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Classical approximation processes in connection with Lax equivalence theorems with orders

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1. Introduction

In this note we continue our previous investigations [8], [10] on Lax equivalence theorems with orders in the setting of linear operators in Banach spaces. There we were concerned (compare also with [18a] as well as with [6] and the literature cited there) with a quantitative description of the approximation of the exact solution $\{E(t); t \geqq 0\}$ of a properly posed initial value problem, being a (continuous) semigroup of class (C_0), by some difference scheme $\{E_\tau^n; n \in \mathbf{P}\}$ constituting a family $(0 \leqq \tau \leqq \delta)$ of discrete semigroups (with $\mathbf{P}$ the set of non-negative integers). According to the hierarchy of the various convergence theorems for families of semigroups as outlined by STRANG [20] (see also [2], [23]), one may then ask whether one can also equip more general theorems than the original Lax one with orders. To this problem Thm. 2 below will give a modest contribution inasmuch as the convergence of a family $\{E_\tau(t); t \geqq 0\}$, $0 \leqq \tau \leqq \delta$ of continuous semigroups towards $\{E(t); t \geqq 0\}$ is considered with orders, but still in the Lax framework.

There is another point which motivated the present studies. In [14] GROETSCH—KING outlined an interesting interconnection between Bernstein polynomials and the convergence of a certain difference scheme (see Sec. 3, Ex. A) which was then continued in [15] with respect to some quantitative results. The procedure, however, looks somewhat isolated so as to be particularly taylored to Bernstein polynomials. Thus the question arises whether there are further classical noncommutative processes in approximation theory of the type

$$\sum f(k/n) Q_{k,n}(x),$$

the convergence of which may be interpreted from this numerical point of view. This is indeed the case and will be worked out explicitly for the familiar Szász—Mirakyan and Baskakov operators. But also the general class of approximation processes as introduced in [18] via the powers of certain functions fit into this program. In fact, it turns out that the procedure and results of [14] may be considered as a genuine application to our previous Lax equivalence theorem with orders.

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In Sec. 2 we first treat two alternative forms of the (discrete) Lax equivalence theorem with orders, extending by the way those of [8], [10] slightly (cf. Thm. 1; 3). Correspondingly, the matter is considered in connection with a continuous version of the theorem of Lax on the convergence of families of semigroups (see Thm. 2; 4). The latter results are obtained by exploiting methods used in [13] to give an elementary proof of a weak (non-order) version of the Trotter theorem. In Sec. 3 the Lax theory for difference schemes (Thm. 1; 3 of Sec. 2) is applied to some examples of the form

$$E_\tau = \sum_{k=0}^{\infty} \varphi_k(\lambda) T_h^k \quad (\lambda := \tau/h),$$

where $T_h f(x) := f(x+h)$. For explicit difference schemes (Ex. A) the series is finite, whereas for implicit difference schemes (Ex. B and C) the series may be infinite (compare [5], [6]). Stability and consistency properties are given in terms of the (positive) functions $\varphi_k(\lambda)$. As mentioned above, special choices of the $\varphi_k(\lambda)$ lead to Bernstein polynomials, Baskakov operators, and the operators of Szász—Mirakyan. In Sec. 4 we consider the same examples from the point of view of the continuous semigroups $\{E_\tau^{t/\tau}; t \geqq 0\}$ which interpolate the discrete ones $\{E_\tau^n; n \in \mathbf{P}\}$ used so far at the grid points $n\tau$. In this situation the continuous versions of the Lax equivalence theorem with orders (cf. Thm. 2; 4 of Sec. 2) may be applied. Finally in Sec. 5, instead of reproducing the Ex. A—C via the interpolating semigroups $\{E_\tau^{t/\tau}; t \geqq 0\}$, one may consider the family of semigroups $\{\exp[t(E_\tau - I)]; t \geqq 0\}$ being a familiar construction in the course of the proof of the original Trotter theorem. In this case one obtains a comparison between a given difference scheme and the corresponding line method which in turn implies a comparison theorem between the Bernstein polynomials and the operators of Szász—Mirakyan.

Summarizing, the applications deliver pointwise direct approximation theorems for the Bernstein polynomials, the Baskakov, and the Szász—Mirakyan operators which are best possible, apart from constants. Though these direct theorems as such are of course well-known, they do not only show interesting interconnections between the Lax theorem in numerical analysis and the convergence of some classical approximation processes but they also indicate that the notions and results of the abstract theory in Sec. 2 seem to be adequate.

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2. General theory

Let X be a Banach space (with norm $\|\cdot\|_X$) and $[X]$ the space of bounded linear operators of X into itself. A (continuous) semigroup $\{E(t); t \geqq 0\} \subset [X]$ (of class (C_0)) is a one parameter family of operators satisfying: $E(0)=I$, the identity, $E(t_1+t_2)=E(t_1)E(t_2)$, and $\lim_{t\to 0+} \|E(t)f-f\|_X=0$ for each $f \in X$. For a semigroup of class (C_0) there exist constants $M \geqq 1$, $\omega \geqq 0$ such that (for the fundamentals of semigroup theory see [7])

$$\|E(t)\|_{[X]} \leqq Me^{\omega t} \quad (t \geqq 0). \tag{2.1}$$

Consider the initial value problem

$$d/dt\, u(t) = Au(t) \quad \text{for} \quad t \geqq 0; \quad u(0)=f \quad \text{for} \quad f \in X, \tag{2.2}$$

where A is a closed linear operator with domain $D(A)$ dense in X and range in X, the given element f describing the initial state. The problem (2.2) is said to be properly (or correctly) posed if there exists a (continuous) semigroup $\{E(t); t \geqq 0\}$ (of class (C_0)) such that each solution of (2.2) is of the form $u(t)=E(t)f$. In this case, A is the infinitesimal generator of the semigroup, i.e. the closed linear operator defined densely in X via

$$Af := \lim_{t\to 0+} t^{-1}[E(t)-I]f,$$

the domain $D(A)$ consisting of all elements $f \in X$ for which the limit exists.

