\right\}$$

where $\mu(h,k)=+1$ or -1. In particular if $q\equiv 1\pmod 4$ and $k=q$, $k_1=1$ then

$$(2.25)\qquad \mu(h,q)=\chi(h)=\left(\frac{h}{q}\right)\ \textit{(the Legendre symbol) if } q \textit{ is prime}$$

$$=1\ \textit{if } q \textit{ is composite.}$$

Proof. We first note that the value of the expression in (2.24) is independent of the solution h^* in (2.23). We have to verify that

$$(2.26)\qquad Q\equiv 0\pmod 4$$

Indeed $Q=\sum_{j=1}^{q-1}j^2\chi(j)\equiv\sum_{j\equiv 1\ (\mathrm{mod}\ 2)}\chi(j)\pmod 4$ and $\sum_{j\equiv 1\ (\mathrm{mod}\ 2)}\chi(j)=0$, trivially from (1.2) if q is even, and from $\sum_{j\equiv 0\ (\mathrm{mod}\ 2)}\chi(j)=\chi(2)\sum_{j=1}^{[q/2]}\chi(j)=0$ if $q\equiv 1\pmod 4$.

Now if $D=q$, $q_1=1$ we can take $\gamma=0$, $\delta=-1$ in (2.0), $h^*=-H$ in (2.23), and (2.22) simplifies to

$$2\sigma^*(h,k_1q)\equiv\frac{6\varphi Q}{gq^2k_1}(h+\zeta h^*)=\frac{6f\varphi Q}{fgq^2k_1}(h+\zeta h^*)\equiv$$

$$\equiv\frac{Q}{2q^2k_1}(h+\zeta h^*)\pmod 1\ \text{by (2.18)}.$$

From here and from $\omega^*(h, k_1 q)=\exp\{2\pi i\zeta\sigma^*(h, k_1 q)\}$ equation (2.24) follows at once.

The sign of $\mu(h, k)$ in (2.24) depends on whether

$$2\sigma^*(h, k_1 q)-\frac{Q}{2q^2 k_1}(h+\zeta h^*) \tag{2.27}$$

is an even or an odd integer. We want to show that if $q\equiv 1 \pmod 4$ then $\mu(h, q)= =\chi(h)=\left(\frac{q}{h}\right)=\left(\frac{h}{q}\right)$ if q is prime, 1 if q is composite.

From (2.4) and (2.21)

(2.28)

$$2\sigma^*(h, k_1 q) = \frac{1}{k}\sum_{r=0}^{k_1-1}\sum_{j=1}^{q-1}\chi(j)\left(\frac{j+rq}{k}-\frac{1}{2}\right)\left(\{h(j+rq)\}_k-\frac{1}{2}k\right) = \frac{1}{k^2}\sum_{j=1}^{k-1} j\chi(j)\{hj\}_k$$

since

$$\sum_{j=1}^{k-1}\chi(j)\{hj\}_k = \chi(h)\sum_{j=1}^{k-1}\chi(hj)\{hj\}_k = \sum_{r=0}^{k_1-1}\sum_{j=1}^{q-1}\chi(j)(rq+j) = 0.$$

Comparing (2.27) and (2.28) we find that

$$\frac{1}{k_1}\sum_{j=1}^{k-1}\chi(j)j\{hj\}_k = \frac{1}{2}Q(h+\zeta h^*)+M(h, k)qk \tag{2.29}$$

for some integer $M(h, k)$. Clearly

$$\mu(h, k) = (-1)^{M(h,k)} \tag{2.30}$$

in (2.24).

Now suppose that $k=q\equiv 1 \pmod 4$. Then by (2.26) $\frac{1}{2}Q(h+\zeta h^*)\equiv 0 \pmod 2$ and hence

$$M(h, q) \equiv \sum_{j=1}^{q-1}\chi(j)j\{hj\}_q \equiv \sum_{\substack{(j,q)=1 \\ j\equiv 1 \,(\mathrm{mod}\, 2)}}^{(q)} \{hj\}_q \pmod 2.$$

Equation (2.25) follows from here, (2.28) and from

Lemma 2.3. *Let q be odd and squarefree, $(h, q)=1$. Set $v=v(h, q)= =\sum_{\substack{(j,q)=1 \\ j\equiv 1 \,(\mathrm{mod}\, 2)}}^{(q)} \{hj\}_q$. Then, $(-1)^{v(h,q)}$ is equal to $\left(\frac{h}{q}\right)$ if q is prime, and to 1 if q is composite.*

Proof. The first half of the lemma is a trivial corollary of Gauss' lemma, (see e.g. BACHMANN [3], p. 266) but we give a direct proof. Each hj, j odd, has a unique odd residue m_j in the interval $-q<m_j<q$, and $m_i=m_j$ if and only if $i=j$ since $ih\equiv -jh \pmod q$, $(h, q)=1$ implies $i+j\equiv 0 \pmod q$ which is impossible since $i+j$ is even and less than $2q$.

Let λ be the number of negative ones among the m_j so that $\nu(h, q) \equiv \equiv \lambda \pmod 2$. Set

$$P = \Pi \{j | j = 1, \ldots, q-1, (j, q) = 1, j \equiv 1 \pmod 2\}.$$

Then

$$\prod_j m_j = (-1)^\lambda p \equiv h^{\frac{1}{2}\varphi(q)} p \pmod q$$

since exactly one of j, $q-j$ is odd hence exactly half of the relatively prime (to q) reduced residues modulo q are odd. Now if q is prime then $h^{1/2\varphi(q)} = h^{1/2(q-1)} \equiv \equiv \left(\frac{h}{q}\right) \pmod q$, giving $(-1)^{\nu(h,q)} = (-1)^\lambda = \left(\frac{h}{q}\right)$. If q is composite and square-free, $q = q_1 \ldots q_r$, $r > 1$ then $\frac{1}{2}\varphi(q)$ is a multiple of $\varphi(q_i)$, $i = 1, \ldots, r$, hence $h^{1/2\varphi(q)} \equiv 1 \pmod{q_i}$, $i = 1, \ldots, r$, $h^{\frac{1}{2}\varphi(q)} \equiv 1 \pmod q$, giving $(-1)^{\nu(h,q)} = 1$.

In the case of even q no simple interpretation of $\mu(h, q)$ has been found. As the case $q = 8$, $Q = 16$ is of special interest, we show:

Lemma 2.4. *Let* $q = 8$, $k_1 \geqq 1$, $(h, 2k_1) = 1$, $\chi(h) = \zeta$. *Then*

$$\mu(h + 2k_1, 8k_1) = \mu(h, 8k_1) \quad \textit{if} \quad k_1 \quad \textit{is odd and} \quad hk_1 \equiv 3 \pmod 4,$$

$$\mu(4k_1 - h, 8k_1) = \mu(h, 8k_1) \quad \textit{if} \quad k_1 \quad \textit{is even.}$$

Proof. Throughout the proof $\chi(j)$ will denote the Kronecker character modulo 8 i.e. $\chi(j) = 1$ for $j \equiv \pm 1 \pmod 8$, $\chi(j) = -1$ for $j \equiv \pm 3 \pmod 8$, $\chi(j) = 0$ for j even. Equation (2.29) now has the form

$$\frac{1}{k_1} \sum_{j=1}^{k-1} \chi(j) j \{hj\}_k = 8(h + \zeta h^*) + 64 k_1 M(h, k), \quad k = 8k_1, \tag{2.31}$$

$hh^* \equiv 1 \pmod{64\, k_1}$.

