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Compact Operator Ranges and Reductive Algebras

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1. Introduction. Let $\mathscr{A}$ be an arbitrary subalgebra of $\mathscr{B}(\mathfrak{H})$ — the algebra of all (bounded) operators on the (complex) Hilbert space $\mathfrak{H}$. A sufficient condition that $\mathscr{A}$ be strongly dense in $\mathscr{B}(\mathfrak{H})$ was found by FOIAŞ [4]; it requires that $\mathscr{A}$ have no invariant operator ranges other than $\{0\}$ and $\mathfrak{H}$. This requirement is stronger than that of (topological) transitivity for $\mathscr{A}$, i.e., the hypothesis that $\mathscr{A}$ has no non-trivial invariant (closed) subspaces. This result was generalized in [5] to the theorem that if every proper operator range invariant under the transitive algebra $\mathscr{A}$ is the range of a compact operator, then $\mathscr{A}$ is strongly dense. One of the purposes of the present paper is to demonstrate that this generalization is not vacuous, and that in fact there exist proper, dense subalgebras of $\mathscr{B}(\mathfrak{H})$ leaving invariant an abundance of compact operator ranges but no other operator ranges.

The second purpose of this paper is to give an extension of the above result to reductive subalgebras of $\mathscr{B}(\mathfrak{H})$, i.e., those algebras whose invariant subspaces are all reducing. The new result, also shown to be non-vacuous, states that if the invariant operator ranges of a reductive algebra $\mathscr{A}$ are all "compact perturbations" of its invariant subspaces, in a certain sense, then $\mathscr{A}$ is strongly dense in a self-adjoint algebra. This also strengthens the theorem in [2] with the same conclusion but requiring that all invariant operator ranges be closed.

Algebras considered will be assumed to contain the identity, although this is not at all essential; the trivial modification necessary for the general case will be obvious to the reader.

2. Algebras with Invariant Compact Operator Ranges. We start with the following lemma whose proof can be found in [4].

Lemma 1. *Let $\mathscr{B}$ be a uniformly closed subalgebra of $\mathscr{B}(\mathfrak{H})$ which leaves the range of an injective operator S invariant. Then there exists $M>0$ such that $\|S^{-1}BS\| \leqq M\|B\|$ for every $B\in\mathscr{B}$.*

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Theorem 2. *Let K be a compact operator with dense range. Let $\mathscr{A}$ be the (transitive) algebra of all operators leaving $K\mathfrak{H}$ invariant. Then every proper operator range invariant under $\mathscr{A}$ is the range of a compact operator.*

(We remark that every $\mathscr{A}$-invariant operator range has to contain $K\mathfrak{H}$ by a result of [7].)

Proof. Assume, with no loss of generality, that $0 \leqq K \leqq 1$. Fix λ with $0<\lambda<1$, and let P_i be the finite-dimensional spectral projection of K corresponding to all the eigenvalues in the interval $(\lambda^i, \lambda^{i-1}]$. Let $\mathscr{T}$ denote the algebra of all upper-block-triangular operators relative to the decomposition $\sum_{i=1}^{\infty} \oplus P_i \mathfrak{H}$ of $\mathfrak{H}$. It follows from the characterization of $\mathscr{A}$ given in [7] that $\mathscr{T} \subseteqq \mathscr{A}$. We must prove that if S is an operator such that $S\mathfrak{H}$ is invariant under $\mathscr{A}$ and $S\mathfrak{H} \neq \mathfrak{H}$, then S is compact. Again we assume, with no loss of generality, that S is positive. This implies, since $S\mathfrak{H}$ is dense in $\mathfrak{H}$, that S is also injective.

