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Concrete representation of related structures of universal algebras. I

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In his recent book [6], I. I. VALUCE quotes without proof a result of A. V. Kuznecov, unpublished up to now. Trying to re-establish the proof, we observed some general facts concerning mutual properties of relations and operations. This enables us to solve several concrete representation problems for related structures of algebras in a uniform way.

The basic propositions of this article are Lemmas 1—5 preceeded by a survey of notions we shall need. Using them we give a simultaneous characterization for related structures of universal algebras (Theorem 6). As special cases of Theorem 6 we get characterizations for the systems of subalgebras of finite direct powers of algebras (G. Grätzer's Problem 19 in [3]; Theorem 7 and 9) and the endomorphism semigroups of algebras (Grätzer's Problem 3 in [3]; Theorem 15; for another solution of this problem, see N. SAUER and M. G. STONE [5]). As corollaries we get Jürgen Schmidt's concrete representation theorem for the subalgebra systems of algebras (see, e.g. [2]) and the Bodnarčuk—Kalužnin—Kotov—Romov theorem for the subalgebra systems of all finite direct powers of finite algebras [1]. Moreover, we characterize the bicentralizers of sets of operations in arbitrary algebras. Then Kuznecov's above mentioned result appears as a special case.

In a forthcoming Part II, we shall apply the method developed here for the representation of other related structures.

Let A be a nonempty set which will be fixed in the sequel. Let O_n $(n=0, 1, 2, \ldots)$ and O denote the set of all n-ary and all finitary operations of A, respectively; furthermore, let $\mathcal{R}_n$ $(n=1, 2, \ldots)$ and $\mathcal{R}$ denote the set of all n-ary and all finitary relations of A, respectively. In general, we shall not distinguish between an operation and the associated relation, i.e., an n-ary operation may be considered as a mapping $f\colon A^n\to A$ and as an $(n+1)$-ary relation $\{(a_1, \ldots, a_n, f(a_1, \ldots, a_n))\,|\,(a_1, \ldots, a_n)\in A^n\}$ as well. Thus we have $O\subseteqq\mathcal{R}$ and $O_n\subseteqq\mathcal{R}_{n+1}$, $n=0, 1, 2, \ldots$. If R is an n-ary relation, we shall often write $R(a_1, \ldots, a_n)$ instead of $(a_1, \ldots, a_n)\in R$.

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We say that an n-ary operation f *preserves* an m-ary relation R, if $R(f(a_{11}, \ldots, a_{1n}), \ldots, f(a_{m1}, \ldots, a_{mn}))$ holds whenever $R(a_{1k}, \ldots, a_{mk})$, $k=1, \ldots, n$, i.e., (R, f) is a subalgebra of the algebra $(A, f)^m$ (the m-th direct power of (A, f)). Remark that the empty set is an n-ary relation for every $n \geqq 1$, and it is preserved by every m-ary operation where $m \geqq 1$. Let f and g be operations of arity n and m, respectively. If M is an $m \times n$ matrix of elements of A, we can apply f [g] to each row [column] of M. Thus we get a column [row] consisting of m [n] elements, which will be denoted by $f(M)$ [$(M)g$]. If for any $m \times n$ matrix M of elements of A, $f((M)g) = (f(M))g$ holds then we say that f and g *commute*. Clearly, two operations commute if and only if any of them preserves the other as a relation. For any set of relations Γ, denote by Γ^* the set of all operations preserving every member of Γ. We call Γ^* the *centralizer* of Γ. If $\Gamma = \Omega$ is a set of operations, then Ω^{**} is called the *bicentralizer* of Γ. The symbol Ω° will denote the set of all relations preserved by every member of Ω. Remark that $\Omega^* = \Omega^\circ \cap O$ for any set of operations Ω.

Let Π be a set of relations of A, i.e., $\Pi \subseteqq \mathcal{R}$. If a relation belongs to Π, we shall call it a *Π-relation*. Let (A, Ω) be an algebra. By the *related structure of type Π* of (A, Ω) (in symbol: $\mathrm{Rel}_\Pi(A, \Omega)$) we mean the set of all Π-relation preserved by every operation of Ω, i.e., $\mathrm{Rel}_\Pi(A, \Omega) = \Omega^\circ \cap \Pi$. Observe that if Π_1 is the set of all n-ary relations of A, Π_2 is the set of all equivalences of A, Π_3 is the set of all unary operations of A, and Π_4 is the set of all bijective unary operations of A, then $\mathrm{Rel}_{\Pi_1}(A, \Omega) = \mathrm{Sub}((A, \Omega)^n)$, $\mathrm{Rel}_{\Pi_2}(A, \Omega) = \mathrm{Con}(A, \Omega)$, $\mathrm{Rel}_{\pi_3}(A, \Omega) = \mathrm{End}(A, \Omega)$ and $\mathrm{Rel}_{\Pi_4}(A, \Omega) = \mathrm{Aut}(A, \Omega)$.

