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Derivations and translations on l-semigroups

H. MITSCH

Introduction. Functions of lattices into themselves have been studied in [4], [7], [9], [10]. With respect to pointwise intersection $\wedge$ and union $\vee$ and to composition of functions $\circ$ the set $(F(L), \wedge, \vee, \circ)$ of all transformations of a lattice $(L, \wedge, \vee)$ forms a "*right-lattice ordered semigroup*" (rl-semigroup; see [3], [4]). This means a set S with three binary operations $\wedge$, $\vee$ and $\cdot$, such that $(S, \cdot)$ is a semigroup, $(S, \wedge, \vee)$ is a lattice and

$$(x\vee y)z = (xz)\vee(yz), \quad (x\wedge y)z = (xz)\wedge(yz) \quad \text{for all} \quad x, y, z\in S.$$

Note that with respect to the order-relation induced by the lattice-operations multiplication satisfies: $x\leqq y\Rightarrow xz\leqq yz$ for each $z\in S$.

Recently SZÁSZ [9], [10] started the investigation of special functions on lattices $(L, \wedge, \vee)$, so-called "*derivations*", motivated by the formal rules of derivations in rings, i.e. transformations φ of L which satisfy

$$\varphi(x\vee y) = \varphi(x)\vee\varphi(y) \quad \text{and} \quad \varphi(x\wedge y) = [\varphi(x)\wedge y]\vee[x\wedge\varphi(y)] \quad \text{for all} \quad x, y\in L.$$

Since in $F(L)$ also the composition of functions is defined, it is natural to consider transformations of the rl-semigroup $(F(L), \wedge, \vee, \circ)$ which satisfy also a formal *chain rule:*

$$\varphi(f\circ g) = [\varphi(f)\circ g]\wedge\varphi(g) \quad \text{for all} \quad f, g\in F(L).$$

For rings with a third operation $\circ$, so-called *composition-rings,* such derivations with chain-rule have been studied — especially for polynomial-rings — in [6], [8].

In the following we suppose S to be a right-lattice ordered semigroup and investigate transformations φ of $(S, \wedge, \vee, \cdot)$ — so-called *C-derivations* — which have the following properties:

$$\left.\begin{aligned} \text{I.}\quad & \varphi(x\vee y) = \varphi(x)\vee\varphi(y) \\ \text{II.}\quad & \varphi(x\wedge y) = [\varphi(x)\wedge y]\vee[x\wedge\varphi(y)] \\ \text{III.}\quad & \varphi(xy) = [\varphi(x)y]\wedge\varphi(y) \end{aligned}\right\} \quad \text{for all} \quad x, y\in S.$$

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Standard examples for S will be the rl-semigroups $(F(L), \wedge, \vee, \circ)$ of all transformations of a lattice L, $(L[x], \wedge, \vee, \circ)$ of all polynomials on L in the indeterminate x and $(P(L), \wedge, \vee, \circ)$ of all polynomial-functions on L (see [3]).

We shall use also the concept of *lattice ordered semigroup* (l-semigroup), which is defined as an rl-semigroup $(S, \wedge, \vee, \cdot)$ satisfying also the two left-distributive laws:

$$x(y\vee z) = (xy)\vee(xz) \quad \text{and} \quad x(y\wedge z) = (xy)\wedge(xz) \quad \text{for all} \quad x, y, z\in S.$$

Note that now multiplication also satisfies: $x\leqq y \Rightarrow zx\leqq zy$ for each $z\in S$; for the general theory see [2].

1. Reduction to translations

The main purpose of this section is to show, that every C-derivation φ on an rl-semigroup S is a special meet-translation [9], i.e. $\varphi(x)=x\wedge a$ for all $x\in S, a\in S$ fixed, if the lattice $(S, \wedge, \vee)$ has a greatest element or if the semigroup $(S, \cdot)$ has an identity.

Properties. Let φ be a C-derivation on S; then

0) $\varphi(x\wedge y)=\varphi(x)\wedge\varphi(y)$ for all $x, y\in S$; $\varphi(x)\leqq x$ for all $x\in S$ ([10]).

1) If S has a least element o, then $\varphi(o)=o$ (by II).

2) If c is a left-zero of $(S, \cdot)$ with least element o, then $\varphi(c)=o$ $(\varphi(c)=\varphi(co)= =\varphi(c)o\wedge\varphi(o)=o)$.

3) If S does not have a least element but admits a left-zero, then there is no C-derivation on S. $(\varphi(c)=\varphi(cx)=\varphi(c)x\wedge\varphi(x)\leqq\varphi(x)\leqq x$ for all $x\in S$ by 0): contradiction.)