Assume S is not compact. Then there is $\varepsilon>0$ and an infinite-dimensional spectral subspace $\mathfrak{M}$ for S such that $S|\mathfrak{M} \geqq \varepsilon$ (and thus $S\mathfrak{M}=\mathfrak{M}$). Now, since the subspace $\sum_{i=n+1}^{\infty} \oplus P_i \mathfrak{H}$ has finite codimension, it interscets $\mathfrak{M}$ nontrivially for every n. Pick a unit vector x in this intersection. Observe that if y is an arbitrary unit vector in $\sum_{i=1}^{n} \oplus P_i \mathfrak{H}$ for any n, then there exists $T \in \mathscr{T}$ with $\|T\|=1$ such that $Tx=y$. This is so because the subset

$$\left(\sum_{i=1}^{n} \oplus P_i\right) \mathscr{T} \left(\sum_{i=n+1}^{\infty} \oplus P_i\right)$$

of $\mathscr{T}$ contains all bounded linar transformations from $\sum_{i=n+1}^{\infty} P_i \mathfrak{H}$ into $\sum_{i=1}^{n} \oplus P_i \mathfrak{H}$, and, in particular, the rank-one operator that sends x to y and $\{x\}^{\perp}$ to $\{0\}$. Hence $y \in TS\mathfrak{H} \subseteqq S\mathfrak{H}$ and

$$\|S^{-1}y\| = \|S^{-1}TS(S^{-1}x\| \leqq \|S^{-1}TS\|/\varepsilon.$$

Since $\|S^{-1}TS\|$ is bounded on the unit ball of $\mathscr{A}$ (Lemma 1), we conclude that S^{-1} is bounded on the dense linear manifold $\bigcup_{n=1}^{\infty} \sum_{i=1}^{n} \oplus P_i \mathfrak{H}$. Thus S^{-1} is bounded and $S\mathfrak{H}=\mathfrak{H}$, which is a contradiction. This completes the proof.

We note here that the algebra $\mathscr{A}$ of the above theorem has many invariant operator ranges, e.g., the range of K^r for $0<r<1$. It also has mutually non-comparable invariant operator ranges (See [4] and [7].)

3. Reductive Algebras. A natural first question on reductive algebras suggested by the above-mentioned result of [5] is: What happens if it is assumed that every proper operator range invariant under the (infinite-dimensional) reductive algebra

$\mathscr{A}$ is a compact operator range? It is very easy to see that such an $\mathscr{A}$ is actually transitive and thus strongly dense by [5]. The next question is: What if we replace "proper" by "non-closed" in the above question? The answer is as expected: Such an algebra will have to be strongly dense in a self-adjoint algebra. But we shall prove a stronger result.

In what follows, by a *compact perturbation* of a subspace $\mathfrak{M}$ of $\mathfrak{H}$ we shall mean the range of any operator of the form $P+K$, where P is the orthogonal projection on $\mathfrak{M}$ and K is a compact operator with $KP=PK=0$. We allow P or K to be trivial. If $P\mathfrak{H}$ and $K\mathfrak{H}$ are both infinite-dimensional, this type of operator ranges are called *class 2b* ranges by Dixmier [1]. (See also [3].) An invariant subspace $\mathfrak{M}$ of an algebra $\mathscr{A}$ is an *atom* if $\mathscr{A}|\mathfrak{M}$ is transitive.

Theorem 3. *Let $\mathscr{A}$ be a reductive algebra on $\mathfrak{H}$ such that every invariant operator range of $\mathscr{A}$ is a compact perturbation of a subspace of $\mathfrak{H}$ (not necessarily invariant under $\mathscr{A}$). Then $\mathfrak{H}$ is a finite direct sum of atoms for $\mathscr{A}$.*