In numerical analysis one is now interested in approximating the family of "exact" operators $\{E(t)\}$ by powers of some finite difference operators $\{E_\tau; 0 \leqq \tau \leqq \delta\} \subset [X]$, in particular to treat the error $\|E_\tau^n f - E(n\tau)f\|_X$ in dependence upon smoothness properties of $f \in X$. In this connection the most important properties of the difference scheme are stability and consistency for which the following definitions (with orders) were used in [8] (see also the literature cited there):

Definition 1. The difference scheme $\{E_\tau; 0 \leqq \tau \leqq \delta\} \subset [X]$ is said to be *consistent of order* $O(\varphi(\tau))$ on the linear manifold $U \subset X$ with respect to the semigroup $\{E(t); t \geqq 0\}$ if there is a constant $C > 0$ such that for all $f \in U$, $t \geqq 0$, $0 \leqq \tau \leqq \delta$

$$\|[E_\tau - E(\tau)]E(t)f\|_X \leqq C\tau\varphi(\tau)e^{\omega t}|f|_U \tag{2.3}$$

where $|f|_U$ denotes a suitable seminorm on U. If U is dense in X and $\varphi(\tau)$ in (2.3) is replaced by $o(1)$, $\tau \to 0+$, the difference scheme is said to be (ordinarily) *consistent.*

Definition 2. The difference scheme $\{E_\tau; 0 \leqq \tau \leqq \delta\} \subset [X]$ is said to be *stable of order* $O(\psi(\tau, 1/n))$ if there is a constant $S > 0$ such that for all $n \in \mathbf{N}$ (=set of natural numbers), $0 \leqq \tau \leqq \delta$

$$\|E_\tau^n\|_{[X]} \leqq S/\psi(\tau, 1/n). \tag{2.4}$$

and (ordinarily) *stable* if the right-hand side of (2.4) is replaced by $O(1)$, $\tau \to 0+$.

Here $\varphi(\tau)$ is some non-negative function on $[0, \delta]$ and $\psi(\tau, y)$ a positive bounded function on $[0, \delta]\times[0, 1]$ monotonely increasing in y and normalized via (cf. (2.1))

(2.5) $$Me^{\omega n\tau} \leqq S/\psi(\tau, 1/n) \quad (0 \leqq \tau \leqq \delta, n\in\mathbf{N}).$$

In this terminology the Lax theorem in its original form reads (see [17], [19])

Theorem L 1 (discrete version). *Given the properly posed initial value problem (2.2) in X and a finite difference scheme $\{E_\tau; 0\leqq\tau\leqq\delta\}$ satisfying the (ordinary) consistency condition, then (ordinary) stability is necessary and sufficient for (ordinary) convergence, i.e. for each $f\in X$*

$$\lim_{j\to\infty} \|E_{\tau_j}^{n_j} f - E(t)f\|_X = 0$$

for each sequence $\{(n_j, \tau_j)\}_{j\in\mathbf{N}}$ with $\tau_j\to 0+$, $n_j\tau_j\to t<\infty$ as $j\to\infty$.

Following [8], [10] one can equip this equivalence theorem with orders, smoothness properties of the element $f\in X$ being measured in terms of the so-called modified K-functional $(t\geqq 0)$

(2.6) $$K(t,f) := K(t,f; X, U) := \inf_{g\in U} \{\|f-g\|_X + t|g|_U\}.$$

This is known to be a continuous and monotonely increasing function of t with $\lim_{t\to 0+} K(t, f)=0$ for all $f\in X$ if U is dense in X. One also has, in view of the definition,

(2.7) $$K(t, f) \leqq \begin{cases} \|f\|_X, & f\in X \\ t|f|_U, & f\in U. \end{cases}$$

Theorem 1. *Let the finite difference scheme $\{E_\tau; 0\leqq\tau\leqq\delta\}$ be consistent of order $O(\varphi(\tau))$ on $U\subset X$ with respect to the semigroup $\{E(t); t\geqq 0\}$. Then the following assertions are equivalent:*

(a) $$\|E_\tau^n f - E(n\tau)f\|_X \leqq \frac{2S}{\psi(\tau, 1/n)} K((C/2)n\tau e^{\omega n\tau}\varphi(\tau), f),$$

(b) $$\|E_\tau^n f - E(n\tau)f\|_X \leqq \frac{2S}{\psi(\tau, 1/n)} \begin{cases} M_f, & f\in X \\ (C/2)n\tau e^{\omega n\tau}\varphi(\tau)|f|_U, & f\in U, \end{cases}$$

(c) $$\|E_\tau^n\|_{[X]} \leqq S/\psi(\tau, 1/n),$$

where M_f is a constant only depending on f (there is a slight abuse of the constants S).

