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A note on congruence extension property

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Congruence extension property (further on CEP) is very useful when trying to find identities for equational classes (see e.g. [3]). However, CEP is not an equational property. Therefore one has to prove it for all members of a given equational class. The only simplification is that subalgebra preserves CEP. The actual aim of this paper is to give an example to prove that homomorphic image does not preserve CEP. We shall put down some other ideas in this topic, too.

We shall use the terminology of [5]. We say that CEP holds for $(\mathbf{A}, \mathbf{B}, \theta)$ if $\mathbf{B}$ is a subalgebra of $\mathbf{A}$, θ is a congruence relation on $\mathbf{B}$ and there exists a congruence relation θ' on $\mathbf{A}$ the restriction of which to $\mathbf{B}$ coincides with θ. The pair $(\mathbf{A}, \mathbf{B})$ satisfies CEP exactly if $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ and for each congruence relation θ on $\mathbf{B}$ CEP holds for $(\mathbf{A}, \mathbf{B}, \theta)$. An algebra $\mathbf{A}$ satisfies CEP iff for all subalgebras $\mathbf{B}$ of $\mathbf{A}$ $(\mathbf{A}, \mathbf{B})$ satisfy CEP. CEP holds for a class of algebras iff each element of this class satisfies CEP.

We define *strong congruence extension property* (further on SCEP) as follows: A quadruple $(\mathbf{A}, \mathbf{B}, \theta, \Phi')$ satisfies SCEP iff $\mathbf{B}$ is a subalgebra of $\mathbf{A}$, θ is a congruence relation on $\mathbf{B}$, Φ' is a congruence relation on $\mathbf{A}$ the restriction of which to $\mathbf{B}$ is contained in θ and there exists a congruence relation θ' on $\mathbf{A}$ containing Φ' the restriction of which to $\mathbf{B}$ coincides with θ. $(\mathbf{A}, \mathbf{B}, \theta)$ satisfies SCEP iff for each congruence relation Φ' on $\mathbf{A}$ the quadruple $(\mathbf{A}, \mathbf{B}, \theta, \Phi')$ satisfies SCEP, provided $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ and the restriction of Φ' to $\mathbf{B}$ is contained in the congruence relation θ of $\mathbf{B}$. We define that a pair $(\mathbf{A}, \mathbf{B})$, an algebra $\mathbf{A}$ and a class of algebras satisfy SCEP as we have defined that for CEP substituting, everywhere, CEP by SCEP, respectively.

Proposition 1. *An algebra* $\mathbf{A}$ *has* SCEP *iff all homomorphic images of* $\mathbf{A}$ *have* CEP.

Proof. Let $\bar{\mathbf{A}}$ be the homomorphic image of $\mathbf{A}$ under the homomorphism φ, $\bar{\mathbf{B}}$ a subalgebra of $\bar{\mathbf{A}}$ and $\bar{\theta}$ a congruence relation on $\bar{\mathbf{B}}$. We define $\mathbf{B}=\varphi^{-1}(\bar{\mathbf{B}})$,

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$\theta=\varphi^{-1}(\bar{\theta})$ and $\Phi'=\operatorname{Ker}\varphi$. SCEP for $(\mathbf{A}, \mathbf{B}, \theta, \Phi')$ yields CEP for $(\bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\theta})$. On the other hand, for given $\mathbf{A}, \mathbf{B}, \theta$ and Φ' CEP for $(\mathbf{A}/\Phi', \mathbf{B}/\Phi', \theta/\Phi')$ implies SCEP for $(\mathbf{A}, \mathbf{B}, \theta, \Phi')$.

Now, we are going to show that SCEP is "more" equational then CEP.

Proposition 2. *If a class K has* SCEP *so does* $HS(K)$.