4) If S has a right-identity e, then $\varphi(x)\leqq\varphi(e)$ for all $x\in S$. $(\varphi(x)=\varphi(xe)= =\varphi(x)e\wedge\varphi(e)\leqq\varphi(e)$ for all $x\in S$.)

5) If S admits a right-identity e and a least element o, then $\varphi(e)=\varphi(o)$ implies $\varphi(x)=o$ for all $x\in S$.

6) If $(S, \cdot)$ is 0-right-simple with o such that $ox=o$ for all $x\in S$, then $\varphi(a)=o$ for an element $a\neq o$ implies $\varphi(x)=o$ for all $x\in S$. (Since $aS=S$, hence for all $x\in S$ there exists a $y\in S$ with $ay=x$; thus $\varphi(x)=\varphi(ay)=\varphi(a)y\wedge\varphi(y)= =oy\wedge\varphi(y)=o$ for all $x\in S$.)

Examples. 1) If S admits a least element o, then $\varphi(x)=o$ for all $x\in S$ is a C-derivation, the *trivial* C-derivation.

2) The identity-function $\varphi(x)=x$ for all $x\in S$ is a C-derivation, iff $xy\leqq y$ for all $x, y\in S$.

3) The constant function $\varphi(x)=a$ for all $x\in S$, $a\in S$ fixed, is a C-derivation,

iff $a=o$ (if $o\in S$ exists) $(a=\varphi(x\wedge x)=\varphi(x)\wedge x\leqq x$ for all $x)$.

4) Concerning meet-translations we know by Corollary 3 of [10]:

Lemma 1.1. *Let S be an rl-semigroup with greatest element i. Then every C-derivation φ on S has the form $\varphi(x)=x\wedge a$ for all $x\in S$ and a suitable element $a\in S$.*

In order to determine the suitable elements $a\in S$ we prove:

Lemma 1.2. *Let S be an rl-semigroup. The function $\varphi(x)=x\wedge a$ for all $x\in S$ is a C-derivation, iff* 1) *a is a neutral element of $(S, \wedge, \vee)$ and* 2) *$xy\wedge a\leqq ay\wedge y$ for all $x, y\in S$.*

Proof. If $a\in S$ satisfies 1), then $\varphi(x)=x\wedge a$ does I and II of the definition, too. If a also satisfies 2), then $\varphi(xy)=xy\wedge a=(xy\wedge a)\wedge(ay\wedge y)=(x\wedge a)y\wedge(a\wedge y)=$ $=\varphi(x)y\wedge\varphi(y)$ for all $x, y\in S$. The converse is clear.

Combining Lemmas 1.1 and 1.2 we get the following

Theorem 1.3. *Let S be an rl-semigroup with greatest element. Then the C-derivations on S are the functions φ of the form $\varphi(x)=x\wedge a$ with a fixed neutral element a of S such that $xy\wedge a\leqq ay\wedge y$ for each pair x, y of elements of S.*

Example. 5) Let S be an rl-semigroup with identity e admitting an invertible element $a\neq e$; then $\varphi(x)=x\wedge e$ is not a C-derivation. (If φ is a C-derivation, then by Lemma 1.2: $xy\wedge e\leqq y$ for all $x, y\in S$; but since $aa'=a'a=e$ for $a'\in S$, this implies in particular $e\leqq a$ and $e\leqq a'$; consequently we get $a'\leqq aa'=e$, thus $a'=a=e$: contradiction.)

Theorem 1.4. *Let S be an rl-semigroup with greatest element i, such that $ix=i$ for all $x\in S$. Then there is no C-derivation on S except the trivial one (if defined).*

Proof. By Theorem 1.3 every C-derivation on S has the form $\varphi(x)=x\wedge a$ such that $xy\wedge a\leqq ay\wedge y$ for all $x, y\in S$. For $x=i$ we get $a\leqq ay\wedge y\leqq y$ for all $y\in S$. If S has a least element o, then $a=o$ and φ is the trivial C-derivation; if not, then we have a contradiction.

The existence of $i\in S$ ensured that every C-derivation is a special meet-translation. The same is true if an identity exists:

Lemma 1.5. *Let S be an rl-semigroup with right-identity e. Then every C-derivation on S has the form $\varphi(x)=x\wedge a$ for all $x\in S$ with $a=\varphi(e)$.*

Proof. Since $x=x\vee(x\wedge e)$, hence $\varphi(x)\geqq x\wedge\varphi(e)$ for all $x\in S$. But $\varphi(x)\leqq$ $\leqq\varphi(e)$ by 4) and $\varphi(x)\leqq x$ for all $x\in S$ by 0); thus $\varphi(x)\leqq x\wedge\varphi(e)$ and equality follows.

