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On automorphism groups of subalgebras of a universal algebra

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Let A be a universal algebra and let Con (A), Sub (A), Aut (A) denote the lattice of all congruences of A, the lattice of all subalgebras of A, and the automorphism group of A, respectively. First in a series of so-called independence results is that of E. T. SCHMIDT [6] asserting that Aut (A) is independent of Sub (A). W. A. LAMPE [5] gave a construction representing any pair of nontrivial algebraic lattices and an arbitrary group as Sub (A), Con (A), and Aut (A) of a finitary algebra A.

Once these results are established, somewhat more detailed investigations of the structures associated with a universal algebra appear to be in order; we would like to formulate further possible questions in this field. For every finitary algebra A there are two obvious homomorphisms H_1: Aut $(A) \to$ Aut (Sub (A)) and H_2: Aut $(A) \to$ Aut (Con (A)) of the respective groups. Given a quintuple (G, L_1, H_1, L_2, H_2) in which G is a group, L_1 and L_2 are algebraic lattices, and H_i: $G \to$ Aut (L_i) are group homomorphisms, one may ask under what circumstances there is an algebra A with Aut $(A) \cong G$, $L_1 \cong$ Sub (A), $L_2 \cong$ Con (A), and H_1, H_2 the two natural homomorphisms as above. [1] states that an arbitrary triple (G, L_1, H_1) is representable in this way. There appears to be no corresponding result for the triple (G, L_2, H_2).

The aim of this note is to prove a partial result concerning the relationship of the subalgebra lattice and the automorphism groups of subalgebras of a finitary algebra. It is well known that automorphism groups of pairs algebra-subalgebra can be chosen arbitrarily, and similar claim is valid for endomorphism monoids as well ([3] and [4], see also [2]). The question we ask is this: what are the systems $(G_x\colon x \in L)$ of groups appearing as automorphism groups of subalgebras of a finitary algebra A whose subalgebra lattice is isomorphic to L?

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To be more precise, let A be a finitary algebra and let

$$H_A\colon \mathrm{Aut}(A) \to \mathrm{Aut}(\mathrm{Sub}(A)) \tag{1}$$

be defined by $(H_A(\alpha))(B)=\alpha^+(B)=\{\alpha(b)\colon b\in B\}$ for $\alpha\in \mathrm{Aut}(A)$ and $B\in \mathrm{Sub}(A)$. H_A is a group homomorphism; if $\mathrm{Aut}_B(A)$ denotes the subgroup of $\mathrm{Aut}(A)$ consisting of all those automorphisms α of A for which $\alpha^+(B)=B$, then

$$\mathrm{Ker}(H_A) \subseteqq \mathrm{Aut}_B(A) \quad \text{for every} \quad B\in \mathrm{Sub}(A). \tag{2}$$

The restriction $R_{AB}(\beta)$ of a $\beta\in \mathrm{Aut}_B(A)$ to B is an automorphism of B and the mapping

$$R_{AB}\colon \mathrm{Aut}_B(A) \to \mathrm{Aut}(B) \tag{3}$$

is a group homomorphism.

We will restrict our attention to the special case

$$\mathrm{Ker}(H_B) = \mathrm{Aut}(B) \quad \text{for all} \quad B\in \mathrm{Sub}(A), \tag{4}$$

that is, it will be assumed that, for every $\alpha\in \mathrm{Aut}(B)$, α^+ acts trivially on $\mathrm{Sub}(B)$ for each $B\in \mathrm{Sub}(A)$. It follows that $\mathrm{Aut}_C(B)=\mathrm{Aut}(B)$ and thus $\mathrm{Aut}(B)$ is the domain of R_{BC} for any pair $C\subseteqq B$ of subalgebras of A.

An algebraic lattice L is isomorphic to the lattice $I(C)$ of all ideals of the join semilattice C of all non-zero compact elements of L. If $J\in I(C)\cong L\cong \mathrm{Sub}(A)$, let A_J denote the subalgebra of A corresponding to the ideal J of C; for a principal ideal $J=(c]$ write A_c instead of A_J. Recall that J is principal if and only if A_J is finitely generated and that $A_J=\bigcup(A_c\colon c\in J)$ for every $J\in I(C)$. It is easy to see that an automorphism $\alpha\colon A_J\to A_J$ acts trivially on $\mathrm{Sub}(A_J)$ if and only if

$$\alpha^+(A_c) = A_c \quad \text{for all} \quad c\in J. \tag{5}$$

Thus the restriction (4) is equivalent to (5) being valid for all $J\in I(C)$. If $c\geqq d$ is a pair of elements of C, let $R_{cd}(\alpha)$ denote the restriction of $\alpha\in \mathrm{Aut}(A_c)$ to A_d. The system of homomorphisms

$$(R_{cd}\colon \mathrm{Aut}(A_c) \to \mathrm{Aut}(A_d),\ c \geqq d \text{ in } C) \tag{6}$$

satisfies

$$\begin{aligned} R_{de}\circ R_{cd} &= R_{ce} \quad \text{for all} \quad c\geqq d\geqq e \text{ in } C,\\ R_{ee} &= \mathrm{id}_{A_e} \quad \text{for all} \quad e\in C \end{aligned} \tag{7}$$

under the restriction (4).