Let $X = \{x_i \mid i \in I\}$ be a set of variables indexed by an arbitrary set I and let Γ be a set of relations of A. If R is a symbol of an n-ary relation in Γ and f, g are symbols of operations of arity m, s that denote a projection or an operation belonging to Γ, respectively, then $R(x_{i_1}, \ldots, x_{i_n})$ and $f(x_{j_1}, \ldots, x_{j_m}) = g(x_{t_1}, \ldots, x_{t_s})$ are said to be formulas of the variable set X over Γ provided $x_{i_1}, \ldots, x_{i_n}, x_{j_1}, \ldots, x_{j_m}, x_{t_1}, \ldots, x_{t_s} \in X$. (Note that we might have formulas of the first kind only, but introducing these two kinds of formulas our considerations became somewhat simpler.) We say that a family $(a_i \mid i \in I) \in A^I$ satisfies the above formulas if $R(a_{i_1}, \ldots, a_{i_n})$, resp. $f(a_{j_1}, \ldots, a_{j_m}) = g(a_{t_1}, \ldots, a_{t_s})$ holds. Consider a triple $\Psi = (\Sigma, X, (x_{i_1}, \ldots, x_{i_n}))$ where $X = \{x_i \mid i \in I\}$ is a set of variables indexed by I, $(x_{i_1}, \ldots, x_{i_n}) \in X^n$, and Σ is a set of formulas of variable set X over Γ. Such a triple will be referred to as a *formula scheme* over Γ. We say that Ψ is finite if both Σ and X are finite. If $\Psi = (\Sigma, X, (x_{i_1}, \ldots, x_{i_n}))$ $(X = \{x_i \mid i \in I\})$ is a formula scheme then we associate with Ψ an n-ary relation R_Ψ defined as follows: $R_\Psi = \{(a_{i_1}, \ldots, a_{i_n}) \mid (a_i \mid i \in I) \in A^I$ and $(a_i \mid i \in I)$ satisfies (every member of) $\Sigma\}$. Then we say that R_Ψ is defined by the formula scheme Ψ.

We say that a formula scheme $\Psi = (\Sigma, X, (x_{i_1}, \ldots, x_{i_n}, x_{i_{n+1}}))$ $(X = \{x_i \mid i \in I\})$

defines the n-ary operation f on $B\subseteq A^n$ if for any $(a_1, \dots, a_n)\in B$, $f(a_1, \dots, a_n)=a_{n+1}$ for some $a_{n+1}\in A$ if and only if $R_\Psi(a_1, \dots, a_n, a_{n+1})$ holds. For $B=A^n$ we say that Ψ defines f. An n-ary operation f is said to be locally definable by a set of relations Γ, if for every finite $B\subseteq A^n$ there exists a formula scheme over Γ defining f on B.

The following lemmas describe the connection between the notions "relations preserved by operations" and "relations defined by formula schemes".

Lemma 1. *Let Γ be a set of relations of A. If a relation R can be defined by a formula scheme over Γ, then $R\in\Gamma^{*0}$.*

Proof. Let $\Psi=(\Sigma, X, (x_{i_1}, \dots, x_{i_n}))$ $(X=\{x_i|i\in I\})$ be a formula scheme over Γ and let f be an m-ary operation preserving all members of Γ. If $R_\Psi=\emptyset$ then f preserves R_Ψ trivially, unless $m=0$. However if $m=0$, i.e., f is a nullary operation then $R(f, \dots, f)$ holds for every $R\in\Gamma$, whence it follows that Σ is satisfied by $(a_i|i\in I)$ where $a_i=f$ for all $i\in I$. Then $R_\Psi(f, \dots, f)$ holds, a contradiction.