Proof. Any infinite chain (under inclusion) of subspaces of $\mathfrak{H}$ contains either a subchain isomorphic to the integers or one anti-isomorphic to the integers. Now pick a maximal chain C of invariant subspaces of $\mathscr{A}$. If C is infinite, then, by the above remark and by the reductivity of $\mathscr{A}$, we obtain infinitely many, mutually orthogonal invariant subspaces for $\mathscr{A}$. If some or all of these subspaces are finite-dimensional, we rearrange them in a double sequence $\mathfrak{N}_{ij}$ and let $\mathfrak{M}_i=\sum_j \oplus \mathfrak{N}_{ij}$. Thus we can assume $\mathfrak{H}=\sum_{i=1}^{\infty}\oplus\mathfrak{M}_i$, where each $\mathfrak{M}_i$ is an infinite-dimensional invariant subspace for $\mathscr{A}$. Then the operator $\sum_{i=1}^{\infty}\oplus(1/i)I_i$, where I_i is the identity on $\mathfrak{M}_i$ commutes with (every member of) $\mathscr{A}$, and thus its range is invariant under $\mathscr{A}$. But this range is not closed and is easily seen not to be a compact perturbation of any subspace, because every eigenvalue $1/i$ has infinite multiplicity. It follows from the hypotheses that C is finite. By the reductivity of $\mathscr{A}$ this chain gives rise to a finite number of atoms $\mathfrak{H}_i$ for $\mathscr{A}$ with $\mathfrak{H}=\sum\oplus\mathfrak{H}_i$.

In the rest of the paper we shall freely use the notation and terminology of [8] with one exception: we do not assume, as part of definition, that a reductive algebra is closed under any topology. The symbol $\mathscr{A}$ will consistently denote a reductive algebra.

We need the following lemmas in the proof of the main result of this section.

Lemma 4. *Let $\mathfrak{H}=\mathfrak{H}_1\oplus\ldots\oplus\mathfrak{H}_m$, where each $\mathfrak{H}_i$ is an atom for $\mathscr{A}$. Let $X\in$* Lat *$\mathscr{A}^{(k)}$ and assume $\mathfrak{X}$ is (a graph subspace) of the form*

$$\mathfrak{X}=\{(C_{11}x\oplus\ldots\oplus C_{1m}x)\oplus\ldots\oplus(C_{k1}x\oplus\ldots\oplus C_{km}x):x\in\mathfrak{D}\},$$

where $\mathfrak{D}$ is a nonzero linear manifold in $\mathfrak{H}_1$, each C_{ij} is a (not necessarily bounded) linear transformation with the common domain $\mathfrak{D}$ and range in $\mathfrak{H}_j$, and C_{11} is the identity on $\mathfrak{D}$. Then there exists bounded linear transformations D_{ij} from $\mathfrak{H}_1$ into $\mathfrak{H}_j$ such that

$$\mathfrak{X} = \{(D_{11}y\oplus\ldots\oplus D_{1m}y)\oplus\ldots\oplus(D_{k1}y\oplus\ldots\oplus D_{km}y): y\in\mathfrak{H}_1\}.$$

Proof. Since $\mathfrak{D}$ is the domain of the closed operator

$$T:\mathfrak{D}\to(\mathfrak{H}_2\oplus\ldots\oplus\mathfrak{H}_m)\oplus[\mathfrak{H}_1\oplus\ldots\oplus\mathfrak{H}_m]^{(k-1)}$$

defined by $Tx=C_{12}x\oplus\ldots\oplus C_{km}x$, it is also the range of a bounded injective operator $S:\mathfrak{H}_1\to\mathfrak{H}_1$ (Theorem 1.1 of [3]). Thus TS is a closed operator defined on $\mathfrak{H}_1$ and hence bounded by the Closed Graph Theorem. It follows that the transformations $D_{ij}=C_{ij}S$ are all bounded on $\mathfrak{H}_1$ and satisfy the requirements of the lemma.

The following lemma is easily verified; its proof is also given, e.g., in [8, Proof of Theorem 9.11].

Lemma 5. *Let $\mathscr{A}$ be a reductive algebra on $\mathfrak{H}=\mathfrak{H}_1\oplus\mathfrak{H}_2$ and let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be atoms for $\mathscr{A}$. Assume T is a bounded linear transformation from $\mathfrak{H}_1$ into $\mathfrak{H}_2$ whose graph is in* Lat $\mathscr{A}$. *Then T is a scalar multiple of an isometry U of $\mathfrak{H}_1$ onto $\mathfrak{H}_2$ and, consequently, $A|\mathfrak{H}_2=U(A|\mathfrak{H}_1)U^*$ for all $A\in\mathscr{A}$.*

