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Attractivity theorems for non-autonomous systems of differential equations

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Dedicated to Professor Béla Szőkefalvi-Nagy on his 65th birthday

1. Introduction

By the classical theorem proved by A. M. LJAPUNOV [1] in 1892 the zero solution of the system $\dot{x}=f(t,x)$ $(t\geqq 0,\ x\in R^n;\ f(t,0)\equiv 0)$ is asymptotically stable provided that there exists a positive definite scalar function $V(t,x)$ tending to zero uniformly in $t\in[0,\infty)$ as $x\to 0$ and having a negative definite derivative $\dot{V}(t,x)$ with respect to the system. Since the early days of stability theory numerous authors have dealt with weakening the conditions of this theorem. There are two main types of attempts.

In theorems belonging to the first type special assumptions are required of the vector field $f(t,x)$ independently of the Ljapunov function $V(t,x)$. The first theorem of this type is due to M. MARAČKOV [2], who assumed $f(t,x)$ to be bounded for all t when x belongs to an arbitrary compact set instead of the condition of $V(t,x)$ tending to 0, uniformly with respect to t, as $x\to 0$. Considering autonomous systems E. A. BARBAŠIN and N. N. KRASOVSKIĬ [3] generalized Ljapunov's theorem to the case when the function $\dot{V}(t,x)$ is not negative definite. By the method of several Ljapunov functions V. M. MATROSOV [4] extended this result to those non-autonomous systems whose right-hand side $f(t,x)$ is bounded for all t when x belongs to an arbitrary compact set. For the systems of the same kind T. YOSHIZAWA [5] and J. P. LASALLE [6] gave sufficient conditions for the attractivity of closed sets. A given set $H\subset R^n$ is called *attractive* if every solution starting from some neighbourhood of H tends to H as $t\to\infty$. In 1976 LaSalle extended his theorem by weakening the condition of boundedness of $f(t,x)$ [7, Th. 1].

Results of the second type are characterized by the fact that the direct conditions on the right-hand side $f(t,x)$ are omitted but certain relations between

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the function $\dot{V}(t, x)$ and the norm $\|f(t, x)\|$ of the right-hand side are required. The most important theorems of this type are due to T. A. BURTON [8] and J. R. HADDOCK [9].

The purpose of this paper is to improve some results of both types in the following two directions. On the one hand, we give the role of $f(t, x)$ to the derivative $\dot{W}(t, x)$ of a function $W: R^n \to R^k$ with respect to the system. On the other hand, in the theorems of the first type we refine the estimates on $\dot{V}(t, x)$ so that we should be able to take into account the finer structure of the "dangerous set" defined by $\dot{V}(t, x)=0$, which depends on the time t in the non-autonomous case. At the end of our paper we give examples to illustrate how our results relate to the above mentioned ones, and applications are given to the study of the asymptotic behaviour of solutions of non-linear second order differential equations.

2. Notations and definitions

The basic differential equation is

$$\dot{x} = f(t, x), \tag{2.1}$$

where $t \in R_+ = [0, \infty)$, and x belongs to the n-dimensional Euclidean space R^n. The function f is defined and continuous on the set $\Gamma^* = R_+ \times G^*$; G^* is an open set in R^n.

Denote by (x, y), $\|x\|$ and $d(x, y)$ the scalar product, norm and distance in R^n, respectively; namely $(x, y) = \sum_{i=1}^{n} x_i y_i$, $\|x\| = (x, x)^{1/2}$ and $d(x, y) = \|x - y\|$. Let R^n_∞ denote the one-point compactification of R^n and define $d(x, \infty) = 1/\|x\|$. For a set $H \subset R^n$ we denote the complement of H by H^c, the closure of H by $\bar{H}$, and the set $H \cup \{\infty\}$ in R^n_∞ by H_∞. For a set $K \subset R^n_\infty$, define $d(x, K) = \inf\{d(x, y): y \in K\}$. If $d(u(t), K) \to 0$ as $t \to \omega - 0$ for a continuous function $u: [0, \omega) \to R^n$, we shall say $u(t) \to K$ as $t \to \omega - 0$.

For $H \subset R^n$, $\varepsilon > 0$ the set $S(H, \varepsilon) = \{x \in R^n: d(x, H) < \varepsilon\}$ is called the ε-neighbourhood of H. We shall need another neighbourhood system. Let a set $G \subset R^n$ and a continuous function $W: G \to R^k$ be given. If $p \in G$ and $\varrho > 0$, we shall use the notation $S^*(p, \varrho) = W^{-1}[S(W(p), \varrho)]$, where $W^{-1}[H]$ denotes the inverse image of $H \subset R^k$ with respect to W.