Proof. We have to prove that if $\mathbf{A}$ satisfies SCEP so do all homomorphic images and subalgebras of $\mathbf{A}$. Proposition 1 takes care of homomorphic images. Now, let $\mathbf{B}$ be a subalgebra of $\mathbf{A}$, $\mathbf{C}$ be a subalgebra of $\mathbf{B}$, Φ' a congruence relation on $\mathbf{B}$ with the restriction Φ to $\mathbf{C}$ and $\theta\geqq\Phi$ be a congruence relation on $\mathbf{C}$. Since CEP holds for $(\mathbf{A}, \mathbf{B}, \Phi')$ there exists a congruence relation Φ'' on $\mathbf{A}$ the restriction of which to $\mathbf{C}$ coincides with Φ. Then SCEP for $(\mathbf{A}, \mathbf{C}, \theta, \Phi'')$ gives us that there exists a congruence relation $\theta''\geqq\Phi''$ on $\mathbf{A}$ the restriction of which to $\mathbf{C}$ equals θ. Hence, for the restriction θ' of θ to $\mathbf{B}$ we have $\theta'\geqq\Phi'$ and the restriction of θ' to $\mathbf{C}$ equals θ proving that $(\mathbf{B}, \mathbf{C}, \theta, \Phi')$ satisfies SCEP.

The next proposition describes a typical situation when SCEP holds.

Proposition 3. *Let $\mathbf{B}$ be a subalgebra of $\mathbf{A}$, Φ' a congruence relation on $\mathbf{A}$ with the restriction Φ to $\mathbf{B}$ and $\theta\geqq\Phi$ a congruence relation on $\mathbf{B}$. If each congruence class of Φ', containes a (nonempty) class of Φ, then* CEP *for* $(\mathbf{A}, \mathbf{B}, \theta)$ *implies* SCEP *for* $(\mathbf{A}, \mathbf{B}, \theta, \Phi')$.

Proof. Actually, CEP for $(\mathbf{A}, \mathbf{B}, \theta)$ is not needed; the statement is an obvious consequence of the first isomorphism theorem.

Theorem. CEP *does not imply* SCEP.

We shall prove the statement by giving an example. The method used can give examples for $4n$-elements algebras with integers n greater than 1. The situation is somewhat more complicated if n is not a prime. The choice $n=2$ has the advantage that there exists a field with eight elements, thus, we can express the functions by the operations of the field, for finite fields are functionally complete.

Example. Let a be a root of the polynomial x^3+x+1 over the two elements field Q_2 and we denote the underlying set of $Q_2(a)$ by A. We define the following algebra:

$$\mathbf{A} = \langle A|0, f, g, p, F\rangle$$

where the operations are given as follows: 0 is a nullary operation assigning the zero element 0; f, g and p are unary operations defined by $f(x)=x+1$, $g(x)=x+a$, $p(x)=x^7$; F is a binary operation defined by $F(x,y)=(x^7+1)\left(a^2((ay)^4+(ay)^2+(ay))\right)$. The elements 0, $f(0)=1$, $g(0)=a$ and $f(g(0))=a+1$ are constants. We denote $B=\{0, 1, a, a+1\}$.

The underlying set of the only proper subalgebra **B** of **A** is B. The only non-trivial congruence θ of **B** consists of the cosets $\{0, 1\}$ and $\{a, a+1\}$. The congruence θ' of **A** consisting of the cosets $\{0, 1, a^2, a^2+1\}$ and $\{a, a+1, a^2+a, a^2+a+1\}$ is an extension of θ. Hence, CEP holds for **A**.

The cosets $U=\{a^2, a^2+a\}$, $V=\{a^2+1, a^2+a+1\}$ and all the singletons disjoint to both U and V form a congruence Φ' of **A** the restriction of which to **B** is ω. Since $\theta' \vee \Phi' = \iota$ we have $\theta' \not\equiv \Phi'$. Thus, $(\mathbf{A}, \mathbf{B}, \theta, \Phi')$, hence **A** do not satisfy SCEP. The details are left to the reader.

Remark. We are going to list some straightforward properties of SCEP to show how close it is to equational classes.

1) Proposition 1 shows that for an equational class CEP implies SCEP.

2) If an element of an equational class has SCEP so do its subdirect irreducible components.

3) If each direct product of subdirect irreducible elements of an equational class has SCEP so does the whole equational class.

4) If each direct product of some subdirect irreducible elements of a congruence distributive equational class has SCEP so does the equational class they generate (comp. [6]).

Problem. Let K be a class of algebras with SCEP. Prove or disprove:

a) K need not be closed under finite direct products.

b) K need not be closed under prime products.

c) Though the class K is closed both under finite direct products and prime products it need not be closed under direct products.

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