Suppose first that k_1 is odd and $k_1 h \equiv 3 \pmod 4$. Then $\chi(h + 2k_1) = \chi(h)$ since $h + (h + 2k_1) = 2(h + k_1) \equiv 0 \pmod 8$, and

$$(h + 2k_1)^* \equiv h^* + 2k_1 \pmod{16} \tag{2.32}$$

as seen from $(h + 2k_1)(h^* + 2k_1) \equiv hh^* + 2k_1(h + h^*) + 4 \equiv 1 \pmod{16}$. Hence

$$8\big(h + 2k_1 + \zeta (h + 2k_1)^*\big) - 8(h + \zeta h^*) \equiv 16 k_1 (1 + \zeta) \pmod{128}. \tag{2.33}$$

Furthermore, writing for the moment j' for $\{hj\}_k$, it is easily seen that

$$\frac{1}{k_1} \sum_{j=1}^{k-1} \chi(j) j \big(\{(h + 2k_1) j\}_k - \{hj\}_k\big) =$$
$$= 2 \sum_{\substack{j \equiv 1 \pmod 4 \\ 0 < j' < 6k_1}} j\chi(j) - 6 \sum_{\substack{(j \equiv 1 \bmod 4) \\ 6k_1 < j' < k_1}} j\chi(j) + 6 \sum_{\substack{j \equiv 3 \pmod 4 \\ 0 < j' < 2k_1}} j\chi(j) - 2 \sum_{\substack{j \equiv 3 \pmod 4 \\ 2k_1 < j' < 8k_1}} j\chi(j) \tag{2.34}$$

where all summations go from $j=1$ to $j=k-1$. For instance

$$\{(h+2k_1)j\}_k = j'+2k_1 \quad \text{if} \quad j \equiv 1 \pmod 4 \quad \text{and} \quad 0 < j'+2k_1 < 8k_1,$$
$$j'-6k_1 \quad \text{if} \quad j \equiv 1 \pmod 4 \quad \text{and} \quad j'+2k_1 > 8k_1, \quad \text{etc.}$$

Now $\sum_{j\equiv 1 \pmod 4} j\chi(j) = -4k_1$, $\sum_{j\equiv 3 \pmod 4} j\chi(j) = 4k_1$, hence the expression in (2.34) is

$$-16k_1 - 8\Big(\sum_{\substack{j\equiv 1 \pmod 4 \\ 6k_1<j'<8k_1}} j\chi(j) - \sum_{\substack{j\equiv 3 \pmod 4 \\ 0<j'<2k_1}} j\chi(j)\Big) =$$
$$= -16k_1 - 16 \sum_{\substack{j\equiv 1 \pmod 4 \\ 6k_1<j'<8k_1}} j\chi(j) + 64k_1 \sum_{\substack{j\equiv 1 \pmod 4 \\ 6k_1<j'<8k_1}} \chi(j)$$

and we have to show, by (2.31) and (2.33), if we denote by S_h the set of residues j for which $6k_1 < \{hj\}_k < 8k_1$, that

$$k_1(2+\zeta) + \sum_{\substack{j\equiv 1 \pmod 4 \\ j\in S_h}} j\chi(j) \equiv 4 \sum_{\substack{j\equiv 1 \pmod 4 \\ j\in S_h}} \chi(j) \pmod 8,$$

or

$$3 \sum_{\substack{j\equiv 1 \pmod 8 \\ j\in S_h}} + \sum_{\substack{j\equiv 5 \pmod 8 \\ j\in S_h}} \equiv k_1(2+\zeta) \pmod 8 \tag{2.35}$$

provided that $hk_1 \equiv 3 \pmod 4$.

Now if $h\equiv 1 \pmod 8$, $k_1 \equiv 3 \pmod 4$ and $(h, 2k_1)=1$, the elements $hj, j\equiv$ $\equiv 1 \pmod 8$, $j\in S_h$ are exactly the elements $\equiv 1 \pmod 8$ between $6k_1$ and $8k_1$, namely $6k_1+7, 6k_1+15, \dots, 8k_1-7$, hence their total number is $\frac{1}{4}(k_1-3)$. Similarly the elements hj, $j\equiv 5 \pmod 8$, $j\in S_h$ are $6k_1+3, 6k_1+11, \dots, 8k_1-3$, and their total number is $\frac{1}{4}(k_1+1)$. Hence the left hand side of (2.33) is $\frac{3}{4}(k_1-3)+$ $+\frac{1}{4}(k_1+1)=k_1-2$ which is $\equiv 3k_1 \pmod 8$ since $k_1\equiv 3 \pmod 4$.

If $h\equiv 3 \pmod 8$, $k_1\equiv 1 \pmod 8$, the elements hj, $j\equiv 1 \pmod 8$, $j\in S_h$ are $6k_1+5, 6k_1+13, \dots, 8k_1-5$, and the elements hj with $j\equiv 5 \pmod 8$, $j\in S_h$ are $6k_1+1, \dots, 8k_1-1$. Hence we get, by counting their respective numbers, $\frac{3}{4}(k_1-1)+$ $+\frac{1}{4}(k_1+3)=k_1$ for both sides of (2.35).

A similar count for $h\equiv 5 \pmod 8$, $k_1\equiv 3 \pmod 4$ gives $\frac{3}{4}(k_1+1)+\frac{1}{4}(k_1-3)=k_1$ and for $h\equiv 7 \pmod 8$, $k_1\equiv 1 \pmod 4$, $\frac{3}{4}(k_1+3)+\frac{1}{4}(k_1-1)=k_1+2$ for the left hand side of (2.35), which agrees with the right hand side in each case. Thus the first half of the Lemma is proved.