Let $x(t)$ be a solution of (2.1) defined on a maximal right interval $[t_0, \omega)$ $(t_0 < \omega \leqq \infty)$. A point p is a *positive limit point* of $x(t)$ if there exists a sequence $\{t_m\}$ such that $t_m \to \omega - 0$ and $x(t_m) \to p$ as $m \to \infty$. The *positive limit set Ω of $x(t)$* is the set of all its positive limit points. If $x(t)$ is bounded and $\Omega \subset G^*$, then $\omega = \infty$, Ω is nonempty, compact, connected and is the smallest closed set that $x(t)$ approaches as $t \to \infty$.

Denote by $C^1(D; R^k)$ the family of all functions $W: D(\subset R^m) \to R^k$ whose components have continuous first partial derivatives. For a function $u \in C^1(R_+ \times R^n; R)$ define the function

$$\dot{u}(t, x) = \sum_{i=1}^{n} \frac{\partial u(t, x)}{\partial x_i} f_i(t, x) + \frac{\partial u(t, x)}{\partial t}$$

which is said to be the *derivative of u with respect to equation* (2.1). The derivative of a vector-function $U \in C^1(R_+ \times R^n; R^k)$ with respect to (2.1) is the vector of the derivatives of the components of U with respect to (2.1).

System (2.1) is non-autonomous, so its solutions $x(t)$ can be represented by the graph $(t, x(t))$ in R^{n+1}. A solution $x(t)$ is said to be in $\Gamma \subset \Gamma^*$ if $(t, x(t)) \in \Gamma$ for all $t \in [t_0, \omega)$. For a given set $\Gamma \subset \Gamma^*$ we shall use the notations

$$G(t) = \{x: (t, x) \in \Gamma\}, \quad G = \bigcup_{t \geqq 0} G(t).$$

Denote by $[a]_+$ and $[a]_-$ the positive and negative part of the real number a, respectively.

Definition 2.1. Let Γ be a subset of Γ^*. We say that $V \in C^1(\Gamma; R)$ is a *Ljapunov function on* Γ if there exists a continuous function $\eta: R_+ \to R_+$ such that

$$\int_0^\infty \eta(t)\, dt < \infty, \quad [\dot{V}(t, x)]_+ \leqq \eta(t) \quad ((t, x) \in \Gamma).$$

Let A be a property concerning the functions V and $\dot{V}$. "Property A is satisfied on the set $t \geqq T, x \in H(\subset R^n)$" if it is satisfied on the subset of $[T, \infty) \times H$ where the Ljapunov function is defined, i.e. on the set $\{(t, x): t \geqq T, x \in H \cap G(t)\}$.

3. Theorems and proofs

In this section we study attractivity conditions of a given set with respect to system (2.1). Namely, we seek conditions assuring that the set contains the positive limit sets of solutions of (2.1).

Assume that we have a Ljapunov function V on Γ and an auxiliary function $W \in C^1(\bar{G} \cap G^*; R^k)$.

Lemma 3.1. *Let $x(t)$ be a solution with maximal right-interval of definition $[t_0, \omega)$, and let $M \subset G$ be a set such that $x(t) \in M$ for $t \in [t_0, \omega)$.*

Suppose that for a point $p \in \bar{G} \cap G^*$ there exist $\delta, \varrho > 0$ and T such that

(i) $V(t, x)$ is bounded from below and

(3.1) *(ii) $\dot{V}(t, x) \leqq -\delta \|\dot{W}(t, x)\| + \eta(t)$*

on the set $t \geqq T, x \in S^*(p, \varrho) \cap M$.

Then either a) $p \notin \Omega$ or b) $\omega = \infty$ and $\Omega \cap G^* \subset W^{-1}[W(p)]$.

Proof. Suppose the contrary of a), i.e. $p \in \Omega$. Since $p \in G^*$ and G^* is open, according to the theorem of continuation of solutions $\omega = \infty$ holds. Suppose b) is false, too. Then there exist $q \in \Omega \cap G^*$ and σ $(0 < \sigma < \varrho/\sqrt{k})$ such that $q \notin S^*(p, \sqrt{k}\sigma)$. Because of $p, q \in \Omega$ there exist a natural number l $(1 \leqq l \leqq k)$ and two sequences $\{t'_m\}$, $\{t''_m\}$ with the following properties:

$$T \leqq t'_1 < t''_1 < \ldots < t'_m < t''_m < \ldots; \quad \lim_{m \to \infty} t'_m = \infty; \tag{3.2}$$

$$\|W(p) - W(x(t))\| < \sigma \sqrt{k} \quad (t'_m \leqq t \leqq t''_m); \tag{3.3}$$

$$|W_l(x(t''_m)) - W_l(x(t'_m))| = \frac{\sigma}{2} \quad (m = 1, 2, \ldots). \tag{3.4}$$