Proof. The implication (a)$\Rightarrow$(b) follows by (2.7). Moreover, by the uniform boundedness principle one may replace the constant M_f by $C_1\|f\|_X$ for some $C_1>0$. Together with (2.1) and (2.5) this shows (b)$\Rightarrow$(c). Concerning the proof (c)$\Rightarrow$(a), in view of the identity

(2.8) $$E_\tau^n g - E(n\tau)g = \sum_{j=0}^{n-1} E_\tau^{n-j-1}[E_\tau - E(\tau)]E(j\tau)g$$

one has for any $g\in U$

$$\|E_\tau^n g - E(n\tau)g\|_X \leqq \sum_{j=0}^{n-1} (S/\psi(\tau,(n-j-1)^{-1}))C\tau\varphi(\tau)e^{\omega j\tau}|g|_U \leqq \frac{SCn\tau}{\psi(\tau,1/n)}e^{\omega n\tau}\varphi(\tau)|g|_U,$$

using stability and consistency with orders. Hence for any $f\in X$ it follows by (2.4), (2.5) that for any $g\in U$

$$\|E_\tau^n f - E(n\tau)f\|_X \leqq \|E_\tau^n(f-g)\|_X + \|E(n\tau)(f-g)\|_X + \|E_\tau^n g - E(n\tau)g\|_X \leqq$$
$$\leqq \left[\frac{S}{\psi(\tau,1/n)} + Me^{\omega n\tau}\right]\|f-g\|_X + \frac{SCn\tau}{\psi(\tau,1/n)}e^{\omega n\tau}\varphi(\tau)|g|_U \leqq$$
$$\leqq \frac{2S}{\psi(\tau,1/n)}\{\|f-g\|_X + (C/2)n\tau e^{\omega n\tau}\varphi(\tau)|g|_U\}.$$

Taking the infimum over all $g\in U$ yields (a). This completes the proof.

Up to this stage we approximated the exact solution $\{E(t); t\geqq 0\}$ by some difference scheme, thus by some family $(0\leqq\tau\leqq\delta)$ of *discrete* semigroups $\{E_\tau^n; n\in\mathbf{P}\}$. Now we want to approximate the "exact" operators by a family of *continuous* semigroups $\{E_\tau(t); t\geqq 0,\ 0\leqq\tau\leqq\delta\}\subset[X]$ of class (C_0). Indeed, the most important properties determining the approximation error $\|E_\tau(t)f - E(t)f\|_X$ are very similar to those given in Def. 1; 2. So one may formulate

Definition 3. The semigroup scheme $\{E_\tau(t); t\geqq 0, 0\leqq\tau\leqq\delta\}$ with infinitesimal generators A_τ is said to be *consistent of order* $O(\varphi(\tau))$ on the linear manifold $U\subset X$ with respect to the semigroup $\{E(t); t\geqq 0\}$ with generator A if $E(t)U\subset \subset D(A)\cap D(A_\tau)$ and there exists a constant $C>0$ such that for all $f\in U$, $t\geqq 0$, $0\leqq\tau\leqq\delta$

(2.9) $$\|[A_\tau - A]E(t)f\|_X \leqq Ce^{\omega t}\varphi(\tau)|f|_U.$$

It is said to be (ordinarily) *consistent* if U is dense in X and $\varphi(\tau)$ in (2.9) is replaced by $o(1)$.

Definition 4. The semigroup scheme $\{E_\tau(t); t\geqq 0, 0\leqq\tau\leqq\delta\}$ is said to be *stable of order* $O(M_\tau e^{\omega_\tau t})$ if there are constants M_τ and ω_τ with $M\leqq M_\tau$ and $\omega\leqq\omega_\tau$ (cf. (2.1)) such that for all $t\geqq 0$, $0\leqq\tau\leqq\delta$

(2.10) $$\|E_\tau(t)\|_{[X]} \leqq M_\tau e^{\omega_\tau t}.$$

It is said to be (ordinarily) *stable* if $M_\tau\leqq M_0<\infty$ and $\omega_\tau\leqq\omega_0<\infty$.

Of course, since $\{E_\tau(t); t\geqq 0\}$ is assumed to be a semigroup of class (C_0) for each $0\leqq\tau\leqq\delta$, property (2.1) always ensures the existence of constants M_τ, ω_τ such that (2.10) holds. So Def. 4 just states that it is appropriate to take $M_\tau e^{\omega_\tau t}$ as a substitute for $S/\psi(\tau, 1/n)$ in (2.4).

In the above terminology one has the following continuous counterpart to Theorem L 1 (cf. [2], [20]):

Theorem L 2 (continuous version). *Let* $\{E_\tau(t);\, t \geqq 0, 0 \leqq \tau \leqq \delta\}$ *be a semigroup scheme (ordinarily) consistent with respect to* $\{E(t);\, t \geqq 0\}$. *Then (ordinary) stability is necessary and sufficient for (ordinary) convergence, i.e. for each* $f \in X$, $t \geqq 0$

$$\lim_{\tau \to 0+} \|E_\tau(t) f - E(t) f\|_X = 0.$$

Again this convergence theorem can be equipped with orders.

Theorem 2. *Let the semigroup scheme* $\{E_\tau(t);\, t \geqq 0,\ 0 \leqq \tau \leqq \delta\}$ *be consistent of order* $O(\varphi(\tau))$ *on* $U \subset X$ *with respect to the semigroup* $\{E(t);\, t \geqq 0\}$. *Then the following assertions are equivalent:*

(a) $$\|E_\tau(t) f - E(t) f\|_X \leqq 2M_\tau e^{\omega_\tau t} K((C/2) t\varphi(\tau), f),$$

(b) $$\|E_\tau(t) f - E(t) f\|_X \leqq 2M_\tau e^{\omega_\tau t} \begin{cases} M_f, & f \in X \\ (C/2) t\varphi(\tau) |f|_U, & f \in U, \end{cases}$$