Suppose next that k_1 is even. Then

$$(4k_1-h)^* \equiv 12k_1 - h^* \pmod{16k_1} \tag{2.36}$$

as seen from $(4k_1-h)(12k_1-h^*)\equiv hh^*-4k_1(3h+h^*)\equiv 1 \pmod{16k_1}$. Hence

$$8\big(4k_1-h+\zeta(4k_1-h)^*\big) \equiv -8(h+\zeta h^*) \pmod{128k_1} \quad \text{if} \quad \zeta = 1$$
$$\equiv -8(h+\zeta h^*)+64k_1 \pmod{128k_1} \quad \text{if} \quad \zeta = -1.$$

Furthermore, if we denote by R_h the subset of odd residues $\{1, 3, \ldots, k-1\}$ for which $\{hj\}_k > 4k_1$,

$$\frac{1}{k_1}\sum_{j=1}^{k-1} \chi(j) j(\{hj\}_k + \{(4k_1-h)j\}_k) = 8 \sum_{j\in R_h} j\chi(j)$$

since $\{hj\}_k + \{(4k_1-h)j\}_k$ is equal to $4k_1$ if $\{hj\}_k < 4k_1$, and to $12k_1$ if $4k_1 < \{hj\}_k < 8k_1 = k$, and since $4\sum_{j=1}^{k-1} j\chi(j) = 0$. To prove the second half of Lemma 2.4 we must therefore show that

$$\sum_{j\in R_h} j\chi(j) \equiv 0 \pmod{16k_1} \quad \text{if} \quad h \equiv \pm 1 \pmod 8$$
$$\equiv 8k_1 \pmod{16k_1} \quad \text{if} \quad h \equiv \pm 3 \pmod 8.$$

Now $0 < j < 4k_1$, $\{hj\}_k > 4k_1 \Rightarrow \{h(4k_1-j)\}_k = 12k_1 - \{hj\}_k > 4k_1$, hence both j and $4k_1 - j$ are in R_h and $j\chi(j) + (4k_1-j)\chi(4k_1-j) = 4k_1\chi(j)$. Similarly $4k_1 < j < 8k_1$ $\{hj\}_k > 4k_1 \Rightarrow \{h(12k_1-j)\}_k = 12k_1 - \{hj\}_k > 4k_1$, and $j\chi(j) + (12k_1-j) = \chi(12k_1-j) = 12k_1\chi(j)$. Hence

$$\sum_{j\in R_h} j\chi(j) = 2k_1 \sum_{\substack{j<4k_1\\ j\in R_h}} \chi(j) + 6k_1 \sum_{\substack{4k_1<j<8k_1\\ j\in R_h}} \chi(j).$$

But $\{h(k-j)\}_k = k - \{hj\}_k$ therefore exactly one of j, $k-j$ is in R_h and since $\chi(j) = \chi(k-j)$, we conclude that among the residues j in R_h exactly half have $\chi(j) = \pm 1$. Hence $\sum_{j\in R_h} \chi(j) = 0$ and we are finished with the proof if we can show that

$$\sum_{\substack{j=1\\ j\in R_h}}^{k_1-1} \chi(j) \equiv 1 - \chi(h) \pmod 4,$$

or, since for $0 < j < 2k_1$, $2k_1 - j \in R_h \Leftrightarrow 2k_1 + j \in R_h$ (the condition for both is $4k_1 < \{hj\}_k < 6k_1$),

$$\sum_{\substack{j=1\\ j\in R_h}}^{2k_1-1} \chi(j) \equiv \frac{1}{2}(1-\chi(h)) \pmod 2.$$

But $\chi(j) \equiv 1 \pmod 2$ hence the last condition is equivalent to

$$\sum_{\substack{j=1\\ j\in R_h}}^{2k_1-1} \equiv \frac{1}{2}(1-\chi(h)) \pmod 2$$

and this again is equivalent to

$$\sum_{\substack{j=1\\ j\in R_h}}^{4k_1-1} 1 \equiv 1 - \chi(h) \pmod 4. \tag{2.37}$$

We formulate the statement in congruence (2.37) as a separate lemma as it has some interest of its own.

Lemma 2.5. *Let* $k=16k'$, $0<h<k$, $(h;k)=1$. *Consider the set* $T_h=\{jh|j=1,3,\dots,8k'-1\}$ *of the first* $4k'$ *odd multiples of* h, *and denote by* N_h *the number of those members of* T_h *whose reduced residues modulo* k *are in the top half of the interval* $(0,k)$, *i.e.* $8k'<\{hj\}_k<16k'$. *Then*

$$N_h \equiv \begin{matrix} 0 \pmod 4 & \text{if} & h\equiv \pm 1 \pmod 8 \\ 2 \pmod 4 & \text{if} & h\equiv \pm 3 \pmod 8. \end{matrix}$$

Proof. The Lemma is not a direct corollary of Gauss' lemma and we give an independent proof. The number of odd multiples of h between $(16r-8)k'$ and $16rk'$, $r=1,2,\dots,\frac{1}{2}(h-1)$ is $\left[\frac{16rk'-h}{2h}\right]-\left[\frac{(16r-8)k'-h}{2h}\right]$ and we must show that

$$\sum_{r=1}^{h-1}(-1)^r\left[\frac{8rk'-h}{2h}\right]\equiv \begin{matrix} 0 \pmod 4 & \text{if} & h\equiv \pm 1 \pmod 8 \\ 2 \pmod 4 & \text{if} & h\equiv \pm 3 \pmod 8. \end{matrix}$$

Set $k'=mh+k_0$, $0<k_0<h$, then

$$\left[\frac{8rk'-h}{2h}\right]=4mr+\left[\frac{8rk_0-h}{2h}\right]$$

and we have to show that for $0<k_0<h$, $(2k_0,h)=1$,

$$\sum_{r=1}^{h-1}(-1)^r\left[\frac{8rk_0-h}{2h}\right]\equiv 1-\chi(h) \pmod 4.$$

The left hand side here is

$$\sum_{r=1}^{h-1}(-1)^r\left\{\frac{4rk_0}{h}-1-\left(\left(\frac{8rk_0-h}{2h}\right)\right)\right\}=\frac{2k_0(h-1)}{2h}-\sum_{r=1}^{h-1}(-1)^r\left(\left(\frac{8rk_0-h}{2h}\right)\right).$$

As r runs through the non-zero residues modulo h, so does rk_0 and the congruence reduces to

$$h\sum_{\lambda=1}^{h-1}(-1)^\lambda\left(\left(\frac{8\lambda-h}{2h}\right)\right)\equiv h(\chi(h)-1)\equiv \chi(h)-1 \pmod 4. \tag{2.38}$$

Break up the summation in (2.38) into

$$\sum_{1\le\lambda\le\left[\frac{h-1}{8}\right]}+\sum_{\left[\frac{h-1}{8}\right]<\lambda\le\left[\frac{3h-1}{8}\right]}+\sum_{\left[\frac{3h-1}{8}\right]<\lambda\le\left[\frac{5h-1}{8}\right]}+\sum_{\left[\frac{5h-1}{8}\right]<\lambda\le\left[\frac{7h-1}{8}\right]}+\sum_{\left[\frac{7h-1}{8}\right]\le\lambda\le h-1}.$$

Then in the i-th sum, $i=1,2,3,4,5$, $\left(\left(\frac{8\lambda-h}{2h}\right)\right)=\frac{4\lambda}{h}-1-t_i$ where the value of t_i is $-1,0,1,2,3$ respectively, and we get for the left side of (2.38)

$$-\sum_1(-1)^\lambda+\sum_3(-1)^\lambda+2\sum_4(-1)^\lambda+3\sum_5(-1)^\lambda \pmod 4 \tag{2.39}$$

where $\sum_1, \ldots, \sum_5$ are summed for the respective ranges in the five sums above. But clearly $\sum_i (-1)^\lambda$ only depends on the residue class of h modulo 16 and therefore it is sufficient to calculate (2.39) for $h=1, 3, 5, 7, 9, 11, 13, 15$. The respective values are 0, 2, 2, 4, 4, 2, 2, 4 and in each case they are congruent to $1-\chi(h)$ modulo 4. This proves Lemma 2.5, and the proof of Lemma 2.4 is complete.