By assumption (ii), for the function $v(t) = V(t, x(t))$ the estimation

$$v(t''_m) - v(t'_m) \leqq -\delta \frac{\sigma}{2} + \int_{t'_m}^{t''_m} \eta(t)\, dt \quad (m = 1, 2, \ldots) \tag{3.5}$$

is satisfied, from which it follows that

$$v(t''_m) \leqq v(t'_1) - m\delta \frac{\sigma}{2} + \int_T^{t''_m} \eta(t)\, dt \to -\infty \quad (m \to \infty),$$

and this contradicts assumption (i).

The lemma is proved.

Remark 3.1. If either the function W is scalar $(k=1)$ or assumptions (i)—(ii) are required on the set $t \geqq T$, $x \in M$, then assumption (ii) may be required of the function $[\dot{W}]_+$ instead of $\dot{W}$. In the first case the statement is unchanged; in the second case it can be stated that either a) $\Omega \cap G^*$ is empty or b) $\omega = \infty$ and there exists a $p \in \bar{M} \cap G^*$ such that $\Omega \cap G^* \subset W^{-1}[W(p)]$.

Indeed, if either the function W is scalar or property (3.3) is not required, then we may also assume property (3.4) is true without the absolute value sign. Then for deduction of inequality (3.5) it is sufficient to require assumption (ii) of the function $[\dot{W}]_+$ instead of $\dot{W}$.

Theorem 3.1. *Let the sets $H \subset R^n$, $M \subset G$ be given and suppose that for any $p \in H^c$ there exist $\varrho(p) > 0$, $\delta(p) > 0$ and $T(p)$ such that assumptions (i)—(ii) in Lemma 3.1 are satisfied on the set $t \geqq T(p)$, $x \in S^*(p, \varrho(p)) \cap M$.*

1) *If $x(t)$ is a solution and $x(t) \in M$ for $t \in [t_0, \omega)$, then either a) $\Omega \cap G^* \subset H$ or b) $\omega = \infty$ and there exists a $d \in R^k$ such that the set $W^{-1}[d] \cap H^c$ is non-empty and $\Omega \cap G^* \subset W^{-1}[d]$.*

2) *If also assumption $\bar{G} \subset G^*$ is satisfied, then either a) $x(t) \to H_\infty$ as $t \to \omega - 0$ or b) $\omega = \infty$ and there exists a $d \in R^k$ such that the set $W^{-1}[d] \cap H^c$ is non-empty and $x(t) \to W^{-1}[d]_\infty$ as $t \to \infty$.*

In the case of a scalar function W $(k=1)$ the statements remain true after replacing function $\dot{W}$ with $[\dot{W}]_+$ in assumption (3.1).

Proof. 1) If the set $\Omega\cap G^*$ is empty, then a) is true. Suppose that it is not empty, and there exists a $p\in\Omega\cap G^*$ such that $p\in H^c$. Then, by Lemma 3.1 (and Remark 3.1) b) is true, namely $d=W(p)$.

2) The statements follow from those under 1) and from the fact that $\Omega\subset G^*$.

Theorem 3.2. *Let the set $M\subset G$ be given, and suppose that there exist $\delta>0$, $T\geqq 0$ such that*

(i) $V(t,x)$ is bounded from below and

(ii) $\dot{V}(t,x)\leqq -\delta\|[\dot{W}(t,x)]_+\|+\eta(t)$

on the set $t\geqq T$, $x\in M$.

If $x(t)$ is any solution and $x(t)\in M$ for $t\in[t_0,\omega)$, then either a) the set $\Omega\cap G^*$ is empty or b) $\omega=\infty$, and there exists a $d\in R^k$ such that $\Omega\cap G^*\subset W^{-1}[d]$.

Proof. Applying Lemma 3.1 and Remark 3.1, the theorem can be proved in the same manner as Th. 3.1.

Our theorems can be used not only for studying stability properties of sets but also for establishing various kinds of asymptotic properties of solutions. For example, let us take $G^*=R^n$, $H=\{0\}$, $W(x)=(x,x)$; furthermore, let $V(t,x)$ be a Ljapunov function on the set $R_+\times R^n$ bounded from below for all $t\in R_+$ when x belongs to an arbitrary compact set. Suppose that for any point $p\neq 0$ there exist $\delta>0$, $\varrho>0$, T such that

$$\dot{V}(t,x)\leqq -\delta\big[(f(t,x),x)\big]_{+(-)}+\eta(t)$$

for $t\geqq T$, $\big|\|x\|-\|p\|\big|<\varrho$, where the symbol $[\cdot]_{+(-)}$ means that either the positive part $[\cdot]_+$ or the negative part $[\cdot]_-$ is considered for all (t,x). By Th. 3.1 these assumptions imply that for any solution $x(t)$ either a) the function $\|x(t)\|$ has a finite limit as $t\to\infty$ or b) $x(t)\to\infty$ as $t\to\omega-0$.