Now suppose $R_\Psi\neq\emptyset$ and let $R_\Psi(a_1^k, \dots, a_n^k)$, $k=1, \dots, m$. Then there exist families $(b_i^k|i\in I)$ satisfying Σ such that $(a_1^k, \dots, a_n^k)=(b_{i_1}^k, \dots, b_{i_n}^k)$, $k=1, \dots, m$. Using the fact that f preserves all relations and commutes with all operations whose symbols occur in Σ, one can observe by routine that $\big(f(b_i^1, \dots, b_i^m)\,|\,i\in I\big)$ satisfies Σ. Hence it follows

$$\big(f(a_1^1, \dots, a_1^m), \dots, f(a_n^1, \dots, a_n^m)\big) = \big(f(b_{i_1}^1, \dots, b_{i_1}^m), \dots, f(b_{i_n}^1, \dots, b_{i_n}^m)\big)\in R_\Psi$$

showing that f preserves R_Ψ. Q.E.D.

Lemma 2. *Let Γ be a set of relations of A. Then for every positive integer n, every finitely generated subalgebra of the algebra $(A, \Gamma^*)^n$ can be defined by a formula scheme over Γ. Moreover, if A is a finite set, then we can choose these formula schemes to be finite.*

Proof. Let T be a finitely generated subalgebra of $(A, \Gamma^*)^n$. If $T=\emptyset$ then Γ^* has no nullary operation. Consider the set of formulas $\Sigma=\{R(x_1, \dots, x_1)|R\in\Gamma\}$. Then there is no element of A satisfying Σ. For if $a\in A$ satisfies Σ then we get $R(a, \dots, a)$ for all $R\in\Gamma$ which implies that $a\in\Gamma^*$, i.e., Γ^* has a nullary operation, a contradiction. Thus the formula scheme $\Psi=(\Sigma, \{x_1\}, (x_1))$ defines $T=\emptyset$, i.e., $R_\Psi=\emptyset=T$. Furthermore, as $R_\Psi=\emptyset$, i.e., there is no element of A satisfying Σ, for any $a\in A$ there is a formula $R_a(x_1, \dots, x_1)\in\Sigma$ such that $R_a(a, \dots, a)$ does not hold. Then the formula scheme $\Psi'=(\Sigma', \{x_1\}, (x_1))$ with $\Sigma'=\{R_a(x_1, \dots, x_1)|a\in A\}$ defines $T=\emptyset$, too. Moreover, if A is a finite set then Ψ' is a finite formula scheme.

Now suppose $T\neq\emptyset$ and the set $\{t_i=(t_{1i}, \dots, t_{ni})|t_i\in A^n, i=1, \dots, s\}$ generates T. Since Γ^* is a clone (i.e., it contains all projections and is closed under superposition), $T=\{f(t_1, \ldots, t_s)|f\in\Gamma^*\cap O_s\}$. We construct a formula scheme Ψ which defines T.

Let X be a set of variables indexed by A^s, i.e., $X=\{x_i|i\in A^s\}$. Consider an arbitrary relation Q from Γ. Let m be the arity of Q. Considering every element of Q as a column vector of length m, every element of Q^s is an $m\times s$ matrix of elements of A. With Q and any matrix $M\in Q^s$ we associate a formula $Q(x_{M_1}, \ldots, x_{M_m})$ of the variable set X, where M_k is the k-th row of M, $k=1, \ldots, m$. Now consider the formula scheme $\Psi=(\Sigma, X, (x_{i_1}, \ldots, x_{i_n}))$ where $X=\{x_i|i\in A^s\}$, $\Sigma=\{Q(x_{M_1}, \ldots, x_{M_m})|Q\in\Gamma$ and $M\in Q^s\}$, and $(i_1, \ldots, i_n)=((t_{11}, \ldots, t_{1s}), \ldots, (t_{n1}, \ldots, t_{ns}))$. We show that T is defined by Ψ, i.e., $T=R_\Psi$. Clearly $R_\Psi=\{(a_{i_1}, \ldots, a_{i_n})|(a_i|i\in A^s)\in A^{A^s}$ and $(a_i|i\in A^s)$ satisfies $\Sigma\}$. Remark, however, that $A^{A^s}=O_s$, and thus we can write $f\in O_s$ instead of $(a_i|i\in A^s)\in A^{A^s}$. Using this notation we get

$$R_\Psi = \{(f(i_1), \ldots, f(i_n))|f\in O_s \text{ and } f \text{ satisfies } \Sigma\} =$$
$$= \{(f(t_{11}, \ldots, t_{1s}), \ldots, f(t_{n1}, \ldots, t_{ns}))|f\in O_s \text{ and } f \text{ satisfies } \Sigma\} =$$
$$= \{f(t_1, \ldots, t_s)|f\in O_s \text{ and } f \text{ satisfies } \Sigma\}.$$

