Necessary and sufficient conditions for imbedding of classes of functions

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Dedicated to Professor Béla Szőkefalvi-Nagy on his 65th birthday

1. Let $\Psi(p)$ $(p>1)$ denote the family of non-negative functions $\psi(u)$ on $(0, \infty)$ such that $\frac{\psi(u)}{u}$ is non-decreasing and $\frac{\psi(u)}{u^p}$ is non-increasing, and let $\Psi := \bigcup_{p>1} \Psi(p)$. Let $\Psi_+(p)$ $(p>1)$ denote the family of non-negative functions $\psi_+(u)$ on $(0, \infty)$ such that $\frac{\psi_+(u)}{u^p}$ is non-decreasing, while for any $p'>p$, $\frac{\psi_+(u)}{u^{p'}}$ is non-increasing, and let $\Psi_+ := \bigcup_{p>1} \Psi_+(p)$; furthermore let $\Psi_-(p)$ $(0<p<1)$ denote the family of non-negative functions $\psi_-(u)$ on $(0, \infty)$ such that $\frac{\psi_-(u)}{u^p}$ is non-increasing, while for any p', $0<p'<p$, $\frac{\psi_-(u)}{u^{p'}}$ is non-decreasing, and let $\Psi_- := \bigcup_{0<p<1} \Psi_-(p)$.

Let $P=P(C)$ $(C\geqq 1)$ denote the family of non-negative and continuous functions $\varrho(u)$ on $(0, \infty)$ which are non-decreasing and satisfy $\varrho(u^2)\leqq C\cdot\varrho(u)$ on $[1, \infty)$, while on $(0, 1]$ are defined by $\varrho(u)=\varrho\left(\frac{1}{u}\right)$, and for 0 by $\varrho(0)=0$; and let $P:=\bigcup_{C\geqq 1} P(C)$.

Let $\Lambda(M)$ denote the family of non-negative monotonic sequences $\lambda=\{\lambda_K\}_1^\infty$ such that $\lambda_{k^2}\leqq M\lambda_k$, and let $\Lambda:=\bigcup_{M>0}\Lambda(M)$, and for $\lambda\in\Lambda$ let $\lambda(u)$ denote the function $\lambda(u)=\sum_{k=1}^{[u]}\frac{\lambda_k}{k}$ for $u\geqq 1$, $\lambda(u)=\lambda\left(\frac{1}{u}\right)$ for $0<u\leqq 1$ and $\lambda(0)=0$.

For $\varrho\in P$ let ϱ_1 and ϱ_2 denote the functions which are equal to $\varrho(u)$ on $1\leqq u<\infty$ and $0\leqq u<1$ respectively, and equal to 0 elsewhere on $[0, \infty)$.

For $\lambda\in\Lambda$ we define the functions λ_1 and λ_2 in an analogous way.

For a non-negative, piecewise continuous function σ on $(0, \infty)$ we denote by $\sigma(L[a, b])$ $(0\leqq a<b\leqq\infty)$ the set of measurable functions f on (a, b) for which

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$\int_a^b \sigma(|f(x)|)\,dx<\infty$. In the case $[a,b]=[0,1]$ we simply write $\sigma(L)$ instead of $\sigma(L[0,1])$.

If φ is a non-negative, continuous and strictly monotonic function on $(0,\infty)$ and if $f\in\varphi(L[a,b])$ then the modulus of continuity of f with respect to φ is defined by

$$\omega_\varphi(\delta;f)=\sup_{0\le h\le\delta}\bar\varphi\Big(\int_a^{b-h}\varphi(|f(x+h)-f(x)|)\,dx\Big)\quad(0<\delta\le b-a)$$

and if $f\in\varphi(L(0,\infty))$ we also define

$$\hat\omega_\varphi(\delta;f)=\bar\varphi\Big(\int_{1/\delta}^{\infty}\varphi(|f(x)|)\,dx\Big),\quad \tilde\omega_\varphi(\delta;f)=\omega_\varphi(\delta;f)+\hat\omega_\varphi(\delta;f),$$

where $\bar\varphi$ denotes the inverse of φ.