From Th. 3.1 by the choice of $W(x)=x$ an important result mentioned in the Introduction follows.

Corollary 3.1. (J. Haddock [9, Th. 3]). *Let $G^*=R^n$, $H\subset R^n$ be a closed set, and $V(t,x)$ be a Ljapunov function on $R_+\times R^n$ bounded from below for all $t\in R_+$ when x belongs to an arbitrary compact set. Suppose that for any $\varepsilon>0$ and any compact set $K\subset R^n$ there exist $\delta(\varepsilon,K)>0$, $T(\varepsilon,K)$ such that*

$$\dot{V}(t,x)\leqq -\delta\|f(t,x)\|+\eta(t) \tag{3.6}$$

on the set $t\geqq T$, $x\in K\cap S^c(H,\varepsilon)$.

If $x(t)$ is any solution, then either a) $x(t)\to H_\infty$ as $t\to\omega-0$ or b) $x(t)\to p$ as $t\to\infty$ for some $p\in H^c$.

In certain cases the fact that assumption (3.6) contains the non-monotonic function $\|\cdot\|$ can cause difficulties. This can be avoided by means of the last statement of Th. 3.1 in the following way: Suppose that $V(t, x)$ is a Ljapunov function on $R_+ \times R^n$, and for any $\varepsilon > 0$ and any $C > 0$ there exist $\delta(\varepsilon, C) > 0$ and $T(\varepsilon, C)$ such that $V(t, x)$ is bounded from below and

$$\dot V(t, x) \leqq -\delta [f_i(t, x)]_{+(-)} + \eta(t) \quad (i = 1, 2, \dots) \tag{3.7}$$

on the set $t \geqq T$, $x \in S^c(H, \varepsilon) \cap \{x \in R^n : |x_i| \leqq C\}$. Then the statement of Cor. 3.1 is true. If (3.7) is satisfied only for a fixed i, then instead of b) it can be stated only $x_i(t) \to p_i$ as $t \to \infty$ (see Th. 3.1, $W(x) = +(-)x_i$).

Having certain "a priori" (independent of the function $V(t, x)$) informations about the function $\dot W(t, x)$, we can replace assumption (ii) in Lemma 3.1 with another one to improve the previous theorems in some respects.

Definition 3.1. A measurable function $\varphi : R_+ \to R$ is said to be *integrally positive* (see [4], [11]) if $\int_I \varphi(t)\,dt = \infty$ holds on every set $I = \bigcup_{m=1}^{\infty} [\alpha_m, \beta_m]$ such that

$$\alpha_m < \beta_m < \alpha_{m+1}, \quad \beta_m - \alpha_m \geqq \delta > 0 \quad (m = 1, 2, \dots).$$

A function $\varphi(t)$ is said to be *integrally negative* if $-\varphi(t)$ is integrally positive.

Lemma 3.2. *Let $x(t)$ be a solution and let $M \subset G$ be an arcwise connected set such that $x(t) \in M$ for all $t \in [t_0, \omega)$.*

Suppose that for a point $p \in \bar G \cap G^*$ there exist $\varrho > 0$ and T such that for any continuous function $u : [T, \infty) \to L = S^*(p, \varrho) \cap M$ the following conditions are satisfied:

(i) $\int_T^t \dot W(s, u(s))\,ds$ *is uniformly continuous,*

(ii) $\dot V(t, u(t))$ *is integrally negative, and*

(iii) $V(t, u(t))$ *is bounded from below on the interval* $[T, \infty)$.

Then $p \notin \Omega$.

Lemma 3.3. *The statement of Lemma 3.2 remains true if conditions* (i)—(ii) *are replaced with the following: for any continuous function $u : [T, \infty) \to L$*

(i′) $\left\| \int_T^{\infty} |\dot W(t, u(t))|\,dt \right\| < \infty,$

(ii′) $\int_T^{\infty} \dot V(t, u(t))\,dt = -\infty.$

Proof of Lemmas 3.2 and 3.3. Assume the contrary, i.e. $p\in\Omega$. Then $\omega=\infty$, and there exists a sequence $\{t_m\}$ such that $t_m\to\infty$ and $x(t_m)\to p$ as $m\to\infty$. On the other hand, however large the time T^* may be, the set $S^*(p,\varrho)$ must not contain the point $x(t)$ for *all* $t\geqq T^*$ because of assumptions (ii) ((ii′), respectively) and (iii). Consequently, in the same manner as in the proof of Lemma 3.1, there are $\sigma>0$, l $(1\leqq l\leqq k)$ and sequences $\{t'_m\}$, $\{t''_m\}$ with properties (3.2)—(3.4). Then we have

$$\left\|\int_{t'_m}^{t''_m} \dot W(t,x(t))\,dt\right\|\geqq\frac{\sigma}{2}\quad(m=1,2,\dots). \tag{3.8}$$

This contradicts (i′); therefore Lemma 3.3 is proved.