3. We shall first use a direct saddle point method to obtain the main asymptotic expression for C_m. From Cauchy's integral formula

$$(3.1) \qquad C_m = \frac{1}{2\pi i} \int_\Gamma t^{-m-1} F(t)\, dt$$

where Γ is any circle of positive radius less than 1 centred at the origin. We set $t=\exp\{-2\pi(\beta-i\theta)\}$,

$$(3.2) \qquad C_m = \int_{-1}^{1} F(e^{-2\pi(\beta-i\theta)}) e^{2\pi m(\beta-i\theta)}\, d\theta, \quad \text{where} \quad \beta = \frac{1}{2q}\sqrt{\frac{Q}{qm}}.$$

This in fact is the saddle point condition as one can show that the derivative of $F(t)t^{-m-1}$ is zero for $t=\exp\left\{-2\pi\left(\beta-i\dfrac{h}{q}\right)\right\}$, $\beta=\dfrac{1}{2q}\sqrt{\dfrac{Q}{qm}}+O(m^{-3/2})$ and for h satisfying $\chi(h)=\zeta$. We omit verification as it will not be needed explicitly.

We break the range of integration up into Farey intervals of order $N=[\beta^{-3/4}]$ For the relevant properties of Farey dissections see [6], Chapter III. Thus

$$(3.3) \qquad C_m = \sum_{(h,k)=1} \int_{I_{h,k}} F(e^{-2\pi(\beta-i\theta)}) e^{2\pi m(\beta-i\theta)}\, d\theta$$

where $I_{h,k}$ is the Farey interval about h/k and the summation extends for $0\leqq h< <k\leqq N$. The Farey intervals with $k=q$, $\zeta\chi(h)=1$ give the dominant terms; however, we require a few lemmas to prove this.

First of all, from Theorem 2.1, letting $\theta=\dfrac{h}{k}+\varphi$, $z=k(\beta-i\varphi)$ it follows that

$$(3.4) \qquad F(e^{-2\pi(\beta-i\frac{h}{k}-i\varphi)}) = \\ = \omega^*(h, k) \exp\left\{\frac{\zeta\pi Q D^2 \chi(hq/D)}{2k^2 q^3} \frac{\beta+i\varphi}{\beta^2+\varphi^2} - \frac{\zeta\pi Q}{2q}(\beta-i\varphi)\right\} \prod_a{}' F_a(\tilde{x}; b, D, \varrho).$$

Lemma 3.1. *There exists a constant $c>0$, independent of the Farey interval $I_{h,k}$ of order N such that*

$$\prod_a{}' F_a(\tilde{x}; b, D, \varrho) = O\{\exp(c\beta^{-1/2})\}$$

on $I_{h,k}$. Furthermore there exists another constant $c'>0$ such that on $I_{h,q}$

$$\prod_a{}' F_a(\tilde{x}; b, D, \varrho) = 1+O\{\exp(-c'\beta^{-1/2})\}.$$

Proof. It is easily seen that

$$|F_a(\tilde{x}; b, D, \varrho)| \leqq \sum_{n=0}^{\infty} p(n)|\tilde{x}|^n$$

where $p(n)$ is the unrestricted partition function of n. Now

$$|\tilde{x}| = \exp\left(-\frac{2\pi}{kK}\frac{\beta}{\beta^2+\varphi^2}\right).$$

Since $k \leqq N = [\beta^{-3/4}]$, $K \leqq qk$ and $|\varphi| \leqq \frac{1}{kN}$ (the length of the Farey interval $I_{h,k}$), we have

$$\frac{2\pi}{kK}\frac{\beta}{\beta^2+\varphi^2} > c_1\beta^{1/2}$$

for some $c_1 > 0$, hence $|\tilde{x}| \leqq \exp(-c_1\beta^{1/2}n)$. It is well known that $p(n) \leqq \exp(c_2 n^{1/2})$ for some $c_2 > 0$. Thus

$$\sum_{n=1}^{\infty} p(n)\exp(-c_1\beta^{1/2}n) = O\{\exp(c_3\beta^{-1/2})\}.$$

This proves the first half of the Lemma.

If $k = q = K$ then

$$\frac{2\pi}{kK}\frac{\beta}{\beta^2+\varphi^2} > 2\pi\beta N^2 > c_4\beta^{-1/2}$$

and $|\tilde{x}| < \exp(-c_4\beta^{-1/2})$, from which the second part of the lemma follows, by the definition (2.1) of F_a. In the following c will denote a suitable positive constant, not necessarily identical with the constant in Lemma 3.1.

Lemma 3.2. *Let* $k \neq q$ *or* $k = q$ *and* $\chi(h) \neq \zeta$. *Then*

$$F\left(e^{-2\pi\left(\beta - i\frac{h}{k} - i\varphi\right)}\right) = O\left(\exp\frac{Q\pi}{4q^3\beta}\right) \quad \textit{for} \quad \varphi \in I_{h,k}.$$

Proof. This follows at once from (3.4) and Lemma 3.1 since $\chi(q/D) = 0$ if $D \neq q$, and if $k \neq q$ then the smallest multiple of q that k can be is $2q$. Hence the expression in (3.4) is $O\left\{\exp\left(\frac{\pi Q}{8q^3\beta} + c\beta^{-1/2}\right)\right\}$. If $k = q$ and $\zeta\chi(h) = -1$ then the expression is $O\{\exp(c\beta^{-1/2})\}$.