It is convenient to introduce another ad-hoc definition: a subspace N of $(\mathfrak{H}_1\oplus\ldots\oplus\mathfrak{H}_m)^{(k)}$ will be called *special* if there exist scalars $\alpha_1,\ldots,\alpha_k$, and an integer i, $1\leqq i\leqq m$ such that

$$\mathfrak{N} = \{(0\oplus\ldots\oplus0\oplus\alpha_1x\oplus0\oplus\ldots\oplus0)\oplus\ldots\oplus(0\oplus\ldots\oplus0\oplus\alpha_kx\oplus0\oplus\ldots\oplus0): x\in\mathfrak{H}_i\},$$

where in each pair of parantheses the component α_jx occurs at the i-th place.

Lemma 6. *Let $\mathscr{A}$ be a reductive algebra on $\mathfrak{H}=\mathfrak{H}_1\oplus\ldots\oplus\mathfrak{H}_m$, where each $\mathfrak{H}_i$, $1\leqq i\leqq m$, is an atom for $\mathscr{A}$, and for no pair (i,j), $i\neq j$, there is an isometry U from $\mathfrak{H}_i$ onto $\mathfrak{H}_j$ such that*

$$A|\mathfrak{H}_j=U(A|\mathfrak{H}_i)U^*, \quad \textit{for all} \quad A\in\mathscr{A}.$$

Suppose also that the only proper operator ranges invariant under $\mathscr{A}|\mathfrak{H}_i$ are ranges of compact operators. Let $\mathfrak{N}\in\mathrm{Lat}\,\mathscr{A}^{(k)}$, so that every $y\in\mathfrak{N}$ has mk components y_j relative to $(\mathfrak{H}_1\oplus\ldots\oplus\mathfrak{H}_m)^{(k)}$. Assume that there is a subset J of the integers $1,\ldots,mk$ such that $y\in\mathfrak{N}$, $y\neq0$ implies $y_j\neq0$ for $j\in J$ and $y_j=0$ otherwise. Then $\mathfrak{N}$ is a special subspace.

Proof. Suppose $\mathfrak{N}\neq\{0\}$; otherwise the conclusion holds. From the hypothesis on $\mathfrak{N}$ one can easily conclude that for any $y\in\mathfrak{N}$, $y\neq0$, all the mk components of y can be uniquely (and hence linearly) determined by any nonzero one. Assume, with no loss of generality, that $y_{11}\neq0$ for all $y\in\mathfrak{N}$. Then Lemma 4 yields

$$\mathfrak{N} = \{(D_{11}x\oplus\ldots\oplus D_{1m}x)\oplus\ldots\oplus(D_{k1}x\oplus\ldots\oplus D_{km}x): x\in\mathfrak{H}_1\},$$

where D_{11} and hence all the nonzero D_{ij} are injective bounded linear transformations, by hypothesis. Since the range of each D_{ij} is invariant under $\mathscr{A}|\mathfrak{H}_j$, it follows that every nonzero D_{ij} is either compact or bijective.

If every D_{ij} is compact, then the operator

$$R: x \to (D_{11}x \oplus \ldots \oplus D_{1m}x) \oplus \ldots \oplus (D_{k1}x \oplus \ldots \oplus D_{km}x)$$

is also compact and thus $\mathfrak{N}$ is the range of a compact operator; since it is closed it must be finite-dimensional, and thus the nonzero D_{ij} are surjective. If $\mathfrak{N}$ is infinite-dimensional, then at least one D_{ij} should be surjective. Hence in all cases we can assume, with no loss of generality again, that D_{11} is surjective. Replacing x by $D_{11}^{-1}y$ and putting $E_{ij} = D_{ij}D_{11}^{-1}$ yields

$$\mathfrak{N} = \{(y \oplus \ldots \oplus E_{1m}y) \oplus \ldots \oplus (E_{k1}y \oplus \ldots \oplus E_{km}y) : y \in \mathfrak{H}_1\}.$$

The proof will be complete if we show that

(a) $E_{ij} = 0$ for $j \neq 1$, and (b) $E_{i1} = \alpha_i I$, $i = 1, 2, \ldots, k$.

