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Dependence of associativity conditions for ternary operations

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1. A ternary operation on a set S is a map on $S \times S \times S$ to S. For the sake of convenience we denote the ternary image of an ordered triple (a, b, c) on S by (a, b, c) itself. If

$$\big((a, b, c)d, e\big) = \big(a, (b, c, d), e\big) = \big(a, b, (c, d, e)\big) \quad \text{for} \quad a, b, c, d, e \in S,$$

we say that the associativity conditions for the sequence of elements a, b, c, d, e, corresponding to the ternary operation $(*, *, *)$ hold [1]. The associativity conditions for ternary operations in S are said to be independent if none of them is implied by the rest [3]. If the associativity conditions are not independent, they are said to be dependent. In other words, the associativity conditions are dependent if for every ternary operation some of the associativity conditions are implied by the rest [3].

2. In an earlier paper [2] we have shown that if a set S consists of more than five elements then the associativity conditions for ternary operations are necessarily independent. In this paper we complete the information by establishing the following theorem:

Theorem. *For a set S having five or less than five elements, the associativity conditions for ternary operations are dependent.*

For proving the theorem, we divide all the associativity conditions of S into two sets P and Q, P being the set of all associativity conditions in which at least two of the five elements are different, and Q the set of those in which all the five elements are the same. To establish our theorem we show that holding of all associativity conditions in P implies at least one of Q. We do this by showing that not holding of all the conditions in Q and holding of all in P lead to a contradiction.

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3. We use the following lemmas in the proof of the theorem. In all the lemmas it is assumed that the associativity conditions in P hold.

Lemma 1. *If $(a, a, a)=b\ (\neq a)$ then $(a, b, b)=(b, a, b)=(b, b, a)$.*

Proof.

$$(a, b, b) = (a, (a, a, a), b) = ((a, a, a), a, b) = (b, a, b),$$
$$(b, a, b) = (b, a, (a, a, a)) = (b, (a, a, a), a) = (b, b, a).$$

Lemma 2. *If $(a, a, a)=b(\neq a)$ and any one of the triples (b, a, a), (a, b, a), (a, a, b) is a, then all of them are a.*

Proof. If $(b, a, a)=a$ then

$$a = (b, a, a) = (b, (b, a, a), a) = (b, b, (a, a, a)) = (b, b, b);$$

if $(a, b, a) = a$ then

$$a = (a, b, a) = (a, b, (a, b, a)) = (a, (b, a, b), a) = (a, (a, b, b), a) =$$
$$= (a, a, (b, b, a)) = (a, a, (a, b, b)) = ((a, a, a), b, b) = (b, b, b),$$

by Lemma 1; while if $(a, a, b)=a$ then

$$a = (a, a, b) = (a, (a, a, b), b) = ((a, a, a), b, b) = (b, b, b).$$

In each case we have got $(b, b, b)=a(\neq b)$ which implies

$$(a, b, a) = (b, a, a) = (a, a, b) = a$$

by Lemma 1. This proves the lemma.

Lemma 3. *If $(a, a, a)=b(\neq a)$ and any one of the triples (b, a, a), (a, b, a), (a, a, b) is b, then all of them are b.*

Proof. If $(b, a, a)=b$ then

$$b = (b, a, a) = ((b, a, a), a, a) = (b, (a, a, a), a) = (b, b, a) = (b, a, b) = (a, b, b)$$

by Lemma 1; if $(a, b, a)=b$ then

$$(b, a, a) = ((a, b, a), a, a) = (a, b, (a, a, a)) = (a, b, b),$$
$$(a, a, b) = (a, a, (a, b, a)) = ((a, a, a), b, a) = (b, b, a)$$

and, again by Lemma 1,

$$b = (a, b, a) = (a, (a, b, a), a) = ((a, a, b), a, a) = ((b, b, a), a, a) = ((a, b, b), a, a) =$$
$$= ((b, a, a), a, a) = (b, a, (a, a, a)) = (b, a, b) = (a, b, b) = (b, b, a);$$

if $(a, a, b)=b$ then

$$b = (a, a, b) = (a, a, (a, a, b)) = (a, (a, a, a), b) = (a, b, b) = (b, a, b) = (b, b, a).$$

Further, if $b=(a, b, b)=(b, a, b)=(b, b, a)$ then

$$(b, a, a) = \big((a, b, b), a, a\big) = \big(a, (b, b, a), a\big) = (a, b, a),$$

$$(a, b, a) = \big(a, (a, b, b), a\big) = \big(a, a, (b, b, a)\big) = (a, a, b).$$

Hence the lemma is proved.

