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# On extendibility of $*$-representations from $*$-ideals

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In this note we give necessary and sufficient conditions for the extension of a $*$-representation from a $*$-ideal in a complex involutory algebra, briefly $*$-algebra, to the whole algebra. One of our conditions, given in the Theorem below, is the same as the one we required in [3], Corollary 2. 8, and [4], Corollary 6, for the existence of such a $*$-representation of the whole algebra on some Hilbert space for which the norms of the representing operators of the elements in the $*$-ideal are equal to the norms of the corresponding representing operators concerning the given $*$-representation. We also give a new proof of our previous result (see [3], Theorem 2. 6; [4], Theorem 4) concerning the extension of $C^*$-semi-norms from $*$-ideals in $*$-algebras to the whole algebras.

Let us be given a $*$-ideal $J$ of a $*$-algebra $A$ over the complex number field $\mathbf{C}$ and a $*$-representation $T$ of $J$ on a Hilbert space $H$, i.e. a $*$-preserving algebra homomorphism $T:J\to B(H)$ of $J$ into the $C^*$-algebra $B(H)$ of all bounded linear operators on $H$. It is natural to ask: when does a $*$-representation $\overline{T}:A\to B(H)$ of $A$ on the same Hilbert space exist, which is an extension of the given $*$-representation $T$, i.e. for which $\overline{T}_b=T_b$ holds whenever $b$ is in $J$.

We answer this question in three ways, giving necessary and sufficient conditions, the first of which is simple enough still it needs no restriction on the algebra unlike in PALMER [2], Theorem 3.1.

LEMMA. *Let $T:J\to B(H)$ be a $*$-representation of a $*$-ideal $J$ in the complex $*$-algebra $A$ on the Hilbert space $H$. Then there exists a $*$-representation $\overline{T}:A\to B(H)$ of $A$ on the same Hilbert space $H$ extending $T$, if and only if*

$$\sup\Big\{\Big\|\sum_n T_{ab_n}x_n\Big\| : b_n\in J,\ x_n\in H,\ \Big\|\sum_n T_{b_n}x_n\Big\|\leqq 1\Big\}<\infty \tag{1}$$

*holds for every $a$ in $A$; here and further on $\sum_n$ denotes a finite sum.*

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Proof. Assume first that such a $*$-representation $\bar{T}$ of $A$ exists. We have then for every $a\in A$, $b_n\in J$ and $x_n\in H$, since $ab_n\in J$,

$$\Big\|\sum_n T_{ab_n}x_n\Big\| = \Big\|\sum_n \bar{T}_{ab_n}x_n\Big\| = \Big\|\bar{T}_a\big(\sum_n \bar{T}_{b_n}x_n\big)\Big\| = \Big\|\bar{T}_a\big(\sum_n T_{b_n}x_n\big)\Big\| \leqq |\bar{T}_a|\Big\|\sum_n T_{b_n}x_n\Big\|;$$

hence (1) is satisfied; here $|\bar{T}_a|$ is the norm of the operator $\bar{T}_a$ on the Hilbert space $H$.

Let

$$H_0 = \{x\in H: T_b x = 0 \text{ for all } b\in J\} = \bigcap\{\operatorname{Ker} T_b : b\in J\}$$

be the maximal closed linear subspace in $H$, the restriction of $T$ to which is a zero $*$-representation $T^0$ of $J$ on $H_0$. Denote by $\bar{T}^0$ the zero $*$-representation of $A$ on $H_0$, the trivial extension of $T^0$ from $J$ to $A$. Denote further

$$H_1 = \{x\in H: (x, y) = 0 \text{ for all } y\in H_0\}$$

the orthogonal complement of $H_0$ in $H$, and $T^1$ the restriction of $T$ to $H_1$ (which is clearly an invariant subspace for $T$). In this way $T$ is the direct sum of a zero and an essential $*$-representation $T^0$ and $T^1$ respectively: $T=T^0\oplus T^1$. Indeed, $G=\{\sum_n T_{b_n}x_n : x_n\in H_1, b_n\in J\}$ is a dense linear manifold in $H_1$, which is invariant for $T$ also, because if an element $x$ in $H_1$ is orthogonal to $G$, then for all $c\in J$

$$\|T_c x\|^2 = (T_c x, T_c x) = (x, T_c^* T_c x) = (x, T_{c^*c}x) = 0,$$

hence $x\in H_0$ and thus $x=0$.