To prove Lemma 3.2 we show that (3.8) contradicts assumptions (i)—(iii), too. Indeed, (3.8) and (i) imply that $t''_m-t'_m\geqq\delta$ for all m with some $\delta>0$. The function $\dot V(t,x(t))$ is integrally negative, consequently

$$V(x(t''_m))\leqq \text{const.}+\sum_{i=1}^{m}\int_{t'_i}^{t''_i}\dot V(t,x(t))\,dt\to-\infty\quad(m\to\infty),$$

which contradicts the boundedness from below of the function $V(t,x(t))$.

The proof of both lemmas is complete.

Remark 3.2. If the function W is scalar $(k=1)$ then assumption (i) (assumption (i′), respectively) may be required of the function $[\dot W]_+$ instead of $\dot W$ ($|\dot W|$, respectively); the statements remain true.

Now suppose that for the derivative of the Ljapunov function V an inequality

$$\dot V(t,x)\leqq\varphi(t)U(x)+\eta(t)\quad((t,x)\in\Gamma) \tag{3.9}$$

holds with continuous functions $\varphi\colon R_+\to R_+$, $U\colon\bar G\cap G^*\to R_-$, $\eta\colon R_+\to R_+$ (the function η is integrable on $[0,\infty)$ by Def. 2.1). Denote by F the so called "dangerous set":

$$F=\{x\in\bar G\cap G^*\colon U(x)=0\},$$

which is closed with respect to G^*.

Theorem 3.3. *Let $M\subset G$ be an arcwise connected set, and suppose that for any $p\in F^c$ there exist $\varrho(p)>0$, $T(p)$ such that:*

(i) $\sup\{U(x)\colon x\in L(p)=S^*(p,\varrho)\cap M\}<0$;

(ii) $\varphi(t)$ is integrally positive;

moreover, for any continuous function $u\colon[T(p),\infty)\to L(p)$

(iii) $\int_T^t\dot W(s,u(s))\,ds$ is uniformly continuous, and

(iv) $V(t, u(t))$ is bounded from below on the interval $[T(p), \infty)$.

If $x(t)$ is any solution and $x(t)\in M$ $(t_0\leqq t<\omega)$, then $\Omega\cap G^*\subset F$.

If also assumption $\bar{G}\subset G^*$ is satisfied, then $x(t)\to F_\infty$ as $t\to\omega-0$.

Theorem 3.4. *The statements of Th. 3.3 remain true if assumptions (ii)—(iii) are replaced with the following ones:*

(ii') $\int\limits_0^\infty \varphi(t)\,dt=\infty;$

(iii') $\left\|\int\limits_T^\infty |\dot{W}(t, u(t))|\,dt\right\|<\infty.$

Proof of Theorems 3.3 and 3.4. Suppose the contrary, i.e. the set $\Omega\cap G^*$ contains a point p not belonging to the set F. Then, by assumptions, there are $\varrho(p)>0$, $\delta(p)>0$, $T(p)$ such that inequality

$$\dot{V}(t, x)\leqq-\delta\varphi(t)+\eta(t)\quad (t\geqq T(p),\ x\in L(p))$$

holds. Hence, using Lemma 3.2 (and Lemma 3.3, respectively) we get $p\notin\Omega$, which contradicts our earlier assumption on p.

$\bar{G}\subset G^*$ implies inclusion $\Omega\subset G^*$. Consequently we have $\Omega\subset F$, so $x(t)\to F_\infty$ as $t\to\omega-0$.

The proof is complete.

Remark 3.3. In case of scalar function $W(k=1)$ the statements of Theorems 3.3 and 3.4 remain true after replacing function $\dot{W}$ (function $|\dot{W}|$, respectively) with $[\dot{W}]_+$ in assumption (iii) (assumption (iii'), respectively).