(c) $$\|E_\tau(t)\|_{[X]} \leqq M_\tau e^{\omega_\tau t}.$$

Proof. By (2.7) we immediately obtain (a)$\Rightarrow$(b). Then by the uniform boundedness principle one may replace the constant M_f by $C_1 \|f\|_X$ which together with (2.1) implies (b)$\Rightarrow$(c). For the proof of (c)$\Rightarrow$(a) it follows that for arbitrary $g \in U \subset$ $\subset D(A) \cap D(A_\tau)$

$$E_\tau(t) g - E(t) g = -\int_0^t \frac{d}{ds} E_\tau(t-s) E(s) g\, ds =$$

$$= \int_0^t [A_\tau E_\tau(t-s) E(s) g - E_\tau(t-s) A E(s) g]\, ds = \int_0^t E_\tau(t-s) [A_\tau - A] E(s) g\, ds$$

which should be compared with (2.8), thus with (2.3) and (2.9), respectively. Hence

$$\|E_\tau(t) g - E(t) g\|_X \leqq \int_0^t M_\tau e^{\omega_\tau (t-s)} C e^{\omega s} \varphi(\tau) |g|_U\, ds \leqq M_\tau C t e^{\omega_\tau t} \varphi(\tau) |g|_U.$$

As in the proof of Thm. 1 we proceed for $f \in X$, $g \in U$

$$\begin{aligned} \|E_\tau(t) f - E(t) f\|_X &\leqq \|E_\tau(t)(f-g)\|_X + \|E(t)(f-g)\|_X + \|E_\tau(t) g - E(t) g\|_X \leqq \\ &\leqq (M_\tau e^{\omega_\tau t} + M e^{\omega t}) \|f-g\|_X + M_\tau C t e^{\omega_\tau t} \varphi(\tau) |g|_U \leqq \\ &\leqq 2M_\tau e^{\omega_\tau t} \{\|f-g\|_X + (C/2) t\varphi(\tau) |g|_U\}. \end{aligned}$$

Taking the infimum over all $g \in U$ completes the proof.

So far Thms. 1, 2 do have the structure of the original Lax equivalence theorem, stating that stability is equivalent to convergence, provided the scheme is consistent. The adequacy of the notions with orders used above may also be illustrated by the fact that the alternative form is valid as well, namely that convergence is equivalent to stability plus consistency, provided some weak additional assumptions are made. First we claim the commutativity of seminorm and semigroup, more specifically, we suppose that $E(t)U\subset U$ and (cf. (2.1))

$$|E(t)g|_U \leqq Me^{\omega t}|g|_U \quad (t\geqq 0) \tag{2.11}$$

for each $g\in U$ (in [18a] problem (2.2) is then said to be strongly correctly posed). For example, inequality (2.11) obviously holds if $|g|_U:=\|A^r g\|_X$, $U=D(A^r)$.

Theorem 3. *Given the finite difference scheme* $\{E_\tau; 0\leqq\tau\leqq\delta\}$ *and the semigroup* $\{E(t); t\geqq 0\}$, *suppose that* (2.11) *be valid and* $\psi(\tau, 1)\geqq C_2>0$ *for all* $0\leqq\tau\leqq\delta$. *Then the following assertions are equivalent:*

(a) $$\|E_\tau^n f-E(n\tau)f\|_X \leqq \frac{2S}{\psi(\tau,1/n)}K((C/2)n\tau e^{\omega n\tau}\varphi(\tau), f),$$

(b) $$\|E_\tau^n f-E(n\tau)f\|_X \leqq \frac{2S}{\psi(\tau,1/n)}\begin{cases}M_f, & f\in X\\ (C/2)n\tau e^{\omega n\tau}\varphi(\tau)|f|_U, & f\in U,\end{cases}$$

(c) (i) $\|E_\tau^n\|_{[X]}\leqq S/\psi(\tau, 1/n)$,

(ii) $\|[E_\tau-E(\tau)]E(t)f\|_X\leqq Ce^{\omega t}\tau\varphi(\tau)|f|_U$ *for all* $f\in U, t\geqq 0, 0\leqq\tau\leqq\delta$.

For a proof one may consult [8].

Theorem 4. *Given the semigroup scheme* $\{E_\tau(t);\ t\geqq 0,\ 0\leqq\tau\leqq\delta\}$ *and the semigroup* $\{E(t); t\geqq 0\}$, *suppose that inequality* (2.11) *be valid and* $M_\tau\leqq M_0<\infty$ *for all* $0\leqq\tau\leqq\delta$. *Then the following assertions are equivalent:*

(a) $$\|E_\tau(t)f-E(t)f\|_X\leqq 2M_\tau e^{\omega_\tau t}K((C/2)t\varphi(\tau), f),$$

(b) $$\|E_\tau(t)f-E(t)f\|_X\leqq 2M_\tau e^{\omega_\tau t}\begin{cases}M_f, & f\in X\\ (C/2)t\varphi(\tau)|f|_U, & f\in U,\end{cases}$$

(c) (i) $\|E_\tau(t)\|_{[X]}\leqq M_\tau e^{\omega_\tau t}$,

(ii) $\|[A_\tau-A]E(t)f\|_X\leqq Ce^{\omega t}\varphi(\tau)|f|_U$ *for all* $f\in U, t\geqq 0, 0\leqq\tau\leqq\delta$.

Proof. In view of the proof of Thm. 2 we only need to show (b)$\Rightarrow$(c, ii). Let $f=E(s)g$ for some $g\in U$, $s\geqq 0$. Then (b) and (2.11) imply

$$\|E_\tau(t)f-E(t)f\|_X\leqq M_\tau e^{\omega_\tau t}Ct\varphi(\tau)|E(s)g|_U\leqq MM_\tau e^{\omega_\tau t}Ct\varphi(\tau)e^{\omega s}|g|_U.$$

Therefore one has

$$\|[A_\tau-A]E(s)g\|_X=\lim_{t\to 0+}\left\|[t^{-1}[E_\tau(t)-I]-t^{-1}[E(t)-I]]E(s)g\right\|_X\leqq$$

$$\leqq MM_\tau C\varphi(\tau)e^{\omega s}|g|_U\leqq C^*\varphi(\tau)e^{\omega s}|g|_U$$

which completes the proof.

