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On normal subgroups of semigroups with identity element

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In the theory of semigroups, normality of subsemigroups has been defined in several different ways. L. RÉDEI [3] has introduced this concept by the following two definitions;

D 1. The subsemigroup N of a semigroup S is called *left normal* if
(i) the partition $S=N\cup a_1N\cup a_2N\cup \dots (a_1, a_2, \dots \in S)$ is compatible, and
(ii) for each i and $n_1, n_2\in N$, $a_in_1=a_in_2$ implies $n_1=n_2$.

Right normality is defined analogously.

D 2. The subsemigroup N of a semigroup S is called *normal,* if it is both right and left normal.

I. PEÁK [2] has modified these definitions by omitting condition (ii). Let us denote the modified definitions by D′ 1 and D′ 2, respectively.

The subgroup N of a semigroup S is called a *normal subgroup* of S if it is a normal subsemigroup in the sense of D 2 or D′ 2, respectively.

The following example shows that Theorem 1 of [2] is false.

Example. Let S be the semigroup of transformations of a set of cardinal 2 into itself.

The mistake in Peák's proof is in the part that (A) implies (B) where he used that N is right normal, too. Therefore, only the following modification of Peák's theorem holds true:

Theorem 1. *Let N be a subgroup of the semigroup S with identity element which contains the identity element of S. Then the following conditions are equivalent:*

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A) *N is normal in the sense of* D′ 2,

B) *for all $a \in S$, $aN=Na$ holds,*

C) *the set of the right cosets of N coincides with the set of the right cosets of N.*

The following Theorem 2 is from [2], but the proof for MN is not correct there.

Theorem 2. *Let S be a semigroup with identity element and let N and M be subgroups of S. If N and M are left normal in the sense of* D′ 1, *then MN is a left normal subgroup of S, and if S is also cancellative, then $M \cap N$ is a left normal subgroup of S in the sense of* D′ 1, *too.*

Theorem 2 can be generalized as follows:

Theorem 3. *Let S be a semigroup and N and M subsemigroups of S containing an identity element. If N and M are left normal in the sense of* D′ 1, *then MN is a left normal subsemigroup of S in the sense of* D′ 1, *and if S is also left cancellative and $M \cap N$ is non-empty, then $M \cap N$ is a left normal subsemigroup of S in the sense of* D′ 1, *too.*

Proof. It is well known that $M \cap N$ is a subsemigroup. If M and N are subgroups then $N \cap M$ is a subgroup. If

$$c \in a(M \cap N)b(M \cap N),$$

then

$$c \in (abM) \cap (abN),$$

and thus there exist an m in M and an n in N such that

$$c = abm = abn.$$

If S has an identity element, then, since N and M are left normal, N and M contain the identity element of S. Since S is left cancellative the last equation implies

$$c \in ab(M \cap N).$$

Let e be the identity element of N and let f be the identity element of M. Since M and N are left normal in the sense of D′ 1, $ef=e$ and $fe=f$ and MN with identity element f is a subsemigroup of S. fN is a left normal subsemigroup of MN in the sense of D′ 1.

If M and N are subgroups of S then fN and MN/fN are groups, therefore MN is a group.

MN is left normal, because if $c \in (aMN)(bMN)$ then $c=amnbm'n'$ holds for some $m, m' \in M$ and $n, n' \in N$. Thus

$$c \in (amN)(bm'N).$$

Since $amebm'e=amfebm'e=ambm'e$,

$$(amN)(bm'N) = ambm'N$$

holds. Since M is left normal too, we have

$$ambm'N = abm''N \subseteqq abMN$$

for $m'' \in M$, and, consequently

$$c \in abMN.$$

If $e=f$ and M and N are subgroups of S then $MN=NM$. It follows that the first assertion of Theorem 2 is true.

Remark 1. If we replace D′ 1 by D 1 in the second assertions of Theorems 2 and 3 then we can omit the condition that S be left cancellative. We introduce an equivalence relation (see LYAPIN [1]):

Let S be a semigroup with identity element and N be a subgroup of S. We say that r is *ϱ_N-equivalent* to s, in symbols $r\varrho_N s$, if there exist elements n, m in N such that $rn=ms$.

The following assertion is a modification of an assertion of PEÁK [2], p. 349.

The partition corresponding to the equivalence relation ϱ_N coincides with the left (right) cosets of N if and only if N is left (right) normal in the sense of D′ 1.

Proof. Suppose that N is left normal in the sense of D′ 1. Any two elements of a left coset of N are ϱ_N-equivalent because $b \in aN$ implies the existence of an element n in N such that

$$an = b = eb, \quad \text{whence} \quad a\varrho_N b.$$

On the other hand, any element c that is ϱ_N-equivalent to a belongs to the left coset aN, because the partition

$$S = N \cup a_1 N \cup a_2 N \cup \dots$$

is compatible.

Conversely, suppose that the partition corresponding to ϱ_N coincides with the left cosets of N. If $c \in N(aN)$ then $c\varrho_N a$. It follows that $c \in aN$. Since $e \in N$, we have $(bN)(aN)=baN$, as we wished to prove.

Peák has also made the following assertion:

Let N run over the set of all subgroups of a semigroup S with identity element, which are left normal in the sense of D′ 1 *and contain the identity element of S. Then either each or none of the factor semigroups S/N is a group.*

Proof. If N is left normal in the sense of $D'1$ and S/N is a group then S is a group.

References

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