Furthermore, an s-ary operation f satisfies Σ if and only if $f\in\Gamma^*$. To show this first suppose that $f\in O_s$ satisfies Σ. Let Q be an arbitrary m-ary relation from Γ, and let $q_j=(q_{1j}, \ldots, q_{mj})\in Q$, $j=1, \ldots, s$. Then from $M=(q_1, \ldots, q_s)\in Q^s$ we get $Q(x_{M_1}, \ldots, x_{M_m})\in\Sigma$, which implies $Q(f(M_1), \ldots, f(M_m))$, i.e., $Q(f(q_{11}, \ldots, q_{1s}), \ldots, f(q_{m1}, \ldots, q_{ms}))$ proving that f preserves Q. Hence $f\in\Gamma^*$. Conversely suppose that $f\in O_s\cap\Gamma^*$ and $Q(x_{j_1}, \ldots, x_{j_m})$ is an arbitrary formula from Σ, where $j_k=(j_{k1}, \ldots, j_{ks})$, $k=1, \ldots, m$. Then the matrix $(j_{kl})_{m\times s}$ is an element of Q^s, i.e., $(j_{1l}, \ldots, j_{ml})\in Q$, $l=1, \ldots, s$. Taking into account that f preserves Q we get that $Q(f(j_{11}, \ldots, j_{1s}), \ldots, f(j_{m1}, \ldots, j_{ms}))$, i.e., $Q(f(j_1), \ldots, f(j_m))$ proving that f satisfies the formula $Q(x_{j_1}, \ldots, x_{j_m})$. Hence f satisfies Σ. This implies $R_\Psi=(f(t_1, \ldots, t_s)|f\in\Gamma^*\cap O_s\}$, and the right side is the same as T.

Now let A be a finite set, and consider the formula scheme Ψ constructed above. For every s-ary operation f that does not satisfy Σ there exists a formula $\mathscr{T}_f\in\Sigma$ such that f does not satisfy $\mathscr{T}_f$. Consider the set of formulas $\Sigma'=\{\mathscr{T}_f|f\in O_s$ and f does not satisfy $\Sigma\}$. It is evident that an s-ary operation satisfies Σ if and only if it satisfies Σ'. Therefore, the formula scheme $\Psi'=(\Sigma', X, (x_{i_1}, \ldots, x_{i_n}))$ where X and $(x_{i_1}, \ldots, x_{i_n})$ are the same as above, defines the relation T. Namely,

$$T = R_\Psi = \{(f(i_1), \ldots, f(i_n))|f\in O_s \quad \text{and} \quad f \text{ satisfies } \Sigma\} =$$
$$= \{(f(i_1), \ldots, f(i_n))|f\in O_s \text{ and } f \text{ satisfies } \Sigma'\} = R_{\Psi'}.$$

Furthermore, from $|X|=|A^s|$ and $|\Sigma'|\leq|O_s|=|A^{A^s}|$ it follows that X and Σ' are finite. Hence Ψ' is a finite formula scheme. Q.E.D.

Lemma 3. *If A is a finite set and a relation can be defined by a formula scheme over a set of relations Γ, then it can be defined by a finite formula scheme over Γ.*

Proof. Suppose an n-ary relation R can be defined by a formula scheme over Γ. From Lemma 1 it follows $R \in \mathrm{Sub}((A, \Gamma^*)^n)$. Applying Lemma 2 we get that R can be defined by a finite formula scheme over Γ. Q.E.D.

Lemma 4. *Let Γ be a set of relations of A. Then a relation R belongs to Γ^{*0} if and only if R is the union of a directed system of relations defined by formula schemes over Γ.*

Proof. First let $R = \bigcup_{i \in I} R_i$ where $(R_i | i \in I)$ is a directed system of relations defined by formula schemes over Γ. Therefore, by Lemma 1, we get that $R_i \in \Gamma^{*0}$, $i \in I$. Furthermore, one can see easily that the union of a directed system of elements of Γ^{*0} belongs to Γ^{*0}.

Now suppose that $R \in \Gamma^{*0}$ is an n-ary relation. Then R is a subalgebra of the algebra $(A, \Gamma^*)^n$. Therefore $R = \bigcup_{i \in I} R_i$ where $(R_i | i \in I)$ is the directed system of the finitely generated subalgebras of $(A, \Gamma^*)^n$ contained in R. In view of Lemma 2, we have that R_i, $i \in I$, can be defined by a formula scheme over Γ. Q.E.D.