Corollary. *If e is the identity of S and $\varphi(x)=x\wedge a$ is a C-derivation, then:* 1) $a\leqq e$; 2) $a^2=a$; 3) $xy=y$ *for all* $y\leqq a\leqq x\leqq e$.

Proof. Since $a=\varphi(e)=e\wedge a$, we have $a\leqq e$; thus $a^2\leqq a$, $ay\leqq y$ for all $y\in S$. By Lemma 1.2: $xy\wedge a\leqq ay\wedge y=ay$ for all $x, y\in S$; for $x=e$ we get $y\wedge a\leqq ay$. For $y=a$ we obtain $a\leqq a^2$, thus $a^2=a$; for $y\leqq a$ we conclude $y\leqq ay$, so that $ay=y$ for all $y\leqq a$. Now if $a\leqq x\leqq e$, then $y=ay\leqq xy\leqq y$ for all $y\leqq a$ and the assertion follows.

Remark. If S is an rl-semigroup with right-identity which is the least element of S, then there is only the trivial C-derivation on S. The same is true in the following case:

Lemma 1.6. *Let S be a left-simple rl-semigroup with right-identity. Then there is no C-derivation on S except the trivial one (if defined).*

Proof. Again for every C-derivation on S we have: $\varphi(x)=x\wedge a$ with $xy\wedge a\leqq y$ for all $x, y\in S$. Since $Sy=S$ for all $y\in S$, for each $y\in S$ there is an $x\in S$ with $xy=e$; thus by Corollary 1) of Lemma 1.5 we conclude $a=e\wedge a\leqq y$ for all $y\in S$ and $a=o$ (if $o\in S$ exists).

Corollary. *Let $S\neq\{e\}$ be an rl-group; then there is no C-derivation on S.*

Proof. Since a semigroup S is a group iff S is left- and right-simple (see [1]), there is at most the trivial C-derivation $\varphi(x)=o$ on S. But an rl-group cannot have a least element o: $o\leqq e$ implies $o^2=o$ and since the only idempotent in S is e, we get $o=e$; thus $e\leqq a$ for all $a\in S$ implies $a^{-1}\leqq e$, so that $a^{-1}=e$ and $a=e$ for all $a\in S$.

Example. 6) Concerning semigroup-left-translations (see [1]) we note the following: if S is a semigroup with left-identity e and φ a mapping of S into itself such that $\varphi(xy)=\varphi(x)y$ for all $x, y\in S$, then for $x=e$ one gets $\varphi(y)=\varphi(e)y$ for all $y\in S$ and $\varphi(x)=ax$ for all $x\in S$.

Lemma 1.7. *Let S be an rl-semigroup with right-identity e. Then the mapping $\varphi(x)=ax$ for all $x\in S$, $a\in S$ fixed, is a C-derivation iff* 1) *$a\in S$ is left-distributive with respect to $\vee$ and* 2) *$ab=a\wedge b$ for all $b\in S$.*

Proof. If $a\in S$ satisfies 1), then $\varphi(x\vee y)=a(x\vee y)=ax\vee ay=\varphi(x)\vee\varphi(y)$ for all $x, y\in S$. If it also satisfies 2), then $\varphi(x\wedge y)=a(x\wedge y)=a\wedge(x\wedge y)=[\varphi(x)\wedge y]\vee\vee[x\wedge\varphi(y)]$ for all $x, y\in S$. Furthermore, since $ax=a\wedge x\leqq a$ implies $axy\leqq ay$ for all $x, y\in S$, it follows: $\varphi(xy)=axy=(ax)y\wedge ay=\varphi(x)y\wedge\varphi(y)$ for all $x, y\in S$. Conversely, let $\varphi(x)=ax$ be a C-derivation; then by I of the definition: $a(x\vee y)==ax\vee ay$ for all $x, y\in S$; by Lemma 1.5 we have $ax=\varphi(x)=x\wedge\varphi(e)=x\wedge a$, that is $ab=a\wedge b$ for all $b\in S$.