3. Applications to specific difference schemes

An example of an initial value problem (2.2) is supplied by the hyperbolic differential equation

$$(3.1)\qquad d/dt\ u(x,t) = d/dx\ u(x,t), \quad x,t \geqq 0; \quad u(x,0) = f(x), \quad x \geqq 0,$$

where f is an element of $X := UCB(\mathbf{R}^+)$, the Banach space of all bounded, uniformly continuous functions on $[0,\infty)$ with $\|f\|_X := \sup_{x\geqq 0} |f(x)|$. This problem is properly posed, the solution operators $E(t)$ being given via

$$(3.2)\qquad u(x,t) = E(t)f(x) = f(x+t), \quad x,t \geqq 0.$$

Let us consider some examples of difference schemes applied in numerical analysis to approximate the exact solution (3.2). We use the notations

$$(3.3)\qquad (T_h f)(x) := f(x+h), \quad (E_\tau u)(x,t) := u(x,t+\tau), \quad \lambda := \tau/h$$

for the translation operator T_h, $h \geqq 0$, the step operator E_τ, $\tau \geqq 0$, and the ratio $\lambda \geqq 0$ of the step sizes, respectively.

Example A. Instead of (3.1) we regard the problem

$$\tau^{-1}[u(x,t+\tau) - u(x,t)] = h^{-1}[u(x+h,t) - u(x,t)].$$

This defines an explicit difference scheme with step operator

$$(3.4)\qquad E_\tau = (1-\lambda)I + \lambda T_h.$$

Example B. If we replace (3.1) by

$$\tau^{-1}[u(x,t+\tau) - u(x,t)] = h^{-1}[u(x+h,t+\tau) - u(x,t+\tau)],$$

the step operator is defined via

$$E_\tau - I = \lambda[T_h - I]E_\tau.$$

This leads to the implicit difference scheme

$$(3.5)\qquad E_\tau = \frac{1}{1+\lambda}\left[I - \frac{\lambda}{1+\lambda}T_h\right]^{-1} = \frac{1}{1+\lambda}\sum_{k=0}^{\infty}\left(\frac{\lambda}{1+\lambda}\right)^k T_h^k.$$

Example C. Replacing only d/dx in (3.1) by the corresponding difference quotient, one has to solve the initial value problem

$$(3.6)\qquad d/dt\, u(x,t) = h^{-1}[u(x+h,t) - u(x,t)].$$

This line or semi-discrete method (cf. [21, p. 545] or [6, p 55]) leads to the step operator (see also Sec. 4)

$$(3.7)\qquad E_\tau = \exp(\tau/h)[T_h - I] = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} T_h^k.$$

Obviously, each of these operators E_τ is of the form

$$(3.8)\qquad E_\tau = \sum_{k=0}^{\infty} \varphi_k(\lambda) T_h^k$$

with certain real-valued functions $\varphi_k(\lambda)$ defined on $[0, \infty)$. To discuss stability and consistency, let us suppose that there exists an interval $J \subset [0, \infty)$ such that for all $\lambda \in J$, $k \in \mathbf{P}$

$$(3.9)\qquad \text{(i)}\quad \varphi_k(\lambda) \geqq 0, \quad \text{(ii)}\quad \sum_{k=0}^{\infty} \varphi_k(\lambda) = 1, \quad \text{(iii)}\quad \sum_{k=0}^{\infty} k\varphi_k(\lambda) = \lambda \quad (\lambda \in J).$$

In particular, (i) assumes the positivity of the operators E_τ which together with (ii) leads to stability since

$$\|E_\tau^n\|_{[X]} \leqq \|E_\tau\|_{[X]}^n = \|E_\tau 1\|_X^n = 1.$$

Concerning consistency let

$$U := UCB^{(2)} := \{f \in X;\ f', f'' \in X\}, \quad |f|_U := \|f''\|_X.$$

Then one has by (3.2), (3.8), (3.9) that for every $f \in U$

$$(3.10)\qquad |E_\tau u(x, t) - E(\tau) u(x, t)| = \left| \sum_{k=0}^{\infty} \varphi_k(\lambda)[u(x+kh, t) - u(x, t+\tau)] \right| =$$

$$= \left| \sum_{k=0}^{\infty} \varphi_k(\lambda) [f(x+t+\tau+(kh-\tau)) - f(x+t+\tau)] \right| =$$

$$= \left| \sum_{k=0}^{\infty} \varphi_k(\lambda) \Big[(kh-\tau) f'(x+t+\tau) + \int_0^{kh-\tau} \int_0^{s} f''(v+x+t+\tau)\, dv\, ds\Big] \right| \leqq$$

$$\leqq \|f''\|_X \sum_{k=0}^{\infty} \varphi_k(\lambda)(kh-\tau)^2/2 = (h^2/2)\sigma(\lambda)\|f''\|_X$$

with second moment $\sigma(\lambda)$ given via

$$\sigma(\lambda) := \sum_{k=0}^{\infty} \varphi_k(\lambda)(k-\lambda)^2 = \sum_{k=0}^{\infty} k^2 \varphi_k(\lambda) - \lambda^2.$$

Before giving an application of Thm. 1, let us recall that for the present choices of spaces X, U one may express the K-functional $K(t, f; UCB, UCB^{(2)})$ equivalently in terms of moduli of continuity. Indeed, one has for any $t \geqq 0$ (cf. [7, p. 192; 258], [9, p. 316])

$$c_1\omega_2(t^{1/2},f) \leqq K(t,f) \leqq c_2\omega_2(t^{1/2},f) \tag{3.11}$$

where the (second) modulus of continuity is defined by

$$\omega_2(t,f) := \sup_{0 \leqq h \leqq t} \|f(x+2h)-2f(x+h)+f(x)\|_X.$$

Thus it follows by Thm. 1, (c)$\Rightarrow$(a), that

Corollary 1. *Concerning the convergence of the difference scheme* (3.8) *towards the exact solution* (3.2) *of the initial value problem* (3.1) *one has*

$$|E_\tau^n f(x)-f(x+n\tau)| \leqq 2c_2\omega_2\big((h/2)[n\sigma(\lambda)]^{1/2},f\big) \tag{3.12}$$

for any $f \in UCB(\mathbf{R}^+)$, $x \geqq 0$, $n \in \mathbf{N}$, $\tau \geqq 0$, *and* $\lambda \in J$.