Lemma 5. *Let Γ be a set of relations of A. Then an operation f belongs to Γ^{**} if and only if f can be defined by Γ locally.*

Proof. First suppose that f is an n-ary operation which is defined by Γ locally. Choose an m-ary operation g from Γ^* and let $M = (a_{kl})_{m \times n}$ be an $m \times n$ matrix of elements of A. According to our assumption, there is a formula scheme Ψ that defines f on

$$B = \{(a_{k1}, \ldots, a_{kn}) | k = 1, \ldots, m\} \cup \{(g(a_{11}, \ldots, a_{m1}), \ldots, g(a_{1n}, \ldots, a_{mn}))\}.$$

Then $R_\Psi(a_{k1}, \ldots, a_{kn}, f(a_{k1}, \ldots, a_{kn}))$ holds, $k = 1, \ldots, m$. Using Lemma 1 we get that $R_\Psi(g(a_{11}, \ldots, a_{m1}), \ldots, g(a_{1n}, \ldots, a_{mn}), g(f(a_{11}, \ldots, a_{1n}), \ldots, f(a_{m1}, \ldots, a_{mn}))$ holds, too, whence

$$f(g(a_{11}, \ldots, a_{m1}), \ldots, g(a_{1n}, \ldots, a_{mn})) = g(f(a_{11}, \ldots, a_{1n}), \ldots, f(a_{m1}, \ldots, a_{mn}))$$

follows, i.e., $f((M)g) = (f(M))g$. Hence f commutes with g showing that $f \in \Gamma^{**}$.

Now suppose that $f \in \Gamma^{**}$ is an n-ary operation and let $B \subseteq A^n$ be a finite set. Considering f as an $(n+1)$-ary relation we have $f \in \Gamma^{*0}$. Therefore, by Lemma 4, we get $f = \bigcup_{i \in I} R_i$ where $(R_i | i \in I)$ is a directed system of ($(n+1)$-ary) relations defined by formula schemes over Γ. As $(R_i | i \in I)$ is a directed system and B is a finite set, $f = \bigcup_{i \in I} R_i$ implies $f | B \subseteq R_{i_0}$ for some $i_0 \in I$. Now let Ψ be a formula scheme over Γ defining R_{i_0}. Then $f | B \subseteq R_{i_0} \subseteq f$ implies

$$f | B = \{(a_1, \ldots, a_n, a_{n+1}) | (a_1, \ldots, a_n) \in B \quad \text{and} \quad (a_1, \ldots, a_n, a_{n+1}) \in R_{i_0} = R_\Psi\}$$

and this means exactly that Ψ defines f on B. Q.E.D.

Theorem 6. *Let $\Gamma_i \subseteq \Pi_i (\subseteq \mathscr{R})$, $i \in I$, be sets of relations of A; furthermore, let $\Omega_j \subseteq \Pi_j (\subseteq \mathscr{R})$, $j \in J$, be sets of such relations which are operations of A. Put $\Gamma = (\bigcup_{i \in I} \Gamma_i) \cup (\bigcup_{j \in J} \Omega_j)$. Then the following two statements are equivalent:*

I. *There exists an algebra (A, Ω) such that $\Gamma_i = \mathrm{Rel}_{\Pi_i}(A, \Omega)$ and $\Omega_j = \mathrm{Rel}_{\Pi_j}(A, \Omega)$ for every $i \in I$ and $j \in J$.*

II. (α) *For every $i \in I$, if a Π_i-relation is the union of a directed system of relations defined by formula schemes over Γ, then it belongs to Γ_i.*

(β) *For every $j \in J$, if a Π_j-relation (operation) can be defined by Γ locally then it belongs to Ω_j.*

Proof. I$\Rightarrow$II. Suppose that $\Gamma_i = \mathrm{Rel}_{\Pi_i}(A, \Omega)$ and $\Omega_j = \mathrm{Rel}_{\Pi_j}(A, \Omega)$ for some algebra (A, Ω) for every $i \in I$ and $j \in J$. First let $i_0 \in I$ and suppose a Π_{i_0}-relation R to be the union of a directed system of relations defined by formula schemes over Γ. Taking into account Lemma 4 and $\Gamma^* \supseteq \Omega$ we have that $R \in \Gamma^{*0} \subseteq \Omega^0$. This fact together with R being a Π_{i_0}-relation shows that $R \in \mathrm{Rel}_{\Pi_{i_0}}(A, \Omega)$. Hence ($\alpha$) holds.

Now let $j_0 \in J$ and suppose a Π_{j_0}-operation f can be defined by Γ locally. Then, by Lemma 5, we have $f \in \Gamma^{**} \subseteq \Omega^* \subseteq \Omega^0$. Hence $f \in \mathrm{Rel}_{\Pi_{j_0}}(A, \Omega)$, i.e., ($\beta$) holds.