If $d, e\in C$, $c=d\vee e$, then $R_{cd}(\alpha)\in \mathrm{Aut}(A_d)$ and $R_{ce}(\alpha)\in \mathrm{Aut}(A_e)$; if both $R_{cd}(\alpha)$ and $R_{ce}(\alpha)$ are identity automorphisms, then α is the identity automorphism of A_c since A_c is generated by $A_d\cup A_e$. Thus $\mathrm{Aut}(A_c)$ is a subgroup of $\mathrm{Aut}(A_d)\times \times \mathrm{Aut}(A_e)$; in other words, $\mathrm{Ker}(R_{cd})\cap \mathrm{Ker}(R_{ce})$ is trivial whenever $c=d\vee e$ in C.

If $J \in I(C)$ is non-principal, then $A_J = \bigcup(A_c : c \in J)$ and, because of (5), each $\alpha \in \operatorname{Aut}(A_J)$ determines a system $(\alpha_c \in \operatorname{Aut}(A_c) : c \in J)$ such that $R_{cd}(\alpha_c) = \alpha_d$ whenever $c \geqq d$ belong to J. Conversely, let $(\alpha_c : c \in J)$ be a system of automorphisms $\alpha_c \in \operatorname{Aut}(A_c)$ such that $R_{cd}(\alpha_c) = \alpha_d$ for all pairs $c \geqq d$ in J. If $d, e \in J$, then $d \vee e = f \in J$ and the equality $\alpha_d(x) = R_{fd}(\alpha_f)(x) = R_{fe}(\alpha_f)(x) = \alpha_e(x)$ holds for all $x \in A_d \cap A_e$. Thus we may define a mapping $\alpha : A_J \to A_J$ by $\alpha(x) = \alpha_c(x)$ for all $x \in A_c$; it is easy to see that α is an automorphism of A_J; α is the identity automorphism if and only if all α_c are identities. $\operatorname{Aut}(A_J)$ is therefore uniquely determined by the system

$$S = (R_{cd} \colon c \geqq d \text{ in } J)$$

of group homomorphisms. S is closed under composition; let $R_c \colon \operatorname{Aut}(A_J) \to \operatorname{Aut}(A_c)$ be the homomorphism that assigns to every $\alpha \in \operatorname{Aut}(A_J)$ its restriction $\alpha_c : A_c \to A_c$. A straightforward argument shows that $\operatorname{Aut}(A_J)$ is isomorphic to the inverse limit of the diagram S with the homomorphisms R_c playing the role of projections.

Now let $L \cong I(C)$ be an algebraic lattice, let G_x be a group for every $x \in L$, and let $r_{cd} : G_c \to G_d$ be a group homomorphism for every pair $c \geqq d$ of elements of C, let r_{cc} be the identity endomorphism of G_c. We say that a system

(8) $$\Sigma = (L, (G_x \colon x \in L), (r_{cd} \colon c \geqq d \text{ in } C))$$

is *representable* if there is a finitary algebra A such that

(9) $$\operatorname{Sub}(A) \cong L,$$

(10) $$\alpha^+(A_y) = A_y \quad \text{for all} \quad y \leqq x \quad \text{and all} \quad \alpha \in \operatorname{Aut}(A_x),$$

(11) $$\operatorname{Aut}(A_x) \cong G_x \quad \text{for every} \quad x \in L,$$

(12) each r_{cd} represents the restriction homomorphism $R_{cd} \colon \operatorname{Aut}(A_c) \to \operatorname{Aut}(A_d)$.

The statement below characterizes representability of Σ.

Theorem. *Σ is representable if and only if*

(a) $r_{de} \circ r_{cd} = r_{ce}$ *for all* $c \geqq d \geqq e$ *in* C,

(b) $\operatorname{Ker}(r_{cd}) \cap \operatorname{Ker}(r_{ce})$ *is trivial whenever* $d \vee e = c$,

(c) *if $x \in L$ is not compact, then G_x is the inverse limit of the diagram* $(r_{cd} : x > c \geqq d,\ c, d \in C)$.