More specifically, this yields for the examples mentioned above:

Example A: In view of (3.4) we see that

$$\varphi_0(\lambda) = 1-\lambda, \quad \varphi_1(\lambda) = \lambda, \quad \varphi_k(\lambda) = 0 \quad \text{for} \quad k \geqq 2.$$

Thus (cf. (3.9) (i)) one has $J=[0, 1]$ and $\sigma(\lambda)=\lambda(1-\lambda)$. Since for $x=0$, $h=1/n$

$$[E_\tau^n f](0) = \sum_{k=0}^{n} \binom{n}{k} (1-\lambda)^{n-k}\lambda^k T_{1/n}^k f(0) = \sum_{k=0}^{\infty} \binom{n}{k}(1-\lambda)^{n-k}\lambda^k f(k/n) = B_n(f,\lambda),$$

Cor. 1 implies the following (pointwise) direct theorem for the Bernstein polynomials.

Corollary 2. *For any function* f, *continuous on* [0, 1], *one has for each* $\lambda \in [0, 1]$, $n \in \mathbf{N}$

$$|B_n(f,\lambda)-f(\lambda)| \leqq c\omega_2\big([\lambda(1-\lambda)/n]^{1/2},f\big).$$

The present procedure to prove this well-known direct estimate (cf. [4, p. 698], [12], and the literature cited there) is essentially contained in [14] (explicitly they prove the Weierstrass convergence theorem for twice differentiable functions, the domain $x, t \in [0, \infty)$ (cf. (3.1)) being replaced by $x, t, x+t \in [0, 1]$). The argument was then refined in [15] in order to obtain an error estimate involving the first modulus of continuity of the first derivative f'.

Example B. In view of (3.5) we see that

$$\varphi_k(\lambda) = \frac{1}{1+\lambda}\left(\frac{\lambda}{1+\lambda}\right)^k \quad (k \geqq 0).$$

Consequently, one has $J=[0,\infty)$ and $\sigma(\lambda)=\lambda(1+\lambda)$. Since for $x=0$, $h=1/n$

$$[E_\tau^n f](0)=\frac{1}{(1+\lambda)^n}\left[\left(I-\frac{\lambda}{1+\lambda}T_h\right)^{-n}f\right](0)=$$

$$=\frac{1}{(1+\lambda)^n}\sum_{k=0}^{n}\binom{n+k-1}{k}\left(\frac{\lambda}{1+\lambda}\right)^k f(k/n)=M_n(f,\lambda),$$

Cor. 1 yields the following direct estimate for the Baskakov operators $M_n(f,\lambda)$.

Corollary 3. *If $f\in UCB(\mathbf{R}^+)$, then for any $n\in\mathbf{N}$, $\lambda\geqq 0$*

$$|M_n(f,\lambda)-f(\lambda)|\leqq c\omega_2([\lambda(1+\lambda)/n]^{1/2},f).$$

As is well-known (cf. [1], [11, p. 39]), this is the correct estimate, apart from constants.

Example C. Here we see from (3.7) that

$$\varphi_k(\lambda)=e^{-\lambda}\lambda^k/k! \quad (k\geqq 0).$$

Thus $J=[0,\infty)$ and $\sigma(\lambda)=\lambda$. Since for $x=0$, $h=1/n$

$$[E_\tau^n f](0)=\exp[n\lambda(T_h-I)]f(0)=e^{-n\lambda}\sum_{k=0}^{\infty}\frac{(n\lambda)^k}{k!}f(k/n)=S_n(f,\lambda),$$

Cor. 1 delivers the following (pointwise) direct estimate for the operators of Szász—Mirakyan.

Corollary 4. *For any $f\in UCB(\mathbf{R}^+)$, $\lambda\geqq 0$, $n\in\mathbf{N}$ one has*

$$|S_n(f,\lambda)-f(\lambda)|\leqq c\omega_2([\lambda/n]^{1/2},f).$$

Again this is the correct estimate apart from constants.

Regarding Cor. 2—4, let us again point out that these (pointwise) direct approximation theorems for the Bernstein polynomials, the Baskakov, and Szász—Mirakyan operators, respectively, are of course well-known. In fact, these results may be obtained even more directly and elementarily exploiting (cf. (3.10)) the second moment of the kernel (cf. [11, p. 39; 244], see also [3] for more intricate results in polynomial weight spaces). Concerning this note, however, they do not only show interesting interconnections between the Lax theorem in numerical analysis and the convergence of some classical approximation processes but also indicate that the notions and results of the abstract theory in Sec. 2 seem to be adequate.