Lemma 4. *Let* $(a, a, a)=b$ *and* $(b, a, a)=c$. *Then*

(i) $(b, b, b)=(a, b, c)=(b, a, c)=(b, c, a)=(c, a, b)=(c, b, a)$ *and* $(c, a, c)=(c, c, a)$. *Moreover,*

(ii) *if* $(a, a, b)=d$ *then* $(b, b, b)=(b, d, a)=(b, a, d)=(d, a, b)$ *and* $(d, a, d)=(d, d, a)$;

(iii) *if* $(a, b, a)=e$ *then* $(b, b, b)=(b, a, e)=(b, e, a)=(a, b, e)$ *and* $(a, e, e)=(b, b, e)$;

(iv) *if* $(a, a, b)=c$ *and* $(a, b, a)=d$ *then* $(b, b, b)=(b, a, d)=(b, d, a)=(a, b, d)$ *and* $(a, d, d)=(b, b, d)$.

Proof. Since $(a, a, a)=b$, by Lemma 1 we have $(a, b, b)=(b, a, b)=(b, b, a)$.

(i)
$$\begin{aligned}
(a, b, c) &= \big(a, b, (b, a, a)\big) = \big((a, b, b), a, a\big) = \big((b, b, a), a, a\big) = \\
&= \big(b, b, (a, a, a)\big) = (b, b, b),\\
(b, a, c) &= \big((a, a, a), a, c\big) = \big(a, (a, a, a), c\big) = (a, b, c) = (b, b, b),\\
(b, c, a) &= \big(b, (b, a, a), a\big) = \big(b, b, (a, a, a)\big) = (b, b, b),\\
(c, a, b) &= \big((b, a, a), a, b\big) = \big(b, (a, a, a), b\big) = (b, b, b),\\
(c, b, a) &= \big(c, (a, a, a), a\big) = \big(c, a, (a, a, a)\big) = (c, a, b) = (b, b, b),\\
(c, a, c) &= \big(c, a, (b, a, a)\big) = \big((c, a, b), a, a\big) = \big((c, b, a), a, a\big) = \\
&= \big(c, (b, a, a), a\big) = (c, c, a).
\end{aligned}$$

(ii)
$$\begin{aligned}
(b, d, a) &= \big(b, (a, a, b), a\big) = \big((b, a, a), b, a\big) = (c, b, a) = (b, b, b),\\
(b, a, d) &= \big(b, a, (a, a, b)\big) = \big(b, (a, a, a), b\big) = (b, b, b),\\
(d, a, b) &= \big((a, a, b), a, b\big) = \big(a, a, (b, a, b)\big) = \big(a, a, (a, b, b)\big) = \\
&= \big((a, a, a), b, b\big) = (b, b, b).\\
(d, a, d) &= \big((a, a, b), a, d\big) = \big(a, a, (b, a, d)\big) = \big(a, a, (b, d, a)\big) = \\
&= \big((a, a, b), d, a\big) = (d, d, a).
\end{aligned}$$

(iii)
$$\begin{aligned}
(b, a, e) &= \big(b, a, (a, b, a)\big) = \big((b, a, a), b, a\big) = (c, b, a) = (b, b, b),\\
(b, e, a) &= \big(b, (a, b, a), a\big) = \big(b, a, (b, a, a)\big) = (b, a, c) = (b, b, b),\\
(a, b, e) &= \big(a, b, (a, b, a)\big) = \big(a, (b, a, b), a\big) = \big(a, (b, b, a), a\big) = \\
&= \big(a, b, (b, a, a)\big) = (a, b, c) = (b, b, b).
\end{aligned}$$

$$(a, e, e) = \big(a, (a, b, a), e\big) = \big(a, a, (b, a, e)\big) = \big(a, a, (a, b, e)\big) =$$
$$= \big((a, a, a), b, e\big) = (b, b, e).$$

(iv) follows immediately from (iii) if e and d are replaced by d and c, respectively.