Proof. We have already seen that (a), (b), (c) are consequences of representability of Σ. To prove the converse, define an algebra A as follows: its underlying set is the disjoint union of all groups G_c for $c \in C$ and its operations are defined by the formulae below.

(13) If $g \in G_c$, define a unary operation $\bar{g}$ by

$$\bar{g}(h) = hg \quad \text{if} \quad h \in G_c;$$
$$\bar{g}(h) = h \quad \text{if} \quad h \notin G_c,$$

(14) If $c>d$ are elements of C, F_{cd} is a unary operation defined as

$$F_{cd}(h) = r_{cd}(h) \quad \text{if} \quad h\in G_c;$$
$$F_{cd}(h) = h \qquad \text{if} \quad h\notin G_c.$$

(15) A single binary operation $*$:

$$g_1 * g_2 = g \quad \text{if} \quad g_1\in G_d, \quad g_2\in G_e, \quad g\in G_c, \quad c = d\vee e, \quad r_{cd}(g) = g_1, \quad r_{ce}(g) = g_2;$$
$$g_1 * g_2 = g_1 \quad \text{otherwise.}$$

Note that (b) implies that $*$ is well-defined.

First we will show that B is a subalgebra of A if and only if B is the (disjoint) union A_I of the groups $G_c(c\in I)$ for some ideal I of C (including $I=\emptyset$); this yields (9) immediately. It is easy to see that each A_I is a subalgebra of A; conversely, if $B\in \mathrm{Sub}\,(A)$, set

$$I = \{c\in C\colon B\cap G_c \neq \emptyset\}.$$

If $I=\emptyset$, then $B=\emptyset$ as well; let $c\in I$ and let $h\in B\cap G_c$. If $g\in G_c$, then $h^{-1}g=k$ belongs to G_c and $\bar{k}(h)=hh^{-1}g=g\in B$. Hence $I=\{c\in C\colon G_c\subseteq B\}$. If $d\in C$ and $d<c\in I$, then $F_{cd}(1_c)=r_{cd}(1_c)=1_d$ for the unit elements $1_c\in G_c$ and $1_d\in G_d$; thus $1_d\in B$, and $d\in I$ as well. $1_c * 1_d=1_{c\vee d}\in B$ whenever $c, d\in I$; hence I is an ideal, and $B=A_I$. A nonempty A_I is finitely generated (one-generated, in fact) if and only if I is a principal ideal.

Let $I=(c]$. $A_I=\bigcup(G_d\colon d\leqq c)$ in this case; for every $g\in G_c$ define a mapping $g^*\colon A_I\to A_I$ by $g^*(h)=r_{cd}(g)\cdot h$ for $h\in G_d$, $d\leqq c$. Observe that $(g_1g_2)^*(h)= =r_{cd}(g_1g_2)\cdot h=r_{cd}(g_1)\cdot r_{cd}(g_2)\cdot h=g_1^*(r_{cd}(g_2)\cdot h)=g_1^*(g_2^*(h))$, and that g^* is the identity mapping on A_I only if $1_c=g^*(1_c)=r_{cc}(g)\cdot 1_c=g$. Hence $g\to g^*$ is a one-to-one homomorphism of G_c into the symmetric group on A_I. To show that $g^*\in \mathrm{Aut}\,(A_I)$, choose a $k\in G_e\,(e\leqq c)$ first. If $h\in G_d\subseteq A_I$, then $\bar{k}(g^*(h))=\bar{k}(r_{cd}(g)\cdot h)= =r_{cd}(g)\cdot h=g^*(h)=g^*(\bar{k}(h))$ if $d\neq e$, and $\bar{k}(g^*(h))=\bar{k}(r_{cd}(g)\cdot h)=r_{cd}(g)\cdot h\cdot k= =g^*(h\cdot k)=g^*(\bar{k}(h))$ if $d=e$. Secondly, let $d>e$ in C. For any $h\in G_f$ with $f\neq d$ we have $g^*(F_{de}(h))=g^*(h)=F_{de}(g^*(h))$. If $f=d$; then $F_{de}(g^*(h))=F_{de}(r_{cd}(g)\cdot h)= =r_{de}(r_{cd}(g)\cdot h)=r_{de}(r_{cd}(g))\cdot r_{de}(h)=r_{ce}(g)\cdot r_{de}(h)=r_{ce}(g)\cdot F_{de}(h)=g^*(F_{de}(h))$ since all r_{cc} are homomorphisms satisfying (a).