Define for an element $a$ in $A$ a linear operator $S_a$ in $H_1$ given on $G$ by

$$S_a\big(\sum_n T_{b_n}x_n\big) = \sum_n T_{ab_n}x_n \quad (b_n\in J,\, x_n\in H_1),$$

We have now to show that $S_a$ is well defined on $G$, that is, $\sum_n T_{b_n}x_n=0$ implies $\sum_n T_{ab_n}x_n=0$. For $y=\sum_n T_{ab_n}x_n$ we have

$$\|y\|^2 = \big(\sum_m T_{ab_m}x_m, \sum_n T_{ab_n}x_n\big) = \sum_{m,n}(T_{ab_m}x_m, T_{ab_n}x_n) =$$

$$= \sum_{m,n}(T_{b_n^*a^*ab_m}x_m, x_n) = \sum_{m,n}(T_{a^*ab_m}x_m, T_{b_n}x_n) = \big(\sum_m T_{a^*ab_m}x_m, \sum_n T_{b_n}x_n\big),$$

hence $y=0$ indeed. But our assumption (1) means then that $S_a$ is a densely defined bounded linear operator in $H_1$, thus it has a unique extension $\bar{T}_a^1$ to $H_1$ in a standard way. For an $a\in J$ we have $S_a x=T_a x$ $(x\in G)$, hence that $\bar{T}_a^1=T_a$. It is now easy to check that $\bar{T}^1: A\to B(H_1)$ is a $*$-representation of $A$ on $H_1$. The linearity and multiplicativity of $\bar{T}^1$ is immediate. On the other hand, if $a\in A$; $b_n, c_m\in J$; $x_n, y_m\in H_1$, then we

$$\big(S_a\big(\sum_n T_{b_n}x_n\big), \sum_m T_{c_m}y_m\big) = \big(\sum_{m,n} T_{ab_n}x_n, T_{c_m}y_m\big) =$$

$$= \sum_{m,n}(x_n, T_{b_n^*a^*c_m}y_m) = \sum_{m,n}(T_{b_n}x_n, T_{a^*c_m}y_m) = \big(\sum_n T_{b_n}x_n, S_{a^*}\big(\sum_m T_{c_m}y_m\big)\big)$$

and thus $(\bar{T}^1_a)^* = \bar{T}^1_{a*}$ as well. Moreover, the extension of $T^1$ to $A$ is unique. Indeed, if $\hat{T}^1$ is an arbitrary $*$-representation of $A$ that is an extension of $T^1$, then for $a \in A$, $b_n \in J$, $x_n \in H_1$ we have

$$\hat{T}^1_a(\sum_n T^1_{b_n} x_n) = \hat{T}^1_a(\sum_n \hat{T}^1_{b_n} x_n) = \sum_n \hat{T}^1_{ab_n} x_n = \sum_n T^1_{ab_n} x_n = S_a(\sum_n T^1_{b_n} x_n).$$

We have finally that $\bar{T} = \bar{T}^0 \oplus \bar{T}^1$ is a $*$-representation of $A$ on the Hilbert space $H$ which extends $T$ and the lemma is proved.

We are now able to improve the above result as follows.

Proposition. *The $*$-representation $T: J \to B(H)$ has an extension, a $*$-representation $\bar{T}: A \to B(H)$ if and only if*

(2) $$\sup \{\|T_{ab} x\| : b \in J,\, x \in H,\, \|T_b x\| \leqq 1\} < \infty \quad \textit{for every } a \in A.$$

Proof. The necessity part is obvious from Lemma as (2) is a specialization of (1).