Let f^* denote a non-increasing function, equidistributed with $|f|$, that is, such that

$$\text{mes}\,\{x\colon x\in[a,b],\ |f(x)|>y\}=\text{mes}\,\{z\colon z\in[a,b],\ f^*(z)>y\}.$$

2. Recently many papers deal with imbedding problems. Among others UL'JANOV [10], [11], [12] gave conditions which assure that a function $f\in L^p$ $(p\ge1)$ should belong to another space L^ν $(\nu>p)$. LEINDLER [2] generalized this result and gave conditions assuring the transition from L^p to $L^p(\ln^+L)^\beta$ and from L^p to $\psi(L)$ where $\psi\in\Psi$; and in [3] the latter results were further generalized. More precisely he proved:

Theorem A. (LEINDLER [3], Theorem 2) *Let φ, $\psi\in\Psi$, and let ϱ be a non-negative, non-decreasing, continuous function with*

$$\sum_{k=m}^{\infty}\frac{\varrho(k)}{k^2}\le K\frac{\varrho(m)}{m}.$$

Then $f\in\varphi(L)$ and

$$\sum_{n=1}^{\infty}\frac{\varrho(n)}{n^2}(\psi\circ\bar\varphi\circ n\varphi)\left[\omega_\varphi\left(\frac1n;f\right)\right]<\infty$$

imply $f\in\psi(L)\varrho(L)$.

STOROŽENKO [9] gave necessary conditions, in terms of the modulus of continuity $\omega_p(\delta;f^*)$, that a function $f\in L^p$ should belong to the class $L^q\varrho_1(L)$ $(q>p)$, where $\varrho\in P$ is absolutely continuous on any interval $(0,A)$, $A\ge1$. She proved:

Theorem B. ([9], Theorem 1) *If $\varrho\in P$ and $f\in L^q\varrho_1(L)$, then*

$$\int_0^1 x^{-q/p}\omega_p^q(x;f^*)\varrho_1\left(\frac1x\right)dx<\infty\quad \textit{for}\quad q>p\ge1 \tag{1}$$

and

$$\int_0^1 x^{-2}\omega_p^p(x;f^*)\varrho_1'\left(\frac{1}{x}\right)dx<\infty \quad \textit{for} \quad q=p\geqq 1. \tag{2}$$

Later LEINDLER [5] gave a generalization of (2) and certain converse of Theorem A which is similar to (1), that is, he proved:

Theorem C. ([5], Theorem 1) *If $\varphi\in\Psi$, $\lambda\in\Lambda$ and $f\in\varphi(L)\lambda_1(L)$ then*

$$\sum_{n=1}^{\infty}\frac{\lambda_n}{n}\varphi\left(\omega_\varphi\left(\frac{1}{n};f^*\right)\right)<\infty.$$

Answering a problem of Leindler we gave a necessary condition in terms of the modulus of continuity $\omega_\varphi(\delta;f^*)$ that f should belong to $\psi(L)\varrho_1(L)$ where $\psi\in\Psi$, $\varrho\in P$. Namely we proved:

Theorem D. ([8]) *Let $\varphi,\psi\in\Psi$ and $\varrho\in P$. Suppose that $\psi\circ\bar{\varphi}$ belongs to Ψ_+. If $f\in\psi(L)\varrho_1(L)$ then we have*

$$\sum_{n=1}^{\infty}\frac{\varrho(n)}{n^2}(\psi\circ\bar{\varphi}\circ n\varphi)\left(\omega_\varphi\left(\frac{1}{n};f^*\right)\right)<\infty.$$

We remark that all of the above mentioned results are valid on the interval [0, 1].

UL'JANOV [12], GAĬMNASAROV [1] and the present author [7] have gave sufficient conditions for imbedding of classes of functions on the interval $(0,\infty)$ which are similar to the results concerning the interval [0, 1].

In this paper we prove a theorem concerning the interval $(0,\infty)$ which is similar to above mentioned results and gives necessary and sufficient conditions for general imbedding problems.