Combining Lemmas 1.5 and 1.7 we get similarly to Theorem 1.3:

Theorem 1.8. *Let S be an rl-semigroup with right-identity. Then the C-derivations on S are the functions φ of the form $\varphi(x)=ax$ with a fixed element $a\in S$ which is left-distributive with respect to $\vee$ such that $ab=a\wedge b$ for all $b\in S$.*

Remark. If S is an rl-semigroup with (right-identity e and) greatest element i, then $\varphi(x)=ax$ such that $ai=i$ is not a C-derivation except $\varphi(x)=x$ (if possible). In fact: if $\varphi(x)=ax=a\wedge x$ for all $x\in S$, then $i=ai=\varphi(i)=a\wedge i=a$ and $\varphi(x)=x$ for all $x\in S$.

For l-semigroups we have:

Theorem 1.9. *Let S be an l-semigroup with identity e, which is the greatest element of S. Then the C-derivations on S are exactly the left-translations $\varphi(x)=ax$ such that $ab=a\wedge b$ for all $b\in S$.*

Proof. On an l-semigroup every function $\varphi(x)=ax$ with $ab=a\wedge b$ for all $b\in S$ is a C-derivation by Lemma 1.7. Conversely, if φ is any C-derivation on S, then by Lemmas 1.2 and 1.5: $\varphi(x)=a\wedge x$ with $xy\wedge a\leqq ay$ for all $x,y\in S$. For $x=e$ we get $y\wedge a\leqq ay$; but $a,y\leqq e$ implies $ay\leqq a$ and $ay\leqq y$, thus $ay\leqq a\wedge y$ and $ay=a\wedge y$ for all $y\in S$. Consequently $\varphi(x)=ax$ for all $x\in S$ with $ab=a\wedge b$ for all $b\in S$.

Corollary. *Let S be a Boolean l-semigroup with identity e (resp. a uniquely complemented l-semigronp with e as greatest element); then the C-derivations on S are exactly the left-translations of S.*

Proof. By the Corollary (resp. Remark) in § 6 of [4] we have in both cases $e=i$ and $xy=x\wedge y$ for all $x,y\in S$.

Returning to general rl-semigroups with identity we show:

Lemma 1.10. *Let S be an rl-semigroup with right-identity e (resp. with greatest element i). Then the set of all C-derivations on S is a commutative, idempotent semigroup with respect to composition of functions: $(\varphi\circ\psi)(x)=\varphi[\psi(x)]$ for all $x\in S$.*

Proof. Let $\varphi(x)=a\wedge x$, $\psi(x)=b\wedge x$ with $a=\varphi(e)$, $b=\psi(e)$ be arbitrary C-derivations on S (see Theorems 1.3 resp. 1.8). Then $(\varphi\circ\psi)(x)=(a\wedge b)\wedge x=c\wedge x$ for all $x\in S$ with $(\varphi\circ\psi)(e)=c\wedge e=c$, since by Corollary 1) to Lemma 1.5: $a\leqq e$, $b\leqq e$, hence $c=a\wedge b\leqq e$. Furthermore, since a and b are neutral, $c=a\wedge b$ is neutral, too. Since $xy\wedge a\leqq ay\wedge y$ and $xy\wedge b\leqq by\wedge y$ for all $x,y\in S$, we get $xy\wedge(a\wedge b)\leqq$ $\leqq(a\wedge b)y\wedge y$ for all $x,y\in S$ and we can apply Lemma 1.2. Trivially we have $(\varphi\circ\psi)(x)=(\psi\circ\varphi)(x)$ and $(\varphi\circ\varphi)(x)=\varphi(x)$ for all $x\in S$.

The results deduced above show, that the class of rl-semigroups which admit non-trivial C-derivations is quite restricted. For concrete examples of rl-semigroups we can prove even more:

Theorem 1.11. *Let $(L, \wedge, \vee)$ be an arbitrary lattice. Then on the rl-semigroups $(F(L), \wedge, \vee, \circ)$ resp. $(P(L), \wedge, \vee, \circ)$ there is no C-derivation except the trivial one (if $o \in L$ exists).*

Proof. We give the proof for $F(L)$. If a least element does not exist, then there is no C-derivation by Property 3. If a least element exists, then for the constant functions $f_a(x)=a, f_o(x)=o$ for all $x \in L$ we have $f_a \circ f_o = f_a$ and $f_o \circ f_a = f_o$ for all $a \in L$. If φ is a C-derivation on $F(L)$, then $\varphi(f_a)=[\varphi(f_a)\circ f_o] \wedge \varphi(f_o) \leqq \varphi(f_o)$ and conversely $\varphi(f_o) \leqq \varphi(f_a)$; thus $\varphi(f_a)=\varphi(f_o)$ for all $a \in L$. Since $F(L)$ has an identity $\mathrm{id}(x)=x$ for all $x \in S$, with respect to $\circ$, we know by Lemma 1.5 that $\varphi(f)=f \wedge \varphi(\mathrm{id})$ for all $f \in F(L)$. Moreover, $\varphi(\mathrm{id}) \leqq \mathrm{id}$ by Property 0). Consequently: $[\varphi(\mathrm{id})](a)=a \wedge [\varphi(\mathrm{id})](a)=f_a(a) \wedge [\varphi(\mathrm{id})](a)=[\varphi(f_a)](a)=[\varphi(f_o)](a) \leqq f_o(a)=o$. Therefore $[\varphi(\mathrm{id})](a)=o$ for all $a \in L$. Thus $\varphi(\mathrm{id})=\theta$, the zero-function on L and $\varphi(f)=f \wedge \theta = \theta$ for all $f \in F(L)$.