Suppose now that (2) holds. We are going to prove that $T$ has an extension $\bar{T}: A \to B(H)$. Let $T^1$ and $H^1$ be as before and write $T^1$ in the form of direct sum of topologically cyclic sub-$*$-representations (see [2]):

$$T^1 = \oplus \{T^\lambda : \lambda \in \Lambda\};\ T^\lambda = T^1|_{H_\lambda},\ T^\lambda : J \to B(H_\lambda)$$

for every index $\lambda$ in $\Lambda$; on a maximal family of pairwise orthogonal $T^1$-invariant subspaces $\{H_\lambda : \lambda \in \Lambda\}$ in $H_1$ (and thus spanning $H_1$) with topological cyclic vectors $x_\lambda \in H_\lambda$ such that $G_\lambda = \{T_b x_\lambda : b \in J\}$ is a dense linear manifold in $H_\lambda$ for each $\lambda \in \Lambda$. An argument similar to that used in Lemma shows, by (2), that for $a \in A$ the linear operator $S^\lambda_a$ in $H_\lambda$ given on $G_\lambda$ by $S^\lambda_a(T_b x_\lambda) = T_{ab} x_\lambda$ $(b \in J)$ is a densely defined bounded linear operator on $H_\lambda$. This has a unique extension $\bar{T}^\lambda_a$ to $H_\lambda$ with norm

$$|\bar{T}^\lambda_a| = \sup \{\|S^\lambda_a(T_b x_\lambda)\| : b \in J,\, \|T_b x_\lambda\| \leqq 1\} =$$
$$= \sup \{\|T_{ab} x_\lambda\| : b \in J,\, \|T_b x_\lambda\| \leqq 1\} \leqq \sup \{\|T_{ab} x\| : b \in J,\, x \in H,\, \|T_b x\| \leqq 1\} < \infty.$$

We thus have the $*$-representations $\bar{T}^\lambda : A \to B(H_\lambda)$ $(\lambda \in \Lambda)$. $\bar{T}^\lambda$ extends $T^\lambda$ in a unique way for all $\lambda$ in $\Lambda$ and thus

$$\bar{T}^1 = \oplus \{\bar{T}^\lambda : \lambda \in \Lambda\},$$

the direct sum of $\bar{T}^\lambda$-s, is a $*$-representation of $A$ on $H_1$ extending $T^1$ uniquely and such that for each $a \in A$

$$|\bar{T}^1_a| = \sup \{|\bar{T}^\lambda_a| : \lambda \in \Lambda\} = \sup \{\|T_{ab} x_\lambda\| : b \in J,\, \lambda \in \Lambda,\, \|T_b x_\lambda\| \leqq 1\}$$
$$\leqq \sup \{\|T_{ab} x\| : b \in J,\, x \in H,\, \|T_b x\| \leqq 1\} < \infty.$$

$\bar{T} = \bar{T}^0 \oplus \bar{T}^1$ is then a $*$-representation of $A$ on $H$ which extends $T$ as well and the proof is complete.

The main result of this note is the following.

**Theorem.** *Let $T: J \to B(H)$ be a $*$-representation of a $*$-ideal in the complex $*$-algebra $A$ on the Hilbert space $H$. There exists then a $*$-representation $\bar{T}: A \to B(H)$, which is an extension of $T$, if and only if*

$$q(a) := \sup\{|T_{ab}| : b \in J, |T_b| \leqq 1\} < \infty \quad \textit{for all } a \textit{ in } A. \tag{3}$$

Proof. Assume first that such a $*$-representation $\bar{T}$ of $A$ exists. Then we have for all $a \in A, b \in J$

$$|T_{ab}| = |\bar{T}_{ab}| = |\bar{T}_a \bar{T}_b| \leqq |\bar{T}_a| \cdot |\bar{T}_b| = |\bar{T}_a| \cdot |T_b|$$

whence $q(a) \leqq |\bar{T}_a|$ follows and (3) is satisfied.