Theorem. *Let $\varphi\in\Psi$, $\varrho\in P$ and $f\in\varphi(L(0,\infty))$.*
If $\psi\in\Psi$ is such that $\psi\circ\bar{\varphi}\in\Psi_+$ then we have

$$f\in\psi(L(0,\infty))\varrho(L(0,\infty)) \tag{3}$$

if and only if

$$\sum_{n=1}^{\infty}\frac{\varrho(n)}{n^2}(\psi\circ\bar{\varphi}\circ n\varphi)\left(\omega_\varphi\left(\frac{1}{n};f^*\right)\right)<\infty. \tag{4}$$

If $\psi\in\Psi$ is such that $\psi\circ\bar{\varphi}\in\Psi_+$ then

$$f\in\psi(L(0,\infty))\varrho(L(0,\infty)) \tag{5}$$

if and only if

$$\sum_{n=1}^{\infty}\varrho(n)\left(\psi\circ\bar{\varphi}\circ\frac{1}{n}\varphi\right)\left(\hat{\omega}_\varphi\left(\frac{1}{n};f^*\right)\right)<\infty. \tag{6}$$

If $\lambda\in\Lambda$ *then*

$$f\in\varphi(L(0,\infty))\lambda(L(0,\infty)) \tag{7}$$

if and only if

$$\sum_{n=1}^{\infty}\frac{\lambda_n}{n}\varphi\left(\tilde{\omega}_{\varphi}\left(\frac{1}{n};f^*\right)\right)<\infty. \tag{8}$$

Corollary. *Let* $f\in L^p(0,\infty)$, $p\geqq 1$, $\varrho\in P$.
If $v>p$ *then* $f\in L^v(0,\infty)\varrho(L(0,\infty))$ *if and only if*

$$\sum_{n=1}^{\infty}\varrho(n)n^{v/p-2}\omega_p^v\left(\frac{1}{n};f^*\right)<\infty.$$

If $1\leqq v<p$ *then* $f\in L^v(0,\infty)\varrho(L(0,\infty))$ *if and only if*

$$\sum_{n=1}^{\infty}\varrho(n)n^{-v/p}\hat{\omega}_p^v\left(\frac{1}{n};f^*\right)<\infty.$$

If $\lambda\in\Lambda$ *then* $f\in L^p\lambda(L)$ *if and only if*

$$\sum_{n=1}^{\infty}\frac{\lambda_n}{n}\tilde{\omega}_p^p\left(\frac{1}{n};f^*\right)<\infty.$$

3. We require the following lemmas

Lemma 1. ([13] p. 29) *If* χ *is a non-negative non-decreasing function on* $(0,\infty)$ *then*

$$\int_0^1(\chi\circ|f|)(x)\,dx=\int_0^1(\chi\circ f^*)(x)\,dx. \tag{9}$$

Lemma 2. ([5], Lemma 2) *If* $\psi\in\Psi$ *and* $\varrho\in P$ *then*

$$\int_0^1((\psi\varrho_1)\circ f^*)(x)\,dx<\infty \tag{10}$$

implies

$$\int_0^1(\psi\circ f^*)(x)\varrho_1\left(\frac{1}{x}\right)dx<\infty. \tag{11}$$

Lemma 3. ([8], Lemma 6) *If* $\psi_+\in\Psi_+$ *and*

$\varrho\in P$, $f(x)\geqq 0$, $F(x)=\int_0^x f(t)\,dt$ *then*

$$\int_0^1\varrho_1\left(\frac{1}{x}\right)\psi_+\left(\frac{F(x)}{x}\right)dx\leqq K(\psi_+)\int_0^1\varrho_1\left(\frac{1}{x}\right)(\psi_+\circ f)(x)\,dx. \tag{12}$$

Lemma 4. *If* $\psi\in\Psi$, $\varrho\in P$ *then*

$$\int_0^\infty ((\psi\varrho)\circ f^*)(x)\,dx<\infty \Leftrightarrow \int_0^\infty ((\psi\varrho)\circ|f|)(x)\,dx<\infty. \tag{13}$$

The proof of this lemma is by an easy application of (9), using the definition of f^* and the properties of ψ and ϱ.

Lemma 5. *If* $\psi\in\Psi$ *and* $f\in\psi(L(0,\infty))$ *then*

$$\psi\left[\omega_\psi\left(\frac{1}{n};f^*\right)\right]\leqq\int_0^{1/n}(\psi\circ f^*)(x)\,dx. \tag{14}$$

The proof is similar to that of Lemma 3 of LEINDLER [5].