For example, by the choice $W(x)=(x, x)$, from Th. 3.3 we get the following: Suppose $V(t, x)$ is a Ljapunov function on $R_+\times R^n$ bounded from below for all $t\in R_+$ when x belongs to an arbitrary compact set. Further, suppose there exist continuous functions $\varphi\colon R_+\to R_+$, $a\colon R_+\to R_+$ such that $a(0)=0$, $a(r)>0$ for $r>0$; φ is integrally positive, and

$$\dot{V}(t, x)\leqq-\varphi(t)a(\|x\|)\quad ((t, x)\in\Gamma=R_+\times R^n).$$

If for any r_1, r_2 $(0<r_1<r_2)$ and for any continuous function $u\colon [T, \infty)\to\to\{x\in R^n\colon r_1\leqq\|x\|\leqq r_2\}$ the function $\int\limits_0^t [(f(s, u(s)), u(s))]_{+(-)}ds$ is uniformly continuous on $[0, \infty)$, then for any solution $x(t)$ either $x(t)\to 0$ or $x(t)\to\infty$ as $t\to\omega-0$.

Similarly to the previous ones, Th. 3.3 yields an important result when $W(x)=x$.

Corollary 3.2. *Suppose the estimation* (3.9) *is satisfied with an integrally positive function* φ. *Further suppose that for any compact set* $K\subset R^n$ *and any continuous function* $u\colon [T, \infty)\to K$ *the function* $\int_T^t f(s, u(s))\,ds$ *is uniformly continuous and* $V(t, u(t))$ *is bounded from below on the interval* $[T, \infty)$. *Then for any solution the inclusion* $\Omega\cap G^*\subset F$ *holds.*

From the point of view of applications the most important case is when the function V is differentiable and f is continuous, and this corollary is an improvement of the LaSalle theorem [7, Th. 1], which can be obtained from it by setting $\varphi(t)\equiv 1$. It will be shown by examples taken from the theory of nonlinear oscillations that even in simple cases it is necessary to introduce the function φ into estimation (3.9).

Remark 3.4. Associate with the functions V and W the set $\mathscr{F}\subset R^n$ defined as follows: $x\in\mathscr{F}$ iff there exists a sequence (t_m, x^m) such that $t_m\to\infty$, $W(x^m)\to W(x)$ and $\dot V(t_m, x^m)-\eta(t)\to 0$ as $m\to\infty$. Similarly as in Th. 3.3, from Lemma 3.2 we can derive a statement assuring the inclusion $\Omega\cap G^*\subset\mathscr{F}$ for any solution of (2.1). In this way it can be generalized a result given by N. Onuchic et al. [13, Th. 1], who introduced the set $\mathscr{F}$ in case of $W(x)=x$, $\eta(t)=0$. Even in this special case, obviously, there are functions f, V, φ and U such that $\mathscr{F}\supset F$, and what is more the set $\mathscr{F}$ is too large to obtain any information about the place of Ω in R^n by the inclusion $\Omega\cap G^*\subset\mathscr{F}$ (e.g. $\dot V(t, x)=\sin^2 t\cdot U(x)$). This fact motivates estimation (3.9). Moreover, if the functions φ and U are chosen in (3.9) "sufficiently well" and φ is bounded, then $\mathscr{F}\supset F$.

Remark 3.5. The key assumption in Th. 3.3 is (iii), which assures the point $x(t)$ not to go away in the same distance from the attractor F infinitely many times within a shorter and shorter time. Even in the special case of $W(x)=x$, the uniform continuity of the function $\int_T^t \sup\{\|f(s, x\|\colon x\in S(p, \varrho)\cap M\}\,ds$ on $[T, \infty)$ is often checked instead of assumption (iii). Before LaSalle's paper [7], in [12] the author used already an assumption equivalent to this one to assure the above mentioned property of the solutions.

4. Applications and examples

I. Let us consider the non-linear differential equation of second order

$$(p(t)\dot x)^{\cdot}+q(t)f(x) = 0 \quad (x\in R), \tag{4.1}$$

where the functions $p, q\colon R_+\to R_+$ are continuously differentiable and $\dot p(t)\geqq 0$,

$\dot{q}(t) \leqq 0$ $(t \in R_+)$; the function $f: R \to R$ is continuous and $xf(x) \geqq 0$ $(x \in R)$; $F(x) = \int\limits_0^x f(s)\,ds$.