II$\Rightarrow$I. Let $\Omega = \Gamma^*$. We shall prove that $\Gamma_i = \mathrm{Rel}_{\Pi_i}(A, \Omega)$ and $\Omega_j = \mathrm{Rel}_{\Pi_j}(A, \Omega)$ for every $i \in I$ and $j \in J$. First choose an arbitrary $i_0 \in I$. The inclusion $\Gamma_{i_0} \subseteq \mathrm{Rel}_{\Pi_{i_0}}(A, \Omega)$ is obvious. Let $R \in \mathrm{Rel}_{\Pi_{i_0}}(A, \Omega)$. Then $R \in \Omega^0 = \Gamma^{*0}$. Therefore, by Lemma 4, we have that R is the union of a directed system of relations defined by formula schemes over Γ. Thus, by the condition (α), $R \in \Gamma_{i_0}$.

Now choose an arbitrary $j_0 \in J$. Again, $\Omega_{j_0} \subseteq \mathrm{Rel}_{\Pi_{j_0}}(A, \Omega)$ is obvious. Let $f \in \mathrm{Rel}_{\Pi_{j_0}}(A, \Omega)$ be a Π_{j_0}-operation. Then $f \in \Omega^* = \Gamma^{**}$. Therefore, by Lemma 5, we get that f can be defined by Γ locally. Thus, by the condition (β), $f \in \Omega_{j_0}$. Q.E.D.

Theorem 7. *Let $(\Gamma_n \mid n = 1, 2, \ldots)$ be a family of sets of relations of A such that Γ_n has n-ary relations only, $n = 1, 2, \ldots$. Then the following two statements are equivalent:*

I. *There exists an algebra (A, Ω) such that $\Gamma_n = \mathrm{Sub}\,((A, \Omega)^n)$, $n = 1, 2, \ldots$.*

II. (α) *For every n, if an n-ary relation can be defined by a formula scheme over $\bigcup_{k=1}^{\infty} \Gamma_k$ then it belongs to Γ_n.*

(β) *For every n, Γ_n is closed under union of directed systems.*

Proof. Put $I = \{1, 2, \ldots\}$, $J = \emptyset$ and, as Π_n, the set of all n-ary relations of A in Theorem 6.

Corollary 8. *If A is a finite set then statement II in Theorem 6 can be replaced by*

II'. *For every n, if an n-ary relation can be defined by a finite formula scheme over* $\bigcup_{k=1}^{\infty} \Gamma_k$ *then it belongs to* Γ_n.

Proof. As A is a finite set, the assumption (β) in Theorem 6 is superfluous and we can apply Lemma 3.

Theorem 9. *Let* Γ *be a set of n-ary relations of A. Then there exists an algebra* (A, Ω) *such that* $\Gamma = \mathrm{Sub}((A, \Omega)^n)$ *if and only if* Γ *is closed under union of directed systems and* Γ *contains every n-ary relation defined by a formula scheme over* Γ.

Proof. Put $I=\{1\}$, $\Gamma_1=\Gamma$, $J=\emptyset$ and, as Π_1, the set of all n-ary relations of A in Theorem 6.

Corollary 10. *Let A be finite and let* Γ *be a set of n-ary relations of A. Then there exists an algebra* (A, Ω) *such that* $\Gamma = \mathrm{Sub}((A, \Omega)^n)$ *if and only if* Γ *contains every n-ary relation defined by a finite formula scheme over* Γ.

Corollary 11. (J. Schmidt) *For a set* Γ *of unary relations of A, there is an algebra* (A, Ω) *such that* $\Gamma = \mathrm{Sub}(A, \Omega)$ *if and only if* Γ *is an algebraic closure system.*

Proof. Suppose that $\Gamma = \mathrm{Sub}(A, \Omega)$ for some algebra (A, Ω). Let $\{R_j \mid j \in J\}$ be a subset of Γ. Then the formula scheme $(\Sigma, \{x_1\}, (x_1))$ with $\Sigma = \{R_j(x_1) \mid j \in J\}$ defines $\bigcap_{j \in J} R_j$. Applying Theorem 9, we get that $\bigcap_{j \in J} R_j \in \Gamma$, i.e., Γ is closed under intersections. This fact together with the conditions of Theorem 9 proves that Γ is an algebraic closure system.