The proof of this Theorem depends essentially on the constant functions on L, which are left-zeroes of the semigroup $(F(L), \circ)$. We can generalize it to left-zero l-semigroups with identity e, that means l-semigroups S, such that $xy=x$ for all $x \neq e, y \in S$ (see [1]) — for example the set of all constant functions on a lattice:

Lemma 1.12. *Let S be a left-zero l-semigroup with identity e. Then there are no C-derivations on S except $\varphi(x)=o$ and $\varphi(x)=x$ for all $x \in S$ (if defined).*

Proof. By Lemma 1.5, $\varphi(x)=x \wedge \varphi(e)$ for all $x \in S$. If there is no least element in S, then by Property 3) there is no C-derivation on S. If there is $o \in S$, then $\varphi(x)=o$ for all $x \neq e$ in S by Property 2) Thus we have to determine only $\varphi(e)$: if $\varphi(e) \neq e$, $\varphi(e)$ is a left-zero of S and $\varphi(e)=\varphi[\varphi(e)]=o$ by Lemma 1.10; if $\varphi(e)=e$, we have for any $x \neq e$: $o=\varphi(x)=x \wedge e$. If e is not the greatest element, then there is an $x > e$ and $o=x \wedge e=e=\varphi(e)$; if e is the greatest element, then $o=x \wedge e=x$ for all $x \neq e$ in S, $S=\{o, e\}$ and $\varphi(x)=x$ for all $x \in S$.

2. Derivations with dual chain-rule

As mentioned above, a large class of rl-semigroups admits only the trivial C-derivation (if defined). Even the standard examples of mappings resp. polynomial-functions on lattices belong to this class. Therefore the abstraction of derivation of functions, which formalizes the rules of differentiating a sum, a product and the composite of functions, turns out to be not very useful. Also if axiom III of a C-derivation is replaced by its dual:

III′. $\varphi(xy)=\varphi(x)y \vee \varphi(y)$ for all $x, y \in S$

we get nothing new. We can show even more:

Theorem 2.1. *Let S be an rl-semigroup with identity e resp. $o=ox$ for all $x\in S$ (if $o\in S$ exists). Then there is no derivation with dual chain-rule except the trivial one (if defined).*

Proof. If S admits no least element and if φ is any mapping satisfying I, II and III′, then $\varphi(x)=\varphi(xe)=\varphi(x)e\vee\varphi(e)\geqq\varphi(e)$ for all $x\in S$; but $\varphi(x)\leqq x$ for all $x\in S$ by Property 0) (valid also in this case) and thus $\varphi(e)$ is the least element of S: contradiction. If S admits o with $ox=o$ for all $x\in S$, then $\varphi(o)=\varphi(ox)==\varphi(o)x\vee\varphi(x)\geqq\varphi(x)$ for all $x\in S$; by Axiom I the mapping φ is order-preserving, hence $\varphi(o)\leqq\varphi(x)$ for all $x\in S$ and $\varphi(x)=\varphi(o)=a$ with some $a\in S$, for all $x\in S$; by Axiom II we have $a=\varphi(x)=\varphi(x\wedge x)=\varphi(x)\wedge x\leqq x$ for all $x\in S$ and $a=o$; thus $\varphi(x)=o$ for all $x\in S$.

Remark. Motivated by the properties of "derivations of formal languages", which are in short lattice-endomorphisms of the l-semigroup of all formal languages on an alphabet X satisfying the dual chain rule III′, the Axiom II of a derivation finally may be replaced by

II′. $\varphi(x\wedge y)=\varphi(x)\wedge\varphi(y)$ for all $x, y\in S$.

Such "derivations" are studied in [5].

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MATH. INSTITUT, UNIVERSITÄT WIEN
STRUDLHOFGASSE 4
1090 WIEN, AUSTRIA