Suppose now that (3) holds for all $a$ in $A$. The quantity $q$ is obviously a semi-norm on $A$. Moreover, if $|T_b| = 0$ for some $b \in J$, then $|T_{ab}| = 0$ for all $a \in A$ since

$$|T_{ab}|^2 = |T^*_{ab} T_{ab}| = |T_{b^* a^*} T_{ab}| = |T_{b^* a^* ab}| = |T_{b^* a^* a} T_b| \leqq |T_{b^* a^* a}| \, |T_b|$$

and thus

$$|T_{ab}| \leqq q(a) |T_b| \quad \text{holds for} \quad b \in J, a \in A.$$

We are now going to show that $q$ has the $C^*$-property:

$$q(a^* a) = (q(a))^2 \quad (a \in A), \tag{4}$$

in other words, $q$ is a $C^*$-semi-norm, and that (3) implies (2) whence our statement follows by the Proposition. For $a \in A$, $b \in J$ we have

$$|T_{ab}|^2 = |T_{b^* a^* ab}| = |T_{b^*} T_{a^* ab}| \leqq |T_{b^*}| \, |T_{a^* ab}| = |T_b| \, |T_{a^* ab}| \leqq q(a^* a) |T_b|^2$$

and thus

$$(q(a))^2 \leqq q(a^* a) \quad (a \in A). \tag{4'}$$

On the other hand, for $a \in A$, $b \in J$ we have

$$|T_{a^* ab}| \leqq q(a^*) |T_{ab}| \leqq q(a^*) q(a) |T_b|.$$

Hence

$$q(a^* a) \leqq q(a^*) q(a) \quad (a \in A). \tag{4''}$$

But (4′) and (4″) together give $q(a) \leqq q(a^*)$ for each $a \in A$; whence by interchanging the rôles of $a$ and $a^*$ we obtain that $q(a) = q(a^*)$ for all $a$ in $A$. We have then by (4′) and (4″)

$$(q(a))^2 \leqq q(a^* a) \leqq q(a^*) q(a) = (q(a))^2$$

whence (4) follows.

We are now able to prove that (2) holds. For if $a \in A$, $b \in J$ and $x \in H$, then we have

$$\|T_{ab} x\|^2 = (T_{ab} x, T_{ab} x) = (T^*_{ab} T_{ab} x, x) = (T_{b^* a^* ab} x, x) =$$
$$= (T_{b^*} T_{a^* ab} x, x) = (T_{a^* ab} x, T_b x) \leqq \|T_{a^* ab} x\| \, \|T_b x\|.$$

Replacing $a$ by $a^* a$ we obtain

$$\|T_{a^* ab} x\|^2 \leqq \|T_{(a^* a)^2 b} x\| \, \|T_b x\|,$$

and by recurrence,

$$\|T_{ab}x\|^{2^{n+1}} \leqq \|T_{(a^*a)^{2^n}}x\|\,\|T_bx\|^{2^{n+1}-1} \quad (n=0,1,2,\ldots)$$

But then we have by (4)

$$\|T_{ab}x\|^{2^{n+1}} \leqq |T_{(a^*a)^{2^n}b}|\,\|x\|\,\|T_bx\|^{2^{n+1}-1} \leqq q\big((a^*a)^{2^n}\big)|T_b|\,\|x\|\,\|T_bx\|^{2^{n+1}-1} =$$
$$= \big(q(a^*a)\big)^{2^n}|T_b|\,\|x\|\,\|T_bx\|^{2^{n+1}-1} = \big(q(a)\big)^{2^{n+1}}|T_b|\,\|x\|\,\|T_bx\|^{2^{n+1}-1}.$$

By letting $n\to\infty$ we obtain

$$\|T_{ab}x\| \leqq q(a)\|T_bx\| \quad \text{for all} \quad a\in A,\ b\in J,\ x\in H;$$

hence by (3)

$$\sup\{\|T_{ab}x\| : b\in J,\ x\in H,\ \|T_bx\|\leqq 1\} \leqq q(a) \quad (a\in A),$$

which finishes the proof.

We get finally a new proof of Theorem 2. 6 in [3] (or Theorem 4 in [4]).

Corollary. *Let $p$ be a $C^*$-semi-norm on a $*$-ideal $J$ of the complex $*$-algebra $A$. There exists then a $C^*$-semi-norm on $A$ which is equal to $p$ on $J$ if and only if*

(5) $$\sup\{p(ab) : b\in J,\ p(b)\leqq 1\} < \infty$$

*holds for all $a$ in $A$.*

Proof. Let $q$ be such a $C^*$-semi-norm. $q$ is also submultiplicative in consequence of our previous result (see [3], Theorem 2. 3 or [4], Theorem 2). We have for every $a\in A$, $b\in J$ that

$$p(ab) = q(ab) \leqq q(a)q(b) = q(a)p(b),$$

proving that (5) is necessary.