Conversely, suppose that Γ is an algebraic closure system. Then Γ is closed under union of directed systems. Now consider a formula scheme $\Psi = (\Sigma, X, (x_1))$ $(X = \{x_i \mid i \in I\})$ over Γ. If $R_\Psi = \emptyset$ then $R_\Psi = \emptyset = \bigcap_{R \in \Gamma} R$. Otherwise, $a \in \bigcap_{R \in \Gamma} R$ implies that $(a_i \mid i \in I)$ where $a_i = a$ for all $i \in I$, satisfies Σ showing $R_\Psi(a)$, a contradiction. Thus $R_\Psi = \emptyset \in \Gamma$. If $R_\Psi \neq \emptyset$, then it is a routine to check that $R_\Psi = \bigcap_{R(x_1) \in \Sigma} R$, i.e., $R_\Psi \in \Gamma$. Thus we get that Γ satisfies the condition of Theorem 9. Q.E.D.

In [1], Kalužnin and his co-workers have given a characterization for the subalgebra system $\bigcup_{n=1}^{\infty} \mathrm{Sub}((A, \Omega)^n)$ of a finite algebra (A, Ω). Now we derive their result from Corollary 8. We need some additional notions and notations.

For an m-ary relation R of A and a permutation τ of the set $\{1, \ldots, m\}$ the *τ-translate* of R is an m-ary relation R^τ of A defined by $R^\tau = \{a_{1\tau}, \ldots, a_{m\tau}) \mid R(a_1, \ldots, a_m)\}$. For any two relations R and T of arity m and n, respectively, the *direct product* of R and T is an $(m+n)$-ary relation $R \times T$ defined by $R \times T = \{(a_1, \ldots, a_{m+n}) \mid R(a_1, \ldots, a_m)$ and $T(a_{m+1}, \ldots, a_{m+n})\}$. If R is an m-ary relation and $1 \leqq i_1 < \ldots < i_t \leqq m$, then the *projection* of R to the coordinates $i_1, \dots, i_t$ is a t-ary relation $R_{i_1, \dots, i_t}$ defined by $R_{i_1, \dots, i_t} = \{(a_{i_1}, \dots, a_{i_t}) | R(a_1, \dots, a_m)\}$. If R is an m-ary relation and Θ is an equivalence relation of the set $\{1, \dots, m\}$, then the *Θ-diagonal* of R is an m-ary relation R_Θ defined by $R_\Theta = \{(a_1, \dots, a_m) | R(a_1, \dots, a_m)$ and $(i\Theta j \Rightarrow a_i = a_j)\}$. Finally, the *n-ary diagonal* D_n is defined by $D_n = \{(a, \dots, a) | a \in A\}$ for any n.

Corollary 12. (V. G. Bodnarčuk, L. A. Kalužnin, V. N. Kotov, V. A. Romov) *If A is a finite set and Γ is a set of relations of A then there exists an algebra (A, Ω) such that $\Gamma = \bigcup_{n=1}^{\infty} \operatorname{Sub}((A, \Omega)^n)$ if and only if all diagonals belong to Γ, and Γ is closed under formation of direct products, as well as arbitrary τ-translates, projections, and Θ-diagonals.*

Proof. By Corollary 8 we have to prove only that a set of relations Γ fulfils the assumptions of the corollary if and only if every relation defined by a finite formula scheme over Γ belongs to Γ.

First suppose that all relations defined by finite formula schemes belong to Γ. Then for any n the formula scheme $(\emptyset, \{x_1\}, (x_1, \dots, x_1))$ defines D_n. If R and T are relations from Γ of arity m and n, respectively, τ is a permutation and Θ is an equivalence relation of the set $\{1, \dots, m\}$ and $1 \leq i_1 < \dots < i_t \leq m$, then the formula schemes

$$(\{R(x_1, \dots, x_m), T(x_{m+1}, \dots, x_{m+n})\}, \{x_1, \dots, x_{m+n}\}, (x_1, \dots, x_{m+n})),$$

$$(\{R(x_1, \dots, x_m)\}, \{x_1, \dots, x_m\}, (x_{1\tau}, \dots, x_{m\tau})),$$

$$(\{R(x_1, \dots, x_m)\}, \{x_1, \dots, x_m\}, (x_{i_1}, \dots, x_{i_t})),$$

and

$$(\{R(x_1, \dots, x_m)\} \cup \{D_2(x_k, x_l) | k\Theta l\}, \{x_1, \dots, x_m\}, (x_1, \dots, x_m))$$

define $R \times T$, R^τ, $R_{i_1, \dots, i_t}$ and R_Θ, respectively.