Lemma 3.2 shows that the total contribution in (3.3) of all the Farey arcs except those with $k = q$ and $\chi(h) = \zeta$ is $O\left(\exp\frac{Q\pi}{4q^3\beta}\right)$. We now evaluate the contribution from those arcs in (3.3) with $k = D = q$ and $\chi(h) = \zeta$. From equation (3.4) and Lemma 3.1 we obtain

$$C_m = \sum_{\chi(h)=\zeta} \exp\left(-2\pi i m \frac{h}{q}\right) \omega^*(h, q) \int_{I_{h,q}} \exp\left\{\frac{\pi Q}{2q^3}\frac{1}{\beta - i\varphi} + \left(2\pi m - \frac{\zeta \pi Q}{2q}\right)(\beta - i\varphi)\right\} d\varphi \tag{3.5}$$
$$+ O\left\{\exp\left(\frac{\pi Q}{2q^3\beta} + 2\pi m\beta - c\beta^{-1/2}\right)\right\} + O\left\{\exp\frac{\pi Q}{4q^3\beta}\right\}.$$

Here $\int\limits_{I_{h,q}}$ can be written as

$$\frac{1}{i}\int\limits_{\beta - i/qN}^{\beta + i/qN} \exp(E_1/w + E_2 w)\, dw, \quad E_1 = \frac{\pi Q}{2q^3}, \quad E_2 = 2\pi m - \frac{\zeta\pi Q}{2q} \tag{3.6}$$

and this can be changed into a contour integral

$$\frac{1}{i}\int\limits_{(0+)} \exp(E_1/w + E_2 w)\, dw$$

with an error $O\{\exp(c\beta N^2)\} = O\{\exp(c\beta^{-1/2})\}$ (see AYOUB [2, p. 185]). Now

$$\frac{1}{i}\int\limits_{(0+)} \exp(E_1/w + E_2 w)\, dw = 2\pi \operatorname*{res}_{w=0}\{\exp(E_1/w + E_2 w)\}\, dw =$$

$$= 2\pi\sqrt{E_1/E_2}\, I_1(2\sqrt{E_1 E_2}) = \frac{2\pi}{q}\sqrt{\frac{Q}{4qm - \zeta Q}} \cdot I_1\left(\frac{\pi}{q^2}\sqrt{Q(4qm - \zeta Q)}\right)$$

where $I_1(t) = \frac{1}{i} J_1(it)$ is the modified Bessel function of order 1. Hence by (3.5)

$$C_m = \frac{2\pi}{q}\sqrt{Q/(4qm - \zeta Q)}\, I_1\left(\frac{\pi}{q^2}\sqrt{Q(4qm - \zeta Q)}\right) \times \sum_{\chi(h)=\zeta} \omega^*(h, q) \exp\left(-2\pi i m \frac{h}{q}\right) +$$
$$+ O\left\{\exp\left(\frac{\pi Q}{2q^3\beta} + 2\pi m\beta - c\beta^{-1/2}\right)\right\}.$$

Using the expression (2.24) for $\omega^*(h, q)$ and the saddle point condition (3.2) we obtain

$$C_m = \frac{2\pi}{q}\sqrt{Q/(4qm - \zeta Q)}\, I_1\left(\frac{\pi}{q^2}\sqrt{Q(4qm - \zeta Q)}\right) \times \tag{3.7}$$
$$\times \sum_{\chi(h)=\zeta} \mu(h, q) \cos\left\{2\pi\left(m\frac{h}{q} - (\zeta h + h^*)\frac{Q}{4q^2}\right)\right\} + O\left\{\exp\left(\frac{2\pi}{q}\sqrt{\frac{mQ}{q}} - cm^{1/4}\right)\right\}$$

for some positive constant c, where $Q = \sum_{j=1}^{q-1} \chi(j) j^2$, $h^* h \equiv 1 \pmod{q^2}$ and $\mu(h, q)$ is given for odd q by equation (2.25), otherwise by (2.29), (2.30).

By the well known asymptotic formula

$$I_1(t) = \frac{1}{\sqrt{2\pi t}} e^t \left(1 + O\left(\frac{1}{t}\right)\right) \quad (t \to \infty)$$

(e.g. [1, formula 9.7.1]) (3.7) reduces to

$$C_m = \frac{Q^{1/4}}{(qm)^{3/4}} \exp\left(\frac{2\pi}{q}\sqrt{\frac{mQ}{q}}\right)\times \tag{3.8}$$
$$\times\left\{\sum_{\substack{h=1 \\ \chi(h)=\zeta}}^{[q/2]} \mu(h,k)\cos\left(2\pi\left(m\frac{h}{q}-(\zeta h+h^*)\frac{Q}{4q^2}\right)\right)+O\left(\frac{1}{\sqrt{m}}\right)\right\}.$$

This shows in particular that the asymptotic expression (3.7) gives C_m with a relative accuracy of $\exp(-cm^{1/4})$, except possibly when

$$\sum_{\chi(h)=\zeta} \mu(h,q)\cos\left\{2\pi\left(m\frac{h}{q}-(\zeta h+h^*)\frac{Q}{4q^2}\right)\right\} = 0.$$

Thus, for Ramanujan's continued fraction ($q=5, \zeta=1, Q=4$) we obtain

$$C_m = \frac{4\pi}{\sqrt{5m-1}} I_1\left(\frac{4\pi}{25}\sqrt{5m-1}\right)\times\left\{\cos\left(\frac{2\pi}{5}\left(m-\frac{2}{5}\right)\right)+O\left(\exp\left(-cm^{1/4}\right)\right)\right\} = \tag{3.9}$$
$$= \frac{\sqrt{2}}{(5m)^{3/4}}\exp\left(\frac{4\pi}{25}\sqrt{5m}\right)\times\left\{\cos\left(\frac{2\pi}{5}\left(m-\frac{2}{5}\right)\right)+O(m^{-1/2})\right\}.$$

When $q=5, \zeta=-1$, we obtain

$$C_m = \frac{\sqrt{2}}{(5m)^{3/4}}\exp\left(\frac{4\pi}{25}\sqrt{5m}\right)\times\left\{\cos\left(\frac{4\pi}{5}\left(m+\frac{3}{20}\right)\right)+O(m^{-1/2})\right\}.$$

In the case of Gordon's continued fraction ($q=8$, $\zeta=1$, $Q=16$) we get $C_m=\frac{1}{2(2m^3)^{1/4}}\exp\left(\frac{\pi}{4}\sqrt{2m}\right)\times\left\{\cos\frac{(m-1)\pi}{4}+O(m^{-1/2})\right\}$ hence the asymptotic term is 0 for $m\equiv 3 \pmod 4$. Similarly if $q=8, \zeta=-1$ then the oscillating part is $\cos\frac{3m\pi}{4}$, hence 0 for $m\equiv 2 \pmod 4$.

4. Finally we consider the representation of C_m as a convergent series. Starting from the integral formula (3.1) we again break the range of integration up into Farey arcs of order N where N is some positive integer. The saddle point condition is now of no help and we take $\exp(-2\pi N^{-2})$ for the radius of the circle Γ. We write (3.1) in the form

$$C_m = \sum_{(h,k)=1} \exp\left(-2\pi i m\frac{h}{k}\right)\int_{I_{h,k}} F\left(e^{2\pi i\frac{h}{k}-2\pi w}\right)e^{2\pi m w}\,d\varphi, \quad w = N^{-2}-i\varphi, \tag{4.1}$$

where $I_{h,k}$ is the Farey interval about h/k and the summation extends over $0\leqq h< <k\leqq N$.