4. Proof of the theorem. We proceed step by step choosing first sets with one element only, then with two elements and so on, and finally with five elements. We note that our hypothesis, not holding of all the associativity conditions in Q, is the same as

$$\text{for all } x \in S, \text{ either } \quad \big((x, x, x), x, x\big) \neq \big(x, (x, x, x), x\big)$$
$$\text{or} \quad \big(x, x, (x, x, x)\big) \neq \big(x, (x, x, x), x\big)$$
$$\text{or} \quad \big((x, x, x), x, x\big) \neq \big(x, x, (x, x, x)\big),$$

which implies in particular that

$$(x, x, x) \neq x. \tag{4.1}$$

Step I. Let S consist of one element only, say, a. Then clearly $(a, a, a) = a$, which contradicts (4.1).

Step II. Let S consist of two distinct elements a, b, say. Then, in view of (4.1), $(a, a, a) = b$ and $(b, b, b) = a$. Hence $\big((a, a, a), a, a\big) = (b, a, a)$, $\big(a, (a, a, a), a\big) = (a, b, a)$, $\big(a, a, (a, a, a)\big) = (a, a, b)$. On the other hand,

$$(b, a, a) = \big(b, (b, b, b), a\big) = \big((b, b, b), b, a\big) = (a, b, a),$$
$$(a, b, a) = \big(a, b, (b, b, b)\big) = \big(a, (b, b, b), b\big) = (a, a, b),$$

which contradicts our hypothesis concerning Q.

Step III. Let S consist of three distinct elements a, b, c, say. Under the hypothesis $(a, a, a) \neq a$, let us denote $b = (a, a, a)$. Further, since all the triples (b, a, a), (a, b, a), (a, a, b) cannot be equal to c (see Step II), at least one of them must be either a or b. However, if any one of them equals a then all equal a by Lemma 2, and if any one of them equals b, all of them get equal to b by Lemma 3. Both cases are in contradiction with the hypothesis as laid down in (4.1).

Step IV. Let S consist of four distinct elements a, b, c, d, say. Then, as in Step III, we fix $b = (a, a, a)$. Further, as demonstrated in Step III, none of the triples (b, a, a), (a, b, a), (a, a, b) can be equal to a or b. Hence it will be sufficient for us to consider the case when two of these triples are equal either to c or d and the third one to d or c, respectively. Thus without loss of generality we can consider the cases

(1) $\quad (b, a, a) = c, \quad (a, a, b) = d \quad$ and $\quad (a, b, a) = c \quad$ or $\quad d,$

(2) $\quad (b, a, a) = c = (a, a, b) \quad$ and $\quad (a, b, a) = d.$

As $(a, a, (b, a, a))=((a, a, b), a, a)=(a, (a, b, a), a)$, we have

(3) in case (1): $(a, a, c)=(d, a, a)\big(=(a, c, a)$ or $(a, d, a)\big)$

and

(4) in case (2): $(a, a, c)=(c, a, a)=(a, d, a)$.

We distinguish four subcases according to the value of (a, a, c).

Step IVA. If $(a, a, c)=a$ then by Lemma 4 (i)

$$b = (a, a, a) = \big(a, a\,(a, a, c)\big) = \big(a, (a, a, a), c\big) = (a, b, c) = (b, b, b)$$

which is in contradiction with (4.1).