Lemma 6. *If the non-negative sequence* $\{a_n\}$ *is quasi-decreasing* $(a_{n+j}\leqq K\cdot a_n$ *for any n and* $j\leqq n)$, *and if* $\{\lambda_k\}$ *is a non-negative sequence and* $\psi_-\in\Psi_-$ *then*

$$\sum_{n=1}^\infty\lambda_n\psi_-\left(\sum_{k=n}^\infty a_k\right)\leqq M\sum_{n=1}^\infty\frac{\psi_-(n\cdot a_n)}{n}\left(n\cdot\lambda_n+\sum_{k=1}^n\lambda_k\right). \tag{15}$$

This is a trivial generalization of the inequality (4) of LEINDLER [4].

Lemma 7. *Let* $\psi\in\Psi$ *and* $\varrho\in P$. *Then we have*

$$\int_0^\infty((\psi\varrho_2)\circ f^*)(x)\,dx<\infty \tag{16}$$

if and only if

$$\int_0^\infty(\psi\circ f^*)(x)\,\varrho_2\left(\frac{1}{x}\right)dx<\infty. \tag{17}$$

Proof. Let $\psi\in\Psi(p)$, $p>1$. (16) implies $\sum_{n=1}^\infty(\psi\circ f^*)(n)<\infty$ and since $(\psi\circ f^*)(n)\downarrow$ we get $(n\psi\circ f^*)(n)=0(1)$, whence

$$\sum_{n=1}^\infty(\psi\circ f^*)(n)\,\varrho_2\left(\frac{1}{n}\right)\leqq K\sum_{n=1}^\infty((\psi(\varrho_2\circ\psi))\circ f^*)(n)=\sum\nolimits_1. \tag{18}$$

Applying the following properties of ϱ_2 and ψ

$$(\varrho_2\circ\psi)(u)\leqq K_1\varrho_2(u^p)\leqq K_1\varrho_2(u)$$

we obtain

$$\sum\nolimits_1\leqq K_3\sum_{n=1}^\infty((\psi\varrho_2)\circ f^*)(n),$$

which by (18) proves the statement (16)⇒(17).

To prove (17)⇒(16) we mention that from the properties of ϱ_2 it follows

$$\sqrt{t}\,\varrho_2(t)\to 0 \quad \text{if} \quad t\to 0_+$$

(see for example [11] p. 664), therefore we can write

$$\sum_{n=1}^{\infty}((\psi\varrho_2)\circ f^*)(n) \leqq K_1 \sum_{(\psi\circ f^*)(n)<n^{-4}} \sqrt{(\psi\circ f^*)(n)}+K_2 \sum_{(\psi\circ f^*)(n)\geqq n^{-4}} (\psi\circ f^*)(n)\,\varrho_2(n^{-4}) \leqq$$

$$\leqq K_3+K_4\sum_{n=1}^{\infty}(\psi\circ f^*)(n)\,\varrho_2(n^{-1}),$$

which gives the statement.

Lemma 8. *If* $\psi_-\in\Psi_-$, $\varrho\in P$ *and* $a_K\geqq 0$ *then*

$$\sum_{n=1}^{\infty}\varrho(n)\psi_-\left(\frac{1}{n}\sum_{k=n}^{\infty}a_k\right)\geqq K\sum_{n=1}^{\infty}\varrho(n)\psi_-(a_n). \tag{19}$$

Proof. Let $\psi\in{}_-\Psi_-(p)$, $0<p<1$. Using

$$\psi_-(tx)=\frac{\psi_-(tx)}{(tx)^p}(tx)^p\geqq\frac{\psi_-(x)}{x^p}t^p x^p=\psi_-(x)t^p \quad \text{if} \quad t\leqq 1,\ x>0; \tag{20}$$

and applying the inequality (see [6], (11))

$$K\cdot\sum_{n=1}^{\infty}\lambda_n\psi\left(\sum_{k=n}^{\infty}a_k\right)\geqq\sum_{n=1}^{\infty}\lambda_n\psi\left(\frac{a_n}{\lambda_n}\sum_{k=1}^{n}\lambda_k\right)$$

with $\lambda_n=\varrho(n)n^{-p}$ and $\psi=\psi_-$ we get for an arbitrary integer l that