Now let $d, e\leqq c$, $f=d\vee e$ and let $h_1\in G_d$, $h_2\in G_e$ be such that there is an $h\in G_f$ with $r_{fd}(h)=h_1$ and $r_{fe}(h)=h_2$. Then $g^*(h_1 * h_2)=g^*(h)=r_{cf}(g)\cdot h$, and $g^*(h_1) * g^*(h_2)=(r_{cd}(g)\cdot h_1) * (r_{ce}(g)\cdot h_2)=(r_{cd}(g)\cdot r_{fd}(h)) * (r_{ce}(g)\cdot r_{fe}(h))= =r_{fd}(r_{cf}(g)\cdot h) * r_{fe}(r_{cf}(g)\cdot h)=r_{cf}(g)\cdot h$ by (15). To deal with the second clause of (15), assume $g^*(h_1)=r_{fd}(k)$ and $g^*(h_2)=r_{fe}(k)$ for a $k=g^*(h_1) * g^*(h_2)$ in G_f. Then $r_{fd}(k)=r_{fd}(r_{cf}(g))\cdot h_1$ and $r_{fe}(k)=r_{fe}(r_{cf}(g))\cdot h_2$ imply that $h_1= =r_{fd}(r_{cf}(g^{-1})\cdot k)$ and $h_2=r_{fe}(r_{cf}(g^{-1})\cdot k)$. Thus $h_1 * h_2=r_{cf}(g^{-1})\cdot k\in G_f$ and $g^*(h_1 * h_2)=r_{cf}(gg^{-1})\cdot k=k=g^*(h_1) * g^*(h_2)$ as required. This proves that $g\to g^*$ is an embedding of G_c into $\mathrm{Aut}\,(A_c)$.

Let $I \neq \emptyset$ be an ideal of C, let $c \in I$, and let $\alpha \in \mathrm{Aut}(A_I)$ be arbitrary. If $I = \{c\}$, then $\alpha \in \mathrm{Aut}(A_c)$ and, in particular, $\alpha(h) = \alpha(\bar{h}(1_c)) = \bar{h}(\alpha(1_c)) = \alpha(1_c) \cdot h$ for every $h \in G_c$. $g = \alpha(1_c) \in G_c$ and $\alpha = g^*$, that is, we know that $\mathrm{Aut}(A_I) \cong G_I$ in this case. If I is not a singleton, then for every $c \in I$ there is a $d \in I$ such that either $c > d$ or $c < d$. Assume that $c \in I$ is not a minimal element, let $d < c$. Note that $G_c = \{h \in A_I \colon F_{cd}(h) \neq h\}$; hence $F_{cd}(\alpha(1_c)) = \alpha(F_{cd}(1_c)) = \alpha(1_d) \neq \alpha(1_c)$ implies $\alpha(1_c) \in G_c$. If, on the other hand, c is minimal in I then there is a $d > c$ in I and $\alpha(1_c) = \alpha(F_{dc}(1_d)) = F_{dc}(\alpha(1_d))$ belongs to G_c since $\alpha(1_d) \in G_d$ by the previous argument. $\alpha(g) = \alpha(\bar{g}(1_c)) = \bar{g}(\alpha(1_c)) = \alpha(1_c) \cdot g \in G_c$ for all $g \in G_c$, $c \in I$. Thus $\alpha^+(A_c) = A_c$ for all $c \in I$ and this implies (10). Denote $g_c = \alpha(1_c)$ for $c \in I$. If $d \leqq c$, then $\alpha(h) = \alpha(\bar{h}(1_d)) = \bar{h}(\alpha(1_d)) = \bar{h}(\alpha(F_{cd}(1_c))) = \bar{h}(F_{cd}(\alpha(1_c))) = \bar{h}(r_{cd}(g_c)) = r_{cd}(g_c) \cdot h$ and $\alpha(h) = g_d \cdot h$ for all $h \in G_d$. Therefore $r_{cd}(g_c) = g_d$ for $c \geqq d$ in I. If $I = (c]$, then $\alpha(h) = g_c^*(h)$ for all $h \in A_c$ and, consequently, $\mathrm{Aut}(A_c) \cong G_c$. This proves (11) for non-zero compact elements of L. If I is not principal, then every $\alpha \in \mathrm{Aut}(A_I)$ determines a system

$$(g_c \in G_c \colon c \in I)$$

such that g_c^* is the restriction of α to A_c. As $r_{cd}(g_c) = g_d$ for all $c \geqq d$ in I, there is a unique $g \in G_I$ whose projection in G_c is g_c. It is now clear that $G_I \cong \mathrm{Aut}(A_I)$ for every ideal I of C.