To prove (a) we note that if $E_{ij} \neq 0$ for some i and j with $j \neq 1$, then $\{x \oplus E_{ij}x : x \in \mathfrak{H}_1\}$ will be an invariant graph subspace for the reductive algebra $\mathscr{A}|(\mathfrak{H}_1 \oplus \mathfrak{H}_j)$ (cf. [8, Lemma 9.1]). Hence, by Lemma 5 there is an isometry U of $\mathfrak{H}_1$ onto $\mathfrak{H}_j$ with $A|\mathfrak{H}_j = U(A|\mathfrak{H}_1)U^*$, contradicting the hypotheses.

To show (b) we observe that $\mathfrak{N} \in \operatorname{Lat} \mathscr{A}^{(k)}$ implies $E_{i1}A_1 = A_1 E_{i1}$ for all A_1 in $\mathscr{A}|\mathfrak{H}_1$. Since $\mathscr{A}|\mathfrak{H}_1$ is strongly dense in $\mathscr{B}(\mathfrak{H}_1)$ (by Theorem 2 of [5]) E_{i1} must be scalar.

Theorem 7. *Let $\mathscr{A}$ be as in the above lemma and k an arbitrary positive integer. Then every invariant subspace of $\mathscr{A}^{(k)}$ is the orthogonal direct sum of (at most mk) special subspaces, and $\mathscr{A}$ is strongly dense in $\mathscr{B}(\mathfrak{H}_1) \oplus \ldots \oplus \mathscr{B}(\mathfrak{H}_m)$.*

Proof. Let $\mathfrak{N} \in \operatorname{Lat} \mathscr{A}^{(k)}$, $\mathfrak{N} \neq \{0\}$. Let y_1 be a nonzero element of $\mathfrak{N}$ with maximal number of zeros among its mk components. Let $\mathfrak{N}_1$ be the invariant subspace of $\mathscr{A}^{(k)}$ generated by y_1, i.e.,

$$\mathfrak{N}_1 = \{A^{(k)}y_1 : A \in \mathscr{A}\}^-.$$

Since $\mathfrak{N}_1 \subseteq \mathfrak{N}$, every nonzero member of $\mathfrak{N}_1$ has the same nonzero components as y_1. Lemma 6 implies that $\mathfrak{N}_1$ is a special subspace of $(\mathfrak{H}_1 \oplus \ldots \oplus \mathfrak{H}_m)^{(k)}$. But every special subspace is invariant under the algebra $[\mathscr{B}(\mathfrak{H}_1) \oplus \ldots \oplus \mathscr{B}(\mathfrak{H}_m)]^{(k)}$ and, hence, so is its orthogonal complement, because this algebra is self-adjoint. Thus $\mathfrak{N}_1^{\perp} \in \operatorname{Lat} \mathscr{A}^{(k)}$ and consequently $\mathfrak{N} \ominus \mathfrak{N}_1 \in \operatorname{Lat} \mathscr{A}^{(k)}$.

If $\mathfrak{N} \ominus \mathfrak{N}_1 \neq \{0\}$, we repeat the process and find a special subspace $\mathfrak{N}_2 \subseteq \mathfrak{N} \ominus \mathfrak{N}_1$ and so on. Since $(\mathfrak{H}_1 \oplus \ldots \oplus \mathfrak{H}_m)^{(k)}$ is the orthogonal direct sum of at most mk special subspaces, it follows that this process terminates after a finite number of steps and we obtain

$$\mathfrak{N} = \mathfrak{N}_1 \oplus \ldots \oplus \mathfrak{N}_r,$$

where each $\mathfrak{N}_i$ is a special subspace invariant not only under $\mathscr{A}^{(k)}$ but actually under $[\mathscr{B}(\mathfrak{H}_1)\oplus\ldots\oplus\mathscr{B}(\mathfrak{H}_m)]^{(k)}$. Thus we have shown that

$$\operatorname{Lat}\mathscr{A}^{(k)}\subseteq\operatorname{Lat}[\mathscr{B}(\mathfrak{H}_1)\oplus\ldots\oplus\mathscr{B}(\mathfrak{H}_m)]^{(k)}$$

for every integer k; it follows from a result of [9] that the strong closure of $\mathscr{A}$ is $\mathscr{B}(\mathfrak{H}_1)\oplus\ldots\oplus\mathscr{B}(\mathfrak{H}_m)$ as asserted.