To evaluate the integral (4.1) we again make use of Iseki's transformation. Let us define r_ν by

$$\prod_a{}' F_a(\tilde{x}; b, D, \varrho) = \sum_{\nu=0}^{\infty} r_\nu \tilde{x}^\nu$$

where $F_a(\tilde{x}; b, D, \varrho)$ is as in (2.1). Then applying Theorem 2.1 to (4.1) with $z=kw$,

$$C_m = \sum_{(h,k)=1} \omega^*(h,k) \exp\left(-2\pi i m \frac{h}{k}\right) \int_{I_{h,k}} \sum_{\nu=0}^{\infty} r_\nu \cdot \exp\left(\frac{2\pi i \nu H}{k}\right) \times$$
$$\times \exp\left\{\left(\frac{\zeta\pi Q D^2}{2k^2 q^3} \chi\left(\frac{hq}{D}\right) - \frac{2\pi\nu}{kK}\right) w^{-1} + \left(2\pi m - \frac{\zeta\pi Q}{2q}\right) w\right\} d\varphi. \tag{4.2}$$

We break the summation over ν into two parts: $\sum_\nu = \sum_{\nu=0}^{\bar{\nu}} + \sum_{\nu>\bar{\nu}}$ where $\bar{\nu}$ is the greatest integer such that

$$\bar{\nu} < \frac{\zeta K Q D^2}{4q^3 k} \chi\left(h\frac{q}{D}\right) = \frac{\zeta D Q}{4q^2} \chi\left(h\frac{q}{D}\right). \tag{4.3}$$

Thus the coefficient of w^{-1} is positive for $\nu = 0, 1, \ldots, \bar{\nu}$, and zero or negative for $\nu > \bar{\nu}$. The sum $\sum_{\nu=0}^{\bar{\nu}}$ is of course empty if $\bar{\nu} < 0$, in particular if $D \neq q$.

Next we split (4.2) in three sums as follows:

(4.4)

$$C_m = \sum_{\substack{k=1\\ D\neq q}}^{N} \sum_k{}' \omega^*(h,k) \exp\left(-2\pi i m \frac{h}{k}\right) \int_{I_{h,k}} \sum_{\nu=0}^{\infty} r_\nu \exp\left(2\pi i \nu \frac{H}{k}\right) \cdot \exp(-E_0/w + F_0 w)\, d\varphi +$$
$$+ \sum_{\substack{k=1\\ D=q}}^{N} \sum_h{}' \omega^*(h,k) \exp\left(-2\pi i m \frac{h}{k}\right) \int_{I_{h,k}} \sum_{\nu>\bar{\nu}} r_\nu \exp\left(2\pi i \nu \frac{H}{k}\right) \exp(-E_1/w + F_0 w)\, d\varphi +$$
$$+ \sum_{\substack{k=1\\ D=q}}^{N} \sum_{\substack{h\\ \chi(h)=\zeta}}{}' \omega^*(h,k) \exp\left(-2\pi i m \frac{h}{k}\right) \int_{I_{h,k}} \sum_{\nu=0}^{\bar{\nu}} r_\nu \exp\left(2\pi i \nu \frac{H}{k}\right) \exp(-E_2/w + F_0 w)\, d\varphi$$

where

$$E_0 = \frac{2\pi\nu}{kK}, \quad E_1 = \frac{2\pi\nu}{k^2} - \frac{\pi\zeta Q}{2qk^2} \chi(h), \quad E_2 = -\frac{2\pi\nu}{k^2} + \frac{\pi Q}{2qk^2},$$
$$F_0 = 2\pi m - \frac{\zeta\pi Q}{2q} \tag{4.5}$$

and $\sum_h{}'$ denotes summation over those h for which $(h,k)=1$, $0<h<k$.

The estimation of the three parts in (4.4) is based upon the following Lemma which will be proved at the end of this section.

Lemma 4.1. *Let m, ν be integers, H as defined in § 2, and $\varepsilon>0$ arbitrary. Then*

$$\sum_h^{(\eta)} \exp\left\{-2\pi i m \frac{h}{k}+2\pi i \nu \frac{H}{k}+2\pi i \sigma^*(h,k)\right\} = O\left\{k^{\frac{2}{3}+\varepsilon} m^{\frac{1}{3}}\right\}, \quad \eta=+1 \quad or \quad -1,$$

where $\sum\limits_h^{(\eta)}$ *denotes summation over those h for which $\chi(h)=\eta\zeta$.*

Consider now the first summation in (4.4). Since the length of the Farey interval $I_{h,k}$ is $\geqq \frac{1}{kN}$ and $k \leqq N$, we have by (4.1) and (4.5)

$$\mathscr{R}(E_0/w) = \mathscr{R}\left(\frac{2\pi\nu}{kKw}\right) \geqq \frac{2\pi\nu}{qk^2}\,\frac{N-2}{N^{-4}+(kN)^{-2}} \geqq \frac{2\pi\nu}{q(k^2N^{-2}+1)} \geqq \frac{\pi\nu}{q}.$$

Thus

$$\int_{I_{h,k}} \exp(-E_0/w+F_0 w)\,d\varphi = O\left\{\frac{1}{kN}\exp\left(-\frac{\pi\nu}{q}+2\pi\frac{m}{N^2}\right)\right\};$$

and upon interchanging the order of summation of h and ν and applying Lemma 4.1 we obtain

$$\sum_{\substack{k=1\\ D\neq q}}^{N}\sum_h{}' \exp(\dots) \int_{I_{h,k}} \dots = O\left\{N^{-1}\sum_{k=1}^{N}\sum_{\nu=0}^{\infty}|r_\nu| e^{-\frac{\pi\nu}{q}+2\pi m N^{-2}} k^{-\frac{1}{3}+\varepsilon} m^{\frac{1}{3}}\right\} =$$

$$= O\left\{e^{2\pi m N^{-2}} m^{\frac{1}{3}} N^{-1}\sum_{k=1}^{N} k^{-\frac{1}{3}+\varepsilon}\right\}$$

since the radius of convergence of the infinite series is 1. Thus for the first summation of (4.4)

$$\sum_{\substack{k=1\\ D\neq q}}^{N}\sum_h{}' \dots = O\left\{e^{2\pi m N^{-2}} N^{-\frac{1}{3}+\varepsilon} m^{\frac{1}{3}}\right\}. \tag{4.6}$$

In a similar manner we obtain for the second summation of (4.4), by the remark after the definition (4.4) of $\bar{\nu}$,

$$\sum_{\substack{k=1\\ D=2}}^{N}\sum_h{}' \dots \int_{I_{h,k}} \sum_{\nu>\bar{\nu}} \dots = O\left\{e^{2\pi m N^{-2}} N^{-\frac{1}{3}+\varepsilon} m^{\frac{1}{3}}\right\}. \tag{4.7}$$

Let us now consider the third (principal) part of (4.4). Define C_m^+ by equation (1.0) with $\zeta=1$. It follows from Lemma 2.1 that

$$r_\nu = c_\nu^+. \tag{4.8}$$

Transforming $\int\limits_{I_{h,k}}$ as in (3.6), we obtain by the method of Ayoub

$$\int_{I_{h,k}} \exp\{E_2/w+F_0 w\}\,d\varphi = \frac{1}{i}\int_{(0+)} \exp\{E_2/w+F_0 w\}\,dw + O\{e^{2\pi m N^{-2}} k^{-1} N^{-1}\}. \tag{4.9}$$