$$\begin{aligned}\sum_{n=1}^{l}\varrho(n)\psi_-\left(\frac{1}{n}\sum_{k=n}^{\infty}a_k\right)&\geqq\sum_{n=1}^{l}\varrho(n)\frac{1}{n^p}\psi_-\left(\sum_{k=n}^{l}a_k\right)\geqq\\&\geqq K_1\sum_{n=1}^{l}\frac{\varrho(n)}{n^p}\psi_-\left(\frac{a_n}{\varrho(n)}n^p\sum_{k=1}^{n}\frac{\varrho(k)}{k^p}\right)=S_1.\end{aligned} \tag{21}$$

Since from the properties of ϱ it follows that

$$\sum_{k=1}^{n}\frac{\varrho(k)}{k^p}\geqq K_2\varrho(n)n^{1-p}$$

we can write

$$S_1\geqq K_3\sum_{n=1}^{l}\frac{\varrho(n)}{n^p}\psi_-(n\cdot a_n)=S_2. \tag{22}$$

By

$$\psi_-(tx)=\frac{\psi_-(tx)}{(tx)^{p'}}(tx)^{p'}\geqq\frac{\psi_-(x)}{x^{p'}}t^{p'}x^{p'}=\psi_-(x)t^{p'} \quad \text{if} \quad x>0,\ t\geqq 1,\ p'<p, \tag{23}$$

choosing p_l' such that $0<p_l'<p$ and $l^{p-p_l'}<2$, we have

$$S_2 \geqq K_4 \sum_{n=1}^{l} \varrho(n) \frac{n^{p_l'}}{n^p} \psi_-(a_n) \geqq K_5 \sum_{n=1}^{l} \varrho(n) \psi_-(a_n). \tag{24}$$

Collecting (21), (22) and (24) we get (19).

Lemma 9. *If $\psi_- \in \Psi_-$, $\varrho \in P$ and $\{a_k\}$ is a non-negative non-increasing sequence then*

$$\sum_{n=1}^{\infty} \varrho(n) \psi_-\left(\frac{1}{n} \sum_{k=n}^{\infty} a_k\right) \leqq K \sum_{n=1}^{\infty} \varrho(n) \psi_-(a_n). \tag{25}$$

Proof. Let $\psi_- \in \Psi_-(p)$, $0<p<1$, and let l be an arbitrary integer and $0<p_l'<p$ such that $l^{p-p_l'}<2$; then applying (15) with $\lambda_n = \varrho(n) n^{-p_l'}$ (20) and (23) we get

$$\sum_{n=1}^{l} \varrho(n) \psi_-\left(\frac{1}{n} \sum_{k=n}^{l} a_k\right) \leqq M \sum_{n=1}^{l} \varrho(n) \frac{1}{n^{p_l'}} \psi_-\left(\sum_{k=n}^{l} a_k\right) \leqq$$

$$\leqq \frac{M_1}{1-p^{l}} \sum_{n=1}^{l} \frac{\psi_-(n \cdot a_n)}{n^{p_l'}} \varrho(n) \leqq \frac{M_1}{1-p'} \sum_{n=1}^{l} \psi_-(a_n) \frac{n^p}{n^{p_l'}} \varrho(n) \leqq M_2 \sum_{n=1}^{l} \psi_-(a_n) \varrho(n).$$

If $l \to \infty$ we obtain (25).

Proof of Theorem.

Implication (4)⇒(3) follows from Theorem 2 of [7], applying Lemma 4.

To prove (3)⇒(4) we apply Lemma 3 and Lemma 5 so we get

$$\sum_{n=1}^{\infty} \frac{\varrho(n)}{n^2} (\psi \circ \bar{\varphi} \circ n\varphi) \left[\omega_\varphi\left(\frac{1}{n}; f^*\right)\right] \leqq \sum_{n=1}^{\infty} \frac{\varrho(n)}{n^2} (\psi \circ \bar{\varphi}) \left(n \cdot \int_0^{1/n} (\varphi \circ f^*)(x)\, dx\right) \leqq$$

$$\leqq K_1 \int_0^1 \varrho\left(\frac{1}{x}\right) (\psi \circ \bar{\varphi}) \left(\frac{1}{x} \int_0^x (\varphi \circ f^*)(t)\, dt\right) dx \leqq K_2 \int_0^1 \varrho\left(\frac{1}{x}\right) (\psi \circ f^*)(x)\, dx.$$

Hence, using Lemmas 2 and 4, the statement follows.