Step IVB. Case (1). If $(a, a, c)=b$ then $(d, a, a)=b$ (by (3)). Furthermore,

$$c = (b, a, a) = \big((d, a, a), a, a\big) = \big(d, a, (a, a, a)\big) = (d, a, b) = (b, b, b)$$

by Lemma 4 (ii) and

$$d = (a, a, b) = \big(a, a, (a, a, c)\big) = \big(a, (a, a, a), c\big) = (a, b, c) = (b, b, b)$$

by Lemma 4 (i).

Case (2). If $(a, a, c)=b$ then

$$c = (a, a, b) = \big(a, a, (a, a, c)\big) = \big(a, (a, a, a), c\big) = (a, b, c) = (b, b, b)$$

and

$$d = (a, b, a) = \big(a, (a, a, c), a\big) = \big((a, a, a), c, a\big) = (b, c, a) = (b, b, b)$$

by Lemma 4 (i). Thus in both the cases $c=d$, which is a contradiction.

Step IVC. If $(a, a, c)=c$ then

$$c = (a, a, c) = \big(a, a, (a, a, c)\big) = \big((a, a, a), a, c\big) = (b, a, c) =$$
$$= \big(b, a, (a, a, c)\big) = \big((b, a, a), a, c\big) = (c, a, c) = (c, c, a)$$

by Lemma 4 (i) and hence

$$c = (c, a, c) = \big((c, c, a), a, c\big) = \big(c, c, (a, a, c)\big) = (c, c, c).$$

But $(c, c, c)=c$ is in contradiction with (4.1).

Step IVD. Case (1). If $(a, a, c)=d$ then $(d, a, a)=d$, too (by (3)), whence

$$d = (d, a, a) = \big((d, a, a), a, a\big) = \big(d, a, (a, a, a)\big) = (d, a, b) =$$
$$= \big((d, a, a), a, b\big) = \big(d, a, (a, a, b)\big) = (d, a, d) = (d, d, a)$$

by Lemma 4 (ii), so that

$$d = (d, a, b) = \big((d, d, a), a, b\big) = \big(d, d, (a, a, b)\big) = (d, d, d).$$

Case (2). If $(a, a, c)=d$ then $(a, d, a)=d$, too (by (4)). Furthermore,

$$d = (a, d, a) = \bigl(a, (a, a, c), a\bigr) = \bigl((a, a, a), c, a\bigr) = (b, c, a) = (b, b, b) = (b, d, a)$$

by Lemma 4 (i), (iv) and thus

$$d = (b, d, a) = \bigl(b, (b, d, a), a\bigr) = \bigl(b, (b, a, d), a\bigr) = \bigl(b, b, (a, d, a)\bigr) =$$
$$= (b, b, d) = (a, d, d) = \bigl(a, d, (a, d, d)\bigr) = \bigl((a, d, a), d, d\bigr) = (d, d, d)$$

by Lemma 4 (iv) again. But $(d, d, d)=d$ is again a contradiction.

Step V. Let S consist of five distinct elements a, b, c, d, e, say. We have to consider only the case when $(b, a, a)=c$, $(a, a, b)=d$ and $(a, b, a)=e$.

As $(a, a, (b, a, a))=((a, a, b), a, a)=(a, (a, b, a), a)$, we have $(a, a, c)=(d, a, a)=(a, e, a)$. If $(a, a, c)=a$ or b or c or d, a contradiction can be established as in the case (1) of Step IV. If $(a, a, c)=(d, a, a)=(a, e, a)=e$ then

$$e = (a, e, a) = \bigl(a, (a, a, c), a\bigr) = \bigl((a, a, a), c, a\bigr) = (b, c, a) = (b, d, a) = (b, e, a)$$

by Lemma 4 (i)—(iii), whence

$$e = (b, e, a) = \bigl(b, (b, d, a), a\bigr) = \bigl(b, b, (d, a, a)\bigr) = (b, b, e) = (a, e, e)$$

by Lemma 4 (iii), implying that

$$e = (a, e, e) = \bigl(a, e, (a, e, e)\bigr) = \bigl((a, e, a), e, e\bigr) = (e, e, e).$$

But $(e, e, e)=e$ is in contradiction with (4.1).

This completes the proof of the theorem.

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