We now consider the most general reductive algebra whose invariant operator ranges are compact perturbations of its invariant subspaces. This can be done, in view of Theorem 3, by allowing isometries of the sort excluded in Lemma 6. Then $\mathscr{A}$ is easily seen to be unitarily equivalent to an algebra (denoted by $\mathscr{A}$ again) of the following form: The underlying space $\mathfrak{H}$ is expressed as $\mathfrak{H}_1^{(p_1)}\oplus\ldots\oplus\mathfrak{H}_r^{(p_r)}$; for each i, $\mathscr{A}|\mathfrak{H}_i^{(p_i)}=\mathscr{A}_i^{(p_i)}$, where $\mathscr{A}_i$ is a transitive algebra on $\mathfrak{H}_i$ whose proper invariant operator ranges are all compact operator ranges. Furthermore for no pair i, j with $i\neq j$, there is an isometry U such that $A_j=UA_iU^*$, $A_i\in\mathscr{A}_i$, $A_j\in\mathscr{A}_j$. (All such unitarily equivalent summands of $\mathscr{A}$ have already been put in the r "bunches".)

Theorem 8. *If all the invariant operator ranges of a reductive algebra $\mathscr{A}$ are compact perturbations of its invariant subspaces, then its strong closure is self-adjoint. More precisely, $\mathscr{A}$ is strongly dense in an algebra of the form*

$$[\mathscr{B}(\mathfrak{H}_1)]^{(p_1)}\oplus\ldots\oplus[\mathscr{B}(\mathfrak{H}_r)]^{(p_r)}$$

modulo a suitable unitary equivalence.

Proof. Let $\mathfrak{H}_i$ and p_i be as in the paragraph preceding the theorem, after a suitable unitarily equivalent form of $\mathscr{A}$ is chosen. Take a "representative of each bunch" and form the subspace

$$\mathfrak{K}=\mathfrak{H}_1\oplus\ldots\oplus\mathfrak{H}_r.$$

Then $\mathscr{A}|\mathfrak{K}$ and $\mathfrak{K}$ satisfy all the hypotheses of theorem 7 and, hence, $\mathscr{A}|\mathfrak{K}$ is strongly dense in $\mathscr{B}(\mathfrak{H}_1)\oplus\ldots\oplus\mathscr{B}(\mathfrak{H}_r)$. It follows that $\mathscr{A}$ is strongly dense in an algebra of the form exhibited above.

We can use Theorem 2 to construct non-trivial examples of reductive algebras satisfying the hypotheses of Theorem 8.

Example. For each i, $1\leq i\leq n$, let $\mathscr{A}_i$ be the algebra of all operators on $\mathfrak{H}$ leaving the range of an injective compact operator K_i invariant. Let $\mathscr{A}$ be the algebra

$$\{A_1\oplus\ldots\oplus A_n: A_i\in\mathscr{A}_i,\quad i=1,\ldots,n\}.$$

It can be verified that $\mathscr{A}$ is reductive and that if $\mathfrak{X}$ is an operator range invariant under $\mathscr{A}$, then $\mathfrak{X}$ is the range of an operator of the form $B_1\oplus\ldots\oplus B_n$, where each B_i is O, I, or a non-zero compact operator (by Theorem 2).

We conclude the paper with a question: can one get a density result for the reductive algebra $\mathcal{A}$ by merely assuming that its invariant operator ranges are all compact perturbations of arbitrary (not necessarily $\mathcal{A}$-invariant) subspaces? We observe that, even in the special case where $\mathcal{A}$ is transitive, the question seems to be non-trivial.

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