Finally, let $c \geqq d \geqq e$ in $C, g \in G_c, k \in G_e$. Then $g^*(k) = r_{ce}(g) \cdot k = r_{de}(r_{cd}(g)) \cdot k = (r_{cd}(g))^*(k)$ and (12) is satisfied as well. This finishes the proof.

Example 1. The set C of nonzero compact elements of an algebraic chain L consists of those $x \in L$ that cover some $y \in L$. If G_c is arbitrary for $c \in C$, $|G_x| = 1$ for $x \notin C$, and if all r_{cd} are constant homomorphisms for $c > d$, then the system Σ is representable. This generalizes the independence of automorphism groups of pairs algebra-subalgebra.

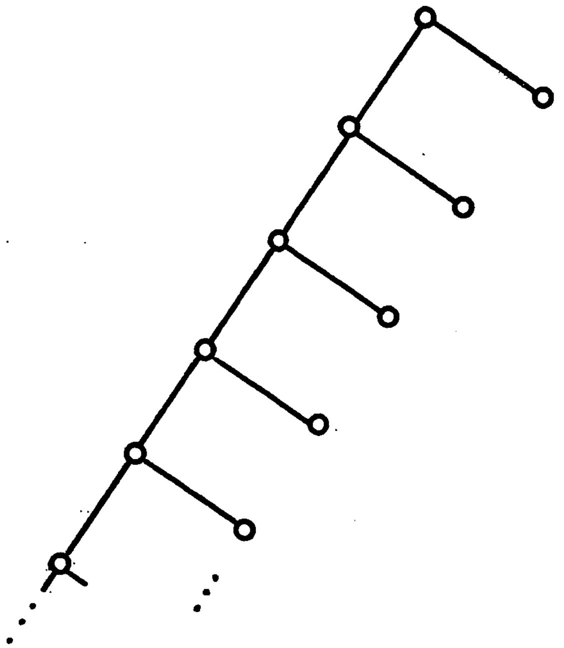
Example 2. Under the restriction (4) assumed throughout this note, the automorphism groups of subalgebras not finitely generated are uniquely determined by the automorphism groups of their finitely-generated subalgebras. A simple example shows that this is not generally the case.

Let L be the chain Z of integers extended by a largest element e and a smallest element z. L is an algebraic chain with $C = Z \cup \{z\}$. Let $G_c = \{1\}$ for $c \in C$ and let $(r_{cd} \colon c \geqq d)$ be the obvious homomorphisms. If $G_e = \{1\}$ as well, then the system Σ formed by these data is representable. On the other hand, if $f \colon Z \to Z$ is defined by $f(n) = n - 1$, then the algebra $A = (Z, f)$ satisfies $\mathrm{Sub}(A) \cong L$, $|\mathrm{Aut}([n])| = 1$ for all nonempty subalgebras $[n] = \{k \colon k \leqq n\}$, while $\mathrm{Aut}(A)$ is isomorphic to the additive group of integers.

Example 3. If $L \cong I(C)$ is an algebraic lattice and if all ideals of C are automorphism-free, then our special-case theorem describes the possible choices of

$(G_x: x \in L)$ completely. This is the case if, for instance, C is the join semilattice indicated by the Figure below.

Note that any non-empty ideal of C that is not a singleton is isomorphic to C; C is automorphism free as a semilattice — which implies (4) for any representable system with $L = I(C)$.



References

[1] E. FRIED—G. GRÄTZER, On automorphisms of the subalgebra lattice induced by the automorphisms of the algebra, *Acta Sci. Math.*, **40** (1978), 49—52.

[2] E. FRIED—J. SICHLER, Homomorphisms of integral domains of characteristic zero, *Trans. Amer. Math. Soc.*, **225** (1977), 163—182.

[3] Z. HEDRLÍN—E. MENDELSOHN, The category of graphs with given subgraph, *Can. J. Math.*, **21** (1969), 1506—1517.

[4] Z. HEDRLÍN—A. PULTR, On full embeddings of categories of algebras, *Illinois J. Math.*, **10** (1966), 392—406.

[5] W. A. LAMPE, The independence of certain related structures of a universal algebra, *Algebra Universalis*, **2** (1972), Part I: 99—112; Part II: 270—283; Part III: 286—295; Part IV: 296—302.

[6] E. T. SCHMIDT: Universale Algebren mit gegebenen Automorphismengruppen und Unteralgebraverbänden, *Acta Sci. Math.*, **24** (1963), 251—254.

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