4. Applications to semigroup schemes

Let us regard Ex. C from another point of view. In order to obtain the difference scheme (3.7) one has to solve the initial value problem (3.6) for one time step $t=\tau$. Looking at the solution of (3.6) for any $t\geqq 0$, however, delivers (continuous) semigroups $\{E_\tau(t); t\geqq 0\}$, $\tau:=\lambda h\geqq 0$. Then (3.6) takes the form

$$d/dt\, E_\tau(t)f = h^{-1}[T_h-I]E_\tau(t)f.$$

Thus the infinitesimal generators A_τ of these semigroups are given via the bounded linear operators

$$A_\tau = h^{-1}[T_h-I] \tag{4.1}$$

so that one has

$$E_\tau(t) = \exp(tA_\tau) = e^{-t/h}\sum_{k=0}^{\infty}\frac{(t/h)^k}{k!}T_h^k. \tag{4.2}$$

Obviously, for any $f\in U\ (:= UCB^{(2)}(\mathbf{R}^+))$

$$A_\tau f(x) = f'(x)+\frac{1}{h}\int_0^h\int_0^s f''(x+v)\,dv\,ds. \tag{4.3}$$

Of course the infinitesimal generator of the solution semigroup $\{E(t); t\geqq 0\}$ (cf. (3.2)) is given via $Af(x)=f'(x)$. Moreover, since the present generators A_τ commute with $E(t)$, one has

$$\|(A_\tau-A)E(t)f\|_X \leqq \|E(t)\|_{[X]}\|A_\tau f-Af\|_X = \|A_\tau f-Af\|_X. \tag{4.4}$$

Therefore the error of consistency (2.9) for the semigroup scheme (4.2) may be estimated by

$$\|(A_\tau-A)E(t)f\|_X \leqq (h/2)\|f''\|_X$$

for any $f\in U$ whereas stability follows from

$$\|E_\tau(t)\|_{[X]} \leqq e^{-t/h}\sum_{k=0}^{\infty}\frac{(t/h)^k}{k!} = 1.$$

Thus an application of Thm. 2, (c)⇒(a), regains Cor. 1; 4, namely (with $K(t,f):= =K(t,f;UCB,UCB^{(2)})$)

Corollary 5. *For the semigroup scheme* (4.2) *one has*

$$|E_\tau(t)f(x)-f(x+t)| \leqq 2K(ht/4,f) \quad \textit{for any} \quad f\in UCB,\ x,t,\tau,h \geqq 0.$$

More generally, given a difference scheme $\{E_\tau; 0\leqq\tau\leqq\delta\}\subset[X]$, to each discrete semigroup $\{E_\tau^n; n\in\mathbf{P}\}$ one may associate a continuous one $\{E_\tau(t); t\geqq 0\}$ according to the formula (cf. (3.7), (4.2))

$$E_\tau(t) := E_\tau^{t/\tau} \tag{4.5}$$

in case the right-hand side can be interpreted suitably. The resulting continuous semigroup then has the interpolation property

$$E_\tau(n\tau) = E_\tau^n. \tag{4.6}$$

Let us continue with considering the matter in connection with Ex. A, B. First recall that for each $\alpha \in \mathbf{R}$

$$(1+u)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} u^k \tag{4.7}$$

absolutely and uniformly for $|u|<1$ (and even for $|u|\leqq 1$ in case $\alpha>0$).

Example A. It follows from (3.4) that

$$E_\tau(t) := E_\tau^{t/\tau} := (1-\lambda)^{t/\tau}\left[I+\frac{\lambda}{1-\lambda}T_h\right]^{t/\tau} = (1-\lambda)^{t/\tau}\sum_{k=0}^{\infty}\binom{t/\tau}{k}\left(\frac{\lambda}{1-\lambda}\right)^k T_h^k, \tag{4.8}$$

the series being convergent in the uniform operator topology for $\lambda\in[0, 1/2]$. For the corresponding infinitesimal generator A_τ one has

$$A_\tau f(x) = f'(x) - \frac{1}{\tau}\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{-\lambda}{1-\lambda}\right)^k \int_0^{kh}\int_0^{s} f''(x+v)\,dv\,ds$$

for any $f\in U$. Therefore (cf. (4.4)) for any $f\in U$, $\lambda\in[0, 1/2)$

$$\|[A_\tau - A]E(t)f\|_X \leqq \frac{h}{2\lambda}\sum_{k=0}^{\infty} k\left|\frac{-\lambda}{1-\lambda}\right|^k \|f''\|_X \leqq \frac{h}{2}\frac{1-\lambda}{1-2\lambda}\|f''\|_X.$$

Concerning stability, for some given $t\geqq 0$ let $m\in\mathbf{P}$ be such that $m\tau\leqq t = m\tau+\eta< <m\tau+\tau$. Then in view of the stability of the explicit difference scheme we see that (cf. (3.9), (4.5), (4.8))

$$\|E_\tau(t)\|_{[X]} \leqq \|E_\tau(m\tau)\|_{[X]}\|E_\tau(\eta)\|_{[X]} \leqq \|E_\tau(\eta)\|_{[X]} =$$

$$= (1-\lambda)^{\eta/\tau}\sum_{k=0}^{\infty}\left|\binom{\eta/\tau}{k}\left(\frac{\lambda}{1-\lambda}\right)^k\right| = 2(1-\lambda)^{\eta/\tau} - (1-2\lambda)^{\eta/t} \leqq 2.$$

Application of Thm. 2, (c)$\Rightarrow$(a), therefore gives

$$|E_\tau(t)f(x) - f(x+t)| \leqq 4K\left(\frac{ht(1-\lambda)}{4(1-2\lambda)}, f\right) \quad (\lambda\in[0, 1/2))$$

which is worse than Cor. 1 or 2, respectively.