Upon interchanging the order of summation of h and v and employing Lemma 4.1 and (4.9), it follows that the third summation of (4.4) is

$$(4.10)\quad \sum_{\substack{k=1\\D=q}}^{N} \sum_{\substack{h\\ \chi(h)=\zeta}}' \cdots \int_{I_{h,k}} \sum_{v=0}^{\bar{v}} \cdots = \sum_{\substack{k=1\\D=q}}^{N} \sum_{v=0}^{\bar{v}} c_v^+ A_k(m, v) L_k(m, v) + O\left\{e^{2\pi m N^{-2}} m^{\frac{1}{3}} N^{-\frac{1}{3}+\varepsilon}\right\}$$

where

$$(4.11)\quad A_k(m, v) = \sum_{\substack{h=1\\(h,kq)=1\\ \chi(h)=\zeta}}^{hq-1} \exp\left\{2\pi i \left(v \frac{H}{kq} - m\frac{h}{kq} + \sigma^*(h, kq)\right)\right\}, \quad hH \equiv -1 \pmod{kq}$$

and

$$L_k(m, v) = \frac{1}{i} \int_{(0+)} \exp\{E_2/w + F_0 w\}\, dw = 2\pi \sqrt{E_2/F_0}\, I_1\left(2\sqrt{F_0 E_2}\right)$$

provided that $F_0 > 0$, i.e. $m > \frac{Q}{4q}$. Hence for $m > \frac{Q}{4q}$

$$(4.12)\quad L_k(m, v) = \frac{2\pi}{kq} (Q-4vq)^{1/2} (4qm - \zeta Q)^{-1/2} I_1\left\{\frac{\pi}{kq^2} (Q-4vq)^{1/2} (4qm-\zeta Q)^{1/2}\right\}.$$

If we let $N \to \infty$, equations (4.4), (4.6), (4.7) and (4.10) yield with Lemma 2.2:

Theorem 4.1. *Let C_m be given by equation* (1.0) *and C_m^+ by* (1.0) *with* $\zeta = 1$. *Let $Q^* = \frac{1}{4} \sum_{j=1}^{q-1} \chi(j) j^2$ and h^* be a solution of $hh^* \equiv 1 \pmod{q^2 k}$. Then for $m > Q^*/q$*

$$(4.13)\quad C_m = \sum_{k=1}^{\infty} \sum_{0 \leq v < Q^*/q} C_v^+ L_k(m, v) A_k^{(\zeta)}(m, v)$$

where $L_k(m, v)$ is given by (4.12),

$$(4.14)\quad A_k^{(\zeta)}(m, v) = \sum_{\substack{h=1\\ \chi(h)=\zeta}}^{kq-1} \mu(h, kq) \cos \frac{2\pi}{kq}\left(mh + vh^* - \frac{Q^*}{q}(\zeta h + h^*)\right)$$

with $\mu(h, kq)$ given by (2.25) *and* (2.29), (2.30) *when* $(h, kq) = 1$, $\mu(h, kq) = 0$ *otherwise.*

The following are the first thirty values of Q^*/q:

q	5	8	12	13	17	21	24	28	29	33	37	40	41	44	53
Q^*/q	$\frac{1}{5}$	$\frac{1}{2}$	1	1	2	2	3	4	3	6	5	7	8	7	7
q	56	57	60	61	65	69	73	76	77	85	88	89	92	93	97
Q^*/q	10	14	12	11	16	12	22	19	12	18	23	26	20	18	34

To prove Lemma 4.1, we estimate

$$\sum_{\substack{h=1\\ \chi(h)=\eta\zeta}}^{k-1} \exp\left\{-2\pi i m\frac{h}{k}+2\pi i v\frac{H}{k}+2\pi i\sigma^*(h,k)\right\} \tag{4.15}$$

by means of Kloosterman sums. Define the trigonometric sum

$$S(u,v;\lambda,\Lambda;r) = \sum_{\substack{0<h<r\\ (h,r)=1\\ h\equiv\lambda\ (\mathrm{mod}\ \Lambda)}} \exp\left\{\frac{2\pi i}{r}(uh+vh^*)\right\} \tag{4.16}$$

for integers $u, v, \lambda, \Lambda, r>0$ where Λ is a positive divisor of r and $hh^*\equiv 1 \pmod r$. It was proven by KLOOSTERMAN [10] that there exists a $\beta>0$ such that with $\varepsilon>0$ arbitrary,

$$S(u,v;\lambda,\Lambda;r) = O\left(r^{1-\beta+\varepsilon}(u,r)^\beta\right). \tag{4.17}$$

According to SALIÉ [12] and DAVENPORT [4], β can be taken as $\beta=\frac{1}{3}$, and we assume this for convenience.

By making use of the expression (2.22) for $\sigma^*(h,k)$, the sum in (4.15) can be written as

$$\sum_{\substack{h=1\\ \chi(h)=\eta\zeta}}^{k-1} \lambda(h,k)\exp\left\{\frac{2\pi i}{gqk}\left[-(3\varphi\delta Q+gqm)h+\delta\left(gqv-3\varphi Q\chi(h)\right)h^*\right]\right\} \quad \text{if} \quad D=q,$$

$$\sum_{\substack{h=1\\ \chi(h)=\eta\zeta}}^{k-1} \lambda(h,k)\exp\left\{\frac{2\pi i}{gDk}\left[-(3\varphi\delta Q+gDm)h+v\delta gDh^*\right]\right\} \quad \text{if} \quad D\neq q$$

where we have taken $H=\delta h^*$, $hh^*\equiv 1 \pmod{gDk}$, by the definition (2.19) of H. The value of

$$\lambda(h,k) = \pm\exp\left\{\pi i\left[\frac{\gamma}{Dq}\sum_{j=1}^{q-1}\chi(j)j\{hj\}_q-\frac{1}{q}\chi(k)\delta_{D,1}\sum_{\substack{r=1\\ 1\leqq\{hr\}_q<q/2}}^{q-1} r\chi(r)\right]\right\}$$

only depends on the residue class to which h and k belong modulo q, provided that we select the solution γ of $\gamma k_1-\delta q_1=1$ in (2.0) always in the interval $0\leqq\gamma<q_1$.

Thus the sum (4.15) splits up in at most q^2 sums of the form $cS(u,v;\lambda,\Lambda;r)$ with $\Lambda=q$, $r=gDk$ and $u=-(gDM+3\varphi\delta Q)$. But $uq=-\left(gDqm+3\varphi Q(\gamma k-D)\right)$ by (2.0) and so $(uq,k)=(gDqm-3\varphi QD,k)\leqq gDqm+3|\varphi QD|$, $(u,r)=O(m)$, and (with $\beta=\frac{1}{3}$)

$$r^{1-\beta+\varepsilon}(u,r)^\beta = O\left(k^{\frac{2}{3}+\varepsilon}m^{\frac{1}{3}}\right).$$

Hence by (4.17) the expression (4.15) itself is $O(k^{2/3+\varepsilon}m^{1/3})$, which is precisely the statement of Lemma 4.1. The proof of Theorem 4.1 is now complete.