Apply Th. 3.2 to the study of asymptotic behaviour of the coordinate x and the momentum $y = p(t)\dot{x}$. In terms of these Hamiltonian variables equation (4.1) has the form

$$\dot{x} = \big(1/p(t)\big)y, \quad \dot{y} = -q(t)f(x). \tag{4.1'}$$

Let us now consider functions

$$V(t, x, y) = \frac{1}{p(t)}\, y^2 + 2q(t)F(x), \quad W(x, y) = xy,$$

whose derivatives by virtue of (4.1′) are

$$\dot{V} = -\frac{\dot{p}(t)}{p^2(t)}\, y^2 + 2\dot{q}(t)F(x), \quad \dot{W} = \frac{1}{p(t)}\, y^2 - q(t)xf(x).$$

On the one hand, if there exist $\gamma_1 > 0$, $T_1 \in R_+$ such that

$$\dot{p}(t)/p(t) \geqq \gamma_1 > 0 \quad (t \geqq T_1), \tag{4.2}$$

then

$$\dot{V} \leqq -\frac{\dot{p}(t)}{p^2(t)}\, y^2 \leqq -\gamma_1 \frac{1}{p(t)}\, y^2 \leqq \gamma_1 [\dot{W}]_+ \quad (t \geqq T_1;\ x, y \in R).$$

On the other hand, if there exist $\gamma_2, \gamma_3 > 0$ and $T_2 \in R$ such that

$$\frac{\dot{q}(t)}{q(t)} \leqq -\gamma_2 < 0 \quad (t \geqq T_2); \quad \gamma_3 F(x) > xf(x) \quad (x \in R), \tag{4.3}$$

then

$$\dot{V} \leqq 2\dot{q}(t)F(x) \leqq -\frac{2\gamma_2}{\gamma_3} \cdot \gamma_3 q(t)F(x) \leqq -\frac{2\gamma_2}{\gamma_3}\, [-W]_+$$

for $t \geqq T_2;\ x, y \in R$.

Applying Th. 3.2 with $M = G = R^2$ and with the functions V, W and V, $-W$, respectively, we get: for any solution of (4.1′) either $|x(t)| + |y(t)| \to \infty$, as $t \to \omega - 0$ or $\omega = \infty$ and $\lim\limits_{t \to \infty} \big(x(t)y(t)\big)$ exists. By the first equation of system (4.1′) $(x^2)^{\cdot} = 2xy/p(t)$, hence, if

$$\int\limits_0^\infty \big(1/p(t)\big)\,dt < \infty, \quad \int\limits_0^\infty q(t)\,dt < \infty, \tag{4.4}$$

then $\lim\limits_{t \to \infty} x(t)$, $\lim\limits_{t \to \infty} y(t)$ exist. Thus, we have:

Suppose either (4.2) *or* (4.3), *and let* $x(t)$ *be any solution of* (4.1) *with the maximal right interval* $[t_0, \omega)$. *Then either a)* $|x(t)|+|p(t)\dot{x}(t)|\to\infty$ *as* $t\to\omega-0$ *or b)* $\omega=\infty$ *and* $\lim_{t\to\infty}(p(t)x(t)\dot{x}(t))$ *exists. If also condition* (4.4) *is satisfied, then in case b) we can state* $\lim_{t\to\infty} x(t)$, $\lim_{t\to\infty}(p(t)\dot{x}(t))$ *exist.*

II. Let us now consider the equation

$$\ddot{x}+a(t)\dot{x}+b(t)f(x)=0 \quad (x\in R), \tag{4.5}$$

where the functions $a\colon R_+\to R$, $f\colon R\to R$ are continuous, $b\colon R_+\to R$ is continuously differentiable. By the aid of Th. 3.3, we seek for conditions which assure that the derivative of any solution of (4.5) tends to 0 as $t\to\infty$.

Introducing the variable $y=\dot{x}$, we can transform equation (4.5) into the system

$$\dot{x}=y, \quad \dot{y}=-b(t)f(x)-a(t)y. \tag{4.5'}$$

Choose the Ljapunov functions

$$V_1(t,x,y)=\frac{y^2}{b(t)}+2F(x); \quad V_2(t,x,y)=\frac{y^2}{2}+b(t)F(x)$$

(see [11]), where $F(x)=\int_0^x f(s)\,ds$, and the auxiliary function $W(x)=(x^2/2, y^2/2)$. Their total derivatives by virtue of (4.5') are

$$\dot{V}_1=-\varphi_1(t)y^2, \quad \varphi_1(t)=\frac{2a(t)}{b(t)}+\frac{\dot{b}(t)}{b^2(t)};$$

$$\dot{V}_2=-a(t)y^2+b(t)F(x), \quad \dot{W}=\left(xy,\ -b(t)yf(x)-a(t)y^2\right).$$

Applying Th. 3.3 with the functions V_1, W and V_2, W, respectively, we obtain the following results: *Suppose the function* $\int_0^t(|a(s)|+|b(s)|)\,ds$ *is uniformly continuous on* R_+.