To prove (6)⇒(5) we apply that from the properties of $\psi \circ \bar{\varphi}$ and ϱ we have

$$\psi(u) \varrho(u) \leqq K \varphi(u) \tag{26}$$

if u is large enough, furthermore applying Lemma 8 we get

$$\int_0^\infty (\psi\circ f^*)(x)\varrho\left(\frac{1}{x}\right)dx = K_1+K_2\sum_{n=1}^\infty \varrho(n)(\psi\circ\bar\varphi\circ\varphi\circ f^*)(n) \leqq$$

$$\leqq K_1+K_3\sum_{n=2}^\infty \varrho(n)(\psi\circ\bar\varphi)\left(\frac{1}{n}\sum_{k=n}^\infty(\varphi\circ f^*)(k)\right)\leqq$$

$$\leqq K_1+K_4\sum_{n=1}^\infty \varrho(n)(\psi\circ\bar\varphi)\left(\frac{1}{n}\hat\omega_\varphi\left(\frac{1}{n};f^*\right)\right).$$

Using Lemmas 4 and 7, and (26) we get that (6)⇒(5).

The proof of (5)⇒(6) runs similarly to that of (6)⇒(5) using Lemma 9 instead of Lemma 8.

To prove (8)⇒(7) we remark, first of all, that from [7] Theorem 1, and from Lemma 4 of the present paper we get

$$\sum_{n=1}^\infty \frac{\lambda_n}{n}\varphi\left(\omega_\varphi\left(\frac{1}{n};f^*\right)\right)<\infty \Rightarrow f\in\varphi(L(0,\infty))\lambda_1(L(0,\infty)). \tag{27}$$

To prove

$$\sum_{n=1}^\infty \frac{\lambda_n}{n}\varphi\left(\hat\omega_\varphi\left(\frac{1}{n};f^*\right)\right)<\infty \Rightarrow f\in\varphi(L(0,\infty))\lambda_2(L(0,\infty)) \tag{28}$$

let $[\alpha_n,\alpha_{n+1})$ denote the interval of values x for which $\frac{1}{(n+1)^2}<f^*(x)\leqq\frac{1}{n^2}$ $(\alpha_0=0)$. If we apply the properties of λ and, furthermore, the property

$$\varphi(tx)\leqq t\varphi(x)\quad (x>0,\ 0<t\leqq 1) \tag{29}$$

of φ then we get

$$\int_0^\infty ((\varphi\lambda_2)\circ f^*)(x)\,dx\leqq K\sum_{n=0}^\infty\int_{\alpha_n}^{\alpha_{n+1}}(\psi\circ f^*)(x)\,dx\sum_{k=1}^n\frac{\lambda_k}{k}\leqq$$

$$\leqq K_1+K_2\sum_{k=1}^\infty\frac{\lambda_k}{k}\left\{\int_{\alpha_k}^k(\varphi\circ f)(x)\,dx+\int_k^\infty(\varphi\circ f^*)(x)\,dx\right\}\leqq$$

$$\leqq K_1+K_3\sum_{k=1}^\infty\frac{\lambda_k}{k}\cdot k\cdot\frac{1}{k^2}+K_4\sum_{k=1}^\infty\frac{\lambda_k}{k}\varphi\left(\hat\omega_\varphi\left(\frac{1}{k};f^*\right)\right)\leqq$$

$$\leqq K_5+K_6\sum_{k=1}^\infty\frac{\lambda_k}{k}\varphi\left(\hat\omega_\varphi\left(\frac{1}{k};f^*\right)\right).$$

Hence we get (28), and by (27) and (28), applying Lemma 4 we have (8)⇒(7).