5. We apply Theorem 4.1 to the case when $q=8$, $Q=16$ and $\zeta=1, m\equiv 3 \pmod 4$ or $\zeta=-1$, $m\equiv 2 \pmod 4$. Suppose first that k is odd and that $hk\equiv 3 \pmod 4$. Then $\chi(h+2k)=\chi(h)$, as in the proof of Lemma 2.4, and we find, by the first case of the Lemma and (2.32)

(5.1) $$\mu(h, 8k)\cos\frac{\pi}{4k}\Big(mh-\frac{1}{2}(\zeta h+h^*)\Big)+\mu(h+2k, 8k)\cos\frac{\pi}{4k}\Big(m(2k+h)-$$
$$-\frac{1}{2}(\zeta(h+2k)+(h+2k)^*)\Big) =$$
$$= \mu(h, 8k)\Big\{\cos\frac{\pi}{4k}\Big(mh-\frac{1}{2}(\zeta h+h^*)\Big)+\cos\frac{\pi}{4k}\Big(mh-\frac{1}{2}(\zeta h+h^*)+$$
$$+k(2m-\zeta-1)\Big) = 0$$

since $m\equiv 3 \pmod 4$ if $\zeta=1$ and $m\equiv 2 \pmod 4$ if $\zeta=-1$. Clearly the co-prime residues h modulo $8k$ with $\chi(h)=\zeta$ can be uniquely grouped in pairs $h, h+2k$ satisfying the condition $kh\equiv 3 \pmod 4$ and each pair of corresponding terms in (4.14) cancels, by (5.1). Therefore $A_k^{(\zeta)}(m, v)=0$ for odd k in (4.14).

Suppose next that k is even. Then $\chi(h)=\chi(4k-h)$ and we find, by the second case of Lemma 2.4 and (2.36)

$$\mu(h, 8k)\cos\frac{\pi}{4k}\Big(mh-\frac{1}{2}(\zeta h+h^*)\Big)+\mu(4k-h, 8k)\cos\frac{\pi}{4k}\Big(m(4k-h)-$$
$$-\frac{1}{2}(\zeta(4k-h)+(4k-h)^*)\Big) =$$
$$= \mu(h, 8k)\Big\{\cos\frac{\pi}{4k}\Big(mh-\frac{1}{2}(\zeta h+h^*)\Big)+\cos\frac{\pi}{4k}\Big(mh-\frac{1}{2}(\zeta h+h^*)+$$
$$+2k(3+\zeta-2m)\Big)\Big\} = 0$$

since m is odd when $\zeta=1$, even when $\zeta=-1$. Thus the terms in the sum (4.14) cancel in pairs and $A_k^{(\zeta)}(m, v)=0$ also for even k. Thus $A_k^{(\zeta)}(m, v)=0$ for all k, and we obtain the following corollary of Theorem 4.1:

Theorem 5.1. *If* $q=8$, $\zeta=1$ *(Gordon's continued fraction) then* $C_m=0$ *in* (1.0) *for all* $m\equiv 3 \pmod 4$. *If* $q=8$, $\zeta=-1$ *then* $C_m=0$ *for all* $m\equiv 2 \pmod 4$.

Another interesting case is $q=12$ when the principal asymptotic term of C_m vanishes for $m\equiv 5 \pmod 6$ if $\zeta=1$ and for $m\equiv 3 \pmod 6$ if $\zeta=-1$. It is quite likely that C_m is zero for these values of m but at present we do not have the appropriate modification of Lemma 2.4. It would be interesting to prove these results independently from the series representation (4.13).

It is easy to give an interpretation of Theorem 5.1 in terms of partitions. Take first Gordon's product

$$F(x) = \frac{(1-x^3)(1-x^{11})\dots(1-x^5)(1-x^{13})\dots}{(1-x)(1-x^9)\dots(1-x^7)(1-x^{15})\dots} = \sum C_m X^m.$$

Since $C_m=0$ for $m\equiv 3 \pmod 4$, we have $F(x)-F(-x)+i(F(ix)-F(-ix))=0$ and upon expressing this equation as a sum of four fractions of products and bringing the fractions to common denominator we obtain for the product

$$G(x) = \prod_{m=0}^{\infty} (1+x^{8m+1})(1-x^{8m+3})(1-x^{8m+5})(1-x^{8m+7})(1+x^{16m+2})(1+x^{16m+14}) =$$
$$= \sum d_n x^n,$$

$G(x)-G(-x)+i(G(ix)-G(-ix))=0$, that is $d_n=0$ for $n\equiv 3 \pmod 4$. Or, if we take the partitions of n into distinct positive integers of the form $8m+1$, $16m+2$, $8m+7$, $16m+14$, $8m+3$, $8m+5$, and if $n\equiv 3 \pmod 4$ then the number of such partitions in which parts $8m\pm 3$ appear an even number of times is the same as the number of those partitions in which parts $8m\pm 3$ appear an odd number of times. By reinterpreting parts $16m+2$ and $16m+14$ as $(8m+1)+(8m+1)$, $(8m+7)+$ $+(8m+7)$ respectively, we obtain

Theorem 5.2. *Denote by Π_n the set of those partitions of n into positive odd parts in which summands* $\equiv\pm 3 \pmod 8$ *appear at most once and summands* $\equiv\pm 1 \pmod 8$ *appear with multiplicity at most three. Then if* $n\equiv 3 \pmod 4$, *exactly half of the partitions belonging to Π_n contain an even (odd) number of summands* $\equiv\pm 3 \pmod 8$.

For instance Π_{19} contains the 14 partitions (1, 1, 1, 3, 13), (1, 1, 1, 5, 11), (1, 1, 1, 7, 9), (1, 1, 17), (1, 1, 3, 5, 9), (1, 5, 13), (1, 9, 9), (1, 1, 3, 7, 7), (1, 3, 15), (1, 7, 11), (3, 5, 11), (3, 7, 9), (5, 7, 7), (19). The first seven of these contain an even number of summands $\equiv\pm 3 \pmod 8$, and the last seven an odd number of such summands.

By turning Gordon's product upside down, a similar theorem is obtained for $n\equiv 2 \pmod 4$, with ± 1 and ± 3 interchanged. For instance in the ten partitions (1, 17), (5, 13), (1, 3, 5, 9), (3, 5, 5, 5), (1, 5, 5, 7), (1, 3, 3, 11), (3, 3, 3, 9), (3, 3, 5, 7), (3, 15), (7, 11) of 18, exactly five contain an even number of summands $\equiv\pm 1 \pmod 8$.

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