1) *If either* $b(t)>0$ *or there exists a* $\gamma>0$ *such that* $b(t)\leqq-\gamma$ *for values of t large enough, and* $\varphi_1(t)$ *is integrally positive, then for any solution* $x(t)$ *of* (4.5) *either a)* $|x(t)|+|\dot{x}(t)|\to\infty$ *as* $t\to\omega-0$ *or b)* $\dot{x}(t)\to 0$ *as* $t\to\infty$.

2) *If* $F(x)\geqq 0$ $(x\in R)$, $b(t)$ *is bounded from below* $(t\in R_+)$, *and* $a(t)$ *is integrally positive, then for any solution* $x(t)$ *of* (4.5) *either a) or b) is satisfied.*

III. Finally, in order to compare our results with those of LaSalle and Haddock, we investigate attractivity properties of the solutions of the linear system

$$\begin{aligned}\dot{x}&=-r(t)x+q(t)y\\ \dot{y}&=-q(t)x-p(t)y\end{aligned} \quad (x,y\in R), \tag{4.6}$$

where $p,q,r\colon R_+\to R$ are continuous, and $p(t)\geqq 0$, $r(t)\geqq 0$ $(t\in R_+)$. Choose the

Ljapunov function $V(x,y)=(x^2+y^2)/2$. Its derivative by virtue of (4.6) $\dot V(t,x,y)=-r(t)x^2-p(t)y^2$ is non-positive, so any solution of (4.6) exists and is bounded on the whole R_+. The LaSalle—Yoshizawa theorem yields the following statement:

A) (J. P. LaSalle [6]: $r(t)\equiv 0$, $q(t)\equiv 1$). If $0<c<p(t)\leqq C$ ($t\in R_+$; $c, C=$ $=$const.), then for every solution of (4.6) $x(t)\to$const., $y(t)\to 0$ as $t\to\infty$.

Haddock deduced from his theorem (see Cor. 3.1 in this paper) the following result:

B) (J. Haddock [10]: $r(t)\equiv 0$). If there exists $\alpha>0$ such that

$$|q(t)|<\alpha p(t)\quad (t\in R_+), \tag{4.7}$$

and $\int_0^\infty p(t)\,dt=\infty$, then for any solution of (4.6) $y(t)\to 0$, $x(t)\to$const. as $t\to\infty$.

Let us now consider the auxiliary function $W(y)=y^2/2$, whose derivative is $\dot W=-q(t)xy-p(t)y^2$, and denote by H the set of the points of x-axis on the plane (x,y). We prove all the conditions of Th. 3.1 are satisfied, provided that (4.7) is true. For any solution $(x(t),y(t))$ of (4.6) there exists a C such that $(x(t),y(t))\in M=\{(x,y):x^2+y^2\leqq C\}$. It is sufficient to show that for every $\varepsilon>0$ there exists a $\delta>0$ such that $(x,y)\in M$, $|y|\geqq\varepsilon$ imply $\dot V(t,x)\leqq -\delta[\dot W(t,x)]_+$ for all $t\in R_+$. Let $\delta=2\varepsilon^2/(\alpha C)$. Then from (4.7) it follows that $-p(t)y^2\leqq$ $\leqq -\delta|q(t)|(x^2+y^2)/2$ $(t\in R_+)$, which implies the desired inequality.

By Th. 3.1, using also the fact that $\lim_{t\to\infty} V(x(t),y(t))$ exists, we obtain the following result:

1) *Suppose* (4.7). *Then both of the components of any solution of* (4.6) *tend to a finite limit as* $t\to\infty$. *If* $\int_0^\infty p(t)\,dt=\infty$ *is also satisfied, then* $x(t)\to$const., $y(t)\to 0$ *as* $t\to\infty$.

It is worth noting that if we applied Haddock's theorem to this case, then in order to get the same result we would also have to require the condition analogous to (4.7) of the function $r(t)$. On the other hand, condition (4.7) is too strong, since it requires much of $q(t)$ *locally* at certain points (e.g. $p(t_0)=0$ implies $q(t_0)=0$), nevertheless the conclusion is only about the *limits* of the solutions. By the aid of Theorems 3.3 and 3.4 ($\varphi(t)=p(t)$, $U=-y^2$, $[\dot W]_+\leqq[q(t)xy]_+$) we do without assumption (4.7):

2) *If* $p(t)$ *is integrally positive and* $\int_0^t |q(s)|\,ds$ *is uniformly continuous on* R_+, *then for any solution of* (4.6) $x(t)\to$const, $y(t)\to 0$ *as* $t\to\infty$.

3) *If* $\int_0^\infty p(t)\,dt=\infty$ *and* $\int_0^\infty |q(t)|\,dt<\infty$, *then for any solution of* (4.6) $x(t)\to\text{const.}$, $y(t)\to 0$ *as* $t\to\infty$.

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