To prove (7)⇒(8) we show that

(30)
$$\sum_{n=1}^{\infty}\frac{\lambda_n}{n}\varphi\left(\omega_\varphi\left(\frac{1}{n};f^*\right)\right)\leqq K\int_0^\infty(\varphi\circ f^*)(x)\lambda_1\left(\frac{1}{x}\right)dx$$

and

(31)
$$\sum_{n=1}^{\infty}\frac{\lambda_n}{n}\varphi\left(\hat{\omega}_\varphi\left(\frac{1}{n};f^*\right)\right)\leqq K\int_0^\infty(\varphi\circ f^*)(x)\lambda_2\left(\frac{1}{x}\right)dx.$$

Now, (30) follows by Lemma 5, namely

$$\sum_{n=1}^{\infty}\frac{\lambda_n}{n}\varphi\left(\omega_\varphi\left(\frac{1}{n};f^*\right)\right)\leqq\sum_{n=1}^{\infty}\frac{\lambda_n}{n}\sum_{k=n}^{\infty}\int_{\frac{1}{k+1}}^{\frac{1}{k}}(\varphi\circ f^*)(x)\,dx\leqq$$

$$\leqq 2\sum_{n=1}^{\infty}\frac{\lambda_n}{n}\sum_{k=n}^{\infty}\frac{1}{k^2}(\varphi\circ f^*)\left(\frac{1}{k}\right)=2\sum_{k=1}^{\infty}\frac{1}{k^2}(\varphi\circ f^*)\left(\frac{1}{k}\right)\lambda_1(k)\leqq$$

$$\leqq K\int_0^1(\varphi\circ f^*)(x)\lambda_1\left(\frac{1}{x}\right)dx.$$

The proof of (31) runs as follows

$$\sum_{n=1}^{\infty}\frac{\lambda_n}{n}\varphi\left(\hat{\omega}_\varphi\left(\frac{1}{n};f^*\right)\right)=\sum_{n=1}^{\infty}\frac{\lambda_n}{n}\sum_{k=n}^{\infty}\int_k^{k+1}(\varphi\circ f^*)(x)\,dx\leqq$$

$$\leqq\sum_{n=1}^{\infty}\frac{\lambda_n}{n}\sum_{k=n}^{\infty}(\varphi\circ f^*)(k)=\sum_{k=1}^{\infty}(\varphi\circ f^*)(k)\lambda_2\left(\frac{1}{k}\right)\leqq$$

$$\leqq 2(\varphi\circ f^*)(x)\lambda_2\left(\frac{1}{x}\right)dx.$$

So from (30) and (31), and applying Lemmas 2, 4 and 7, we get (7)⇒(8).

Thus our Theorem is completely proved.

References

[1] Г. Гаймназаров, Теорема вложения для $L_p(-\infty, \infty)$ классов функций, *Изв. ВУЗ*, сер. мат., **4** (119) (1972), 44—54.

[2] L. Leindler, On embedding of classes H_p^ω, *Acta Sci. Math.*, **31** (1970), 13—31.

[3] L. Leindler, On imbedding theorems, *Acta Sci. Math.*, **34** (1973), 231—244.

[4] L. Leindler, Inequalities of the Hardy—Littlewood type, *Analysis Math.*, **2** (1972), 117—123.

[5] L. Leindler, Necessary conditions for imbedding of classes of functions, *Analysis Math.*, **1** (1975), 55—61.

[6] J. Németh, Generalization of the Hardy—Littlewood inequality. II, *Acta Sci. Math.*, **35** (1973), 128—134.

[7] J. Németh, Note on imbedding theorems, *Publ. Math. Debrecen* (to appear).

[8] J. Németh, Necessary condition for imbedding of classes of functions, *Analysis Math.*, **3** (1977), 213—219.

[9] Э. А. Стороженко, Необходимые и достаточные условия для вложения некоторых классов функций, *Изв. АН СССР*, сер. мат., **37** (1973), 386—398.

[10] П. Л. Ульянов, Теоремы вложения и наилучшие приближения, *Докл. АН СССР*, **184** (1968), 1044—1047.

[11] П. Л. Ульянов, Вложение некоторых классов функций H_p^ω, *Изв. АН СССР*, **32** (1968), 649—686.

[12] П. Л. Ульянов, Теоремы вложения и соотношения мещду наилучшими приближениями в разных метриках, *Мат. Сб.*, **81** (1971), 104—131.

[13] A. Zygmund, *Trigonometric series*. I (Cambridge, 1959).

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