To show that (5) is sufficient consider $J_p=\{b\in J: p(b)=0\}$. Since $p$ is automatically submultiplicative and the $*$-operation is isometric with respect to $p$ also, $J_p$ is in fact a $*$-ideal in $J$ such that the completion $B_p$ of the quotient algebra $J/J_p$ with respect to the quotient norm and with natural involution is a $C^*$-algebra. The classical Gelfand—Naimark theorem assures a canonical isometrical $*$-representation of $B_p$ on some Hilbertspace $H$: $\overline{T}^p: B_p\to B(H)$. From the restriction $T^p$ of $\overline{T}^p$ to $J/J_p$ we get in a standard way a $*$-representation $T: J\to B(H)$ of $J$ on $H$ such that $T_b=T^p_{b+J_p}$ and $|T_b|=p(b+J_p)=p(b)$ holds for all $b\in J$. As a consequence (5) implies (3) for this $*$-representation $T$ and the statement follows from the Theorem.

Remark. Our result on automatic submultiplicativity of a $C^*$-semi-norm, mentioned above, is an improvement of a recent result due to ARAKI and ELLIOTT concerning the definition of $C^*$-algebras (see Theorem 1 in [1]).

For the following two remarks we are indebted to Dr. J. Szűcs. First, in the proof of the Lemma $S_a$ is well defined simply because (1) implies for $a\in A$, $b\in J$, $x\in H$

$$\sum_n T_{ab_n}x_n=0 \quad \text{provided} \quad \sum_n T_{b_n}x_n=0$$

since for each $t>0$, $0=\sum_n T_{tb_n}x_n=t\cdot\sum T_{b_n}x_n$ so that by (1)

$$\sup\{t\|\sum_n T_{ab_n}x_n\| : t>0\}=\sup\{\|\sum_n T_{atb_n}\| : t>0\}<\infty.$$

Secondly, (3) implies (2) as an easy application of Kaplansky's Density Theorem shows. For if $a\in A$, $b\in J$, $x\in H_1$ and $\|T_bx\|\leqq 1$ then it is enough to show $\|T_{ab}x\|\leqq q(a^*)$ or equivalently

$$|(T_{ab}x, y)|\leqq q(a^*)\|y\| \quad \text{for each} \quad y\in H.$$

$\{T_b^1 : b\in J\}\subset B(H_1)$ is a $*$-algebra of bounded linear operators on $H_1$ such that $\{T_b^1x : b\in J, x\in H_1\}$ spans $H_1$, its double commutant $N$ is a von Neumann algebra containing the identity operator on $H_1$. But $\{T_b^1 : b\in J\}$ is strongly dense in $N$ hence by Kaplansky's theorem the strong closure of the unit ball in $\{T_b^1 : b\in J\}$ contains $N_1$, the unit ball of $N$, especially the identity operator. Hence for a fixed $y\in H_1$ there exists $\{b_n\}_{n=1}^{\infty}\subset J$ with $|T_{b_n}|\leqq 1$ such that $\|T_{b_n}y-y\|\to 0$. We then obtain for $a\in A$, $b\in J$, $x\in H_1$ that

$$\begin{aligned}|(T_{ab}x, y)| &\leqq \sup_n|(T_{ab}x, T_{b_n}y)|=\sup_n|(T_{b_n^*ab}x, y)|=\\ &=\sup_n|(T_{(a^*b_n)^*}T_bx, y)|=\sup_n|(T_bx, T_{a^*b_n}y)|\leqq\sup_n\|T_{a^*b_n}y\|\leqq\\ &\leqq\sup_n|T_{a^*b_n}|\|y\|\leqq\|y\|q(a^*)\sup_n|T_{b_n}|\leqq\|y\|q(a^*),\end{aligned}$$

and thus the required inequality follows.

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