Conversely, suppose that Γ satisfies the assumptions of the corollary and let $\Psi = (\Sigma, X, (x_{i_1}, \dots, x_{i_n}))$ $(X = \{x_i | i \in I\})$ be a finite formula scheme over Γ. We have to prove that R_Ψ can be got from Γ in a finite number of steps by formation of directed products, τ-translates, projections, and Θ-diagonals. Concerning Ψ, we can assume w.l.o.g. that every component of $(x_{i_1}, \dots, x_{i_n})$ occurs in some formula of Σ, otherwise we can add the formulas $D_2(x_{i_1}, x_{i_1}), \dots, D_2(x_{i_n}, x_{i_n})$ to Σ. Furthermore, we can assume that $(x_{i_1}, \dots, x_{i_n})$ has pairwise distinct components, otherwise we can consider the formula scheme $\Psi = (\Sigma', X', (y_1, \dots, y_n))$ where $X' = X \cup \{y_1, \dots, y_n\} (X \cap \{y_1, \dots, y_n\} \neq \emptyset)$ and $\Sigma' = \Sigma \cup \{D_2(x_{i_1}, y_1), \dots, D_2(x_{i_n}, y_n)\}$. Clearly $R_\Psi = R_{\Psi'}$. Finally, we can also assume that Σ has formulas of the form $R(x_{j_1}, \dots, x_{j_m})$ $(R \in \Gamma)$ only. Otherwise, if a formula ε of the form $f(x_{t_1}, \dots, x_{t_s}) = g(x_{k_1}, \dots, x_{k_r})$ belongs to Σ, then replace ε by the formulas

$f(x_{t_1}, \ldots, x_{t_s}) = y_\varepsilon$ and $g(x_{k_1}, \ldots, x_{k_r}) = y_\varepsilon$. Considering f and g as $(s+1)$-ary and $(r+1)$-ary relations, respectively, these formulas have the form we required. Thus we get a set of formulas Σ''. Then the formula scheme $\Psi'' = (\Sigma'', X'', (x_{i_1}, \ldots, x_{i_n}))$ with $X'' = X \cup \{y_\varepsilon | \varepsilon \in \Sigma$ and ε is of the form $f = g\}$ defines R_Ψ.

Now suppose that Ψ has these properties. Then let

$$\Sigma = \{R_1(y_1^1, \ldots, y_{n_1}^1), \ldots, R_s(y_1^s, \ldots, y_{n_s}^s)\}, \quad y_k^l \in X, \quad l = 1, \ldots, s, \quad k = 1, \ldots, n_l.$$

Consider the formula scheme $\Phi = (\Sigma, X, (y_1^1, \ldots, y_{n_1}^1, \ldots, y_1^s, \ldots, y_{n_s}^s))$. Observe that R_Ψ can be got from R_Φ by formation of a suitable projection and τ-translate. Furthermore, let Θ be an equivalence of the set $\left\{1, \ldots, \sum_{k=1}^{s} n_k\right\}$ defined as follows: $j \Theta l$ if and only if the j-th and l-th components of $(y_1^1, \ldots, y_{n_1}^1, \ldots, y_1^s, \ldots, y_{n_s}^s)$ are equal, $j, l = 1, \ldots, \sum_{k=1}^{s} n_k$. Now it is a routine to verify that R_Φ equals the Θ-diagonal of $R_1 \times \ldots \times R_s$. Q.E.D.

Theorem 13. *If Ω is a set of operations of A, then $\Omega = \Omega^{**}$ if and only if Ω contains every operation defined by Ω locally.*

It follows from Lemma 5 immediately.

Corollary 14. (A. V. Kuznecov) *If A is a finite set, then $\Omega = \Omega^{**}$ for some set of operations Ω if and only if every operation defined by a finite formula scheme over Ω belongs to Ω.*

Proof. If A is a finite set, an operation f locally definable by Ω can be defined by a formula scheme over Ω. Lemma 3 shows that we can restrict ourselves to finite formulas. It remains to apply Theorem 13.

Theorem 15. *For a set E of transformations of A there exists an algebra (A, Ω) such that $E = \mathrm{End}\,(A, \Omega)$ if and only if E contains every transformation defined by E locally.*

Proof. Put $I = \emptyset$, $J = \{1\}$, $\Omega_1 = E$ and, as Π_1, the set of all unary operations in Theorem 6.

Corollary 16. *If A is a finite set, then for a set E of transformations of A there exists an algebra (A, Ω) such that $E = \mathrm{End}\,(A, \Omega)$ if and only if E contains every transformation defined by a finite formula scheme over E.*

Proof. We can proceed similarly as it was done in the proof of Corollary 12.

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