Example B. The interpolating semigroups (4.5) for the difference operators E_τ of (3.5) are given by

$$E_\tau(t) := E_\tau^{t/\tau} := \left(\frac{1}{1+\lambda}\right)^{t/\tau}\left(I - \frac{\lambda}{1+\lambda}T_h\right)^{-t/\tau} = \tag{4.9}$$

$$= \left(\frac{1}{1+\lambda}\right)^{t/\tau}\sum_{k=0}^{\infty}\binom{-t/\tau}{k}\left(\frac{-\lambda}{1+\lambda}\right)^k T_h^k$$

the series being convergent in $[X]$ for each $\lambda \geqq 0$. For the infinitesimal generators A_τ one has

$$A_\tau f(x) = f'(x) + \frac{1}{\tau} \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{\lambda}{1+\lambda}\right)^k \int_0^{kh} \int_0^s f''(x+v)\, dv\, ds$$

for any $f \in U$. It follows that for any $f \in U$, $\lambda \geqq 0$

$$\|[A_\tau - A]E(t)f\|_X \leqq \frac{h}{2\lambda} \sum_{k=0}^{\infty} k \left(\frac{\lambda}{1+\lambda}\right)^k \|f''\|_X = \frac{h}{2}(1+\lambda)\|f''\|_X.$$

Since $\|E_\tau(t)\|_{[X]} \leqq 1$, one has for the semigroup scheme (4.9) that

$$|E_\tau(t)f(x) - f(x+t)| \leqq 2K\big(ht(1+\lambda)/4, f\big)$$

which reproduces the results of Cor. 1; 3 (upon setting $t = n\tau = nh\lambda$).

Summarizing, Thm. 2; 4 seem to be more appropriate for line methods (cf. treatment of Ex. C) whereas Thm. 1; 3 seem to be more suitable for genuine difference schemes.

5. A comparison theorem

In the course of the proof of the familiar Trotter theorem (cf. [22], [16, p. 507 ff]) one makes use of just another method (than (4.5)) to associate a semigroup scheme $\{\tilde{E}_\tau(t); t \geqq 0\}$ to some given difference scheme $\{E_\tau\}$, $0 < \tau \leqq \delta$. Indeed, with the step operator $E_\tau \in [X]$ one also has $B_\tau := (E_\tau - I)/\tau \in [X]$ so that via $(t \geqq 0)$

$$\tilde{E}_\tau(t) := \exp(tB_\tau) := \sum_{k=1}^{\infty} (t^k/k!) B_\tau^k \tag{5.1}$$

there is defined a (continuous) semigroup of class (C_0) for each $0 < \tau \leqq \delta$. Though $\{\tilde{E}_\tau(t); t \geqq 0\}$ does not have the interpolation property (4.6), one has (cf. [16, p. 508]):

Lemma 1. *With* $\{E_\tau\} \subset [X]$, $0 < \tau \leqq \delta$, *let* $\tilde{E}_\tau(t)$ *be given via* (5.1). *If there exist constants* M_τ *such that* $\|E_\tau^n\|_{[X]} \leqq M_\tau$ *uniformly for* $n \in \mathbf{P}$, *then also* $\|\tilde{E}_\tau(t)\|_{[X]} \leqq M_\tau$ *uniformly for* $t \geqq 0$ *and*

$$\|E_\tau^n f - \tilde{E}_\tau(n\tau)f\|_X \leqq (1/2) M_\tau n\tau^2 \|B_\tau^2 f\|_X \quad \textit{for every} \quad f \in X, n \in \mathbf{P}, 0 < \tau \leqq \delta.$$

This may be interpreted as a comparison theorem between a given difference scheme and the corresponding line method. Whenever a discretization of (2.2) is given via

$$\frac{1}{\tau}[E_\tau - I]u(t) = B_\tau u(t)$$

with difference scheme $\{E_\tau; 0<\tau\leqq\delta\}$ satisfying the stability condition $\|E_\tau^n\|_{[X]}\leqq M_\tau$, then the approximation error can be estimated according to

$$\|E_\tau^n f - E(n\tau)f\|_X \leqq \|\tilde{E}_\tau(n\tau)f - E(n\tau)f\|_X + (1/2)M_\tau n\tau^2 \|B_\tau^2 f\|_X \tag{5.2}$$

where $\tilde{E}_\tau(t):=\exp(tB_\tau)$ denotes the line method defined by

$$d/dt\, u(t) = B_\tau u(t). \tag{5.3}$$

Concerning Ex. A, the operators B_τ are given by

$$B_\tau := \frac{1}{\tau}[E_\tau - I] = \frac{1}{h}[T_h - I]$$

which are just the infinitesimal generators of the semigroup scheme in Ex. C. Thus $\tilde{E}_\tau(t)$ is equal to $E_\tau(t)$ from (4.2). Since $\|B_\tau^2 f\|_X\leqq\|f''\|_X$ for any $f\in U$, in view of the stability and Lemma 1 this leads to (cf. proof of Thm. 1, (c)$\Rightarrow$(a))

$$\|E_\tau^n f - E_\tau(n\tau)f\|_X \leqq 2K(n\tau^2/4, f)$$

with E_τ from (3.4) and $E_\tau(t)$ from (4.2). Therefore, proceeding as in the previous sections, one obtains

Corollary 6. *For $f\in UCB$ one has the following comparison estimate between the Bernstein polynomials and operators of Szász—Mirakyan:*

$$|B_n(f,\lambda) - S_n(f,\lambda)| \leqq 2K(\lambda^2/4n, f) \quad \textit{for all} \quad \lambda\in[0,1], n\in\mathbb{N}.$$

Thus, though the individual operators behave like $O(\lambda)$ at $\lambda=0+$, their difference behaves like $O(\lambda^2)$.

Added in proof: For a (parallel to [14]) concrete discussion of the pure convergence of the Bernstein and Baskakov operators in connection with the explicit and implicit difference scheme of Ex. A, B, respectively, see also G. C. Papanicolau, *Amer. Math. Monthly,* **82** (1975), 674—676.

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