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On an extension of semigroups

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1. Since the appearance of N. R. REILLY's paper [13] in 1966 a number of structure theorems has been proved for regular semigroups. In the paper [13] it is proved that a semigroup is a $\mathcal{D}$-simple regular ω-semigroup if and only if it is isomorphic to a Bruck semigroup over a group ([12]). This result was generalized by B. P. KOČIN ([4]) and W. D. MUNN ([9]) by showing that a semigroup is a simple regular ω-semigroup if and only if it is a Bruck semigroup over a finite chain of groups. The structure of a 0-$\mathcal{D}$-simple orthodox semigroup the subsemigroup of idempotents of which is isomorphic to the direct product of a descending ω-chain and a rectangular 0-band whose non-zero idempotents form a subsemigroup, has been described by G. LALLEMENT and M. PETRICH in [6].

In order to generalize these constructions we define the concept of the (0-) extension of a semigroup Σ by a semigroup S. The sets of nonzero elements of S and Σ will be denoted by S_0 and Σ_ω, their zero elements by o and ω, respectively. Let $S_0^{(2)}$ be the subset of $S_0 \times S_0$ consisting of all those pairs (s, t) of elements for which $st \in S_0$. Let C be a cancellative monoid. Its identity element will be denoted by 1. Let $f, g\colon S_0^{(2)} \to C$ be a pair of functions with the following properties:

$$f_{r,s} f_{rs,t} = f_{r,st}, \tag{1}$$

$$g_{r,s} f_{rs,t} = f_{s,t} g_{r,st}, \tag{2}$$

$$g_{rs,t} = g_{s,t} g_{r,st} \tag{3}$$

whenever $rst \in S_0$. Moreover, let a homomorphism $\varkappa$ of C into the endomorphism monoid of Σ be given.

Definition. Define a multiplication on the set $S_0 \times \Sigma_\omega \cup 0$ by

$$(s, \sigma)(t, \tau) = \begin{cases} (st, \sigma(f_{s,t}\varkappa)\tau(g_{s,t}\varkappa)) & \text{if} \quad st \in S_0 \quad \text{and} \quad \sigma(f_{s,t}\varkappa)\tau(g_{s,t}\varkappa) \in \Sigma_\omega \\ 0 \quad \text{otherwise,} \end{cases}$$

$$(s, \sigma)0 = 0(s, \sigma) = 0 \cdot 0 = 0.$$

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The groupoid obtained in this way is a *semigroup*, denoted by $\mathscr{S}^0(S, \Sigma, C, f, g, \varkappa)$ and called a *0-extension of Σ by S over C.*

If none of S and Σ has zero elements then $\mathscr{S}^0(S, \Sigma, C, f, g, \varkappa)\backslash 0$ is a *semigroup*. This will be denoted by $\mathscr{S}(S, \Sigma, C, f, g, \varkappa)$ and called the *extension of Σ by S over C.*

For example, the Bruck semigroup $\mathscr{B}(\Sigma, \pi)$ over the monoid Σ is the extension $\mathscr{S}(B, \Sigma, N^0, f^*, g^*, \varkappa)$ of Σ by the bicyclic semigroup B, where N^0 is the additive monoid of nonnegative integers, $B \cong N^0 \times N^0$ with the multiplication defined by

$$(m, n)(p, q) = \big(m+p-\min(n, p), n+q-\min(n, p)\big),$$

$f^*, g^*: B\times B \to N^0$ are defined as follows:

$$f^*_{(m,n),(p,q)} = p-\min(n, p), \quad g^*_{(m,n),(p,q)} = n-\min(n, p),$$

and $\varkappa$ is the homomorphism of N^0 into the monoid of endomorphisms of Σ mapping k into π^k. Note that the functions f^* and g^* have the properties (1)—(3).

It is clear that it suffices to investigate the properties of the semigroup $\mathscr{S}^0(S, \Sigma, C, f, g, \varkappa)$ because the properties of $\mathscr{S}(S, \Sigma, C, f, g, \varkappa)$ can be deduced from those of $\mathscr{S}^0(S, \Sigma, C, f, g, \varkappa)$.

Define an equivalence relation $\mathscr{C}$ on $\mathscr{S}^0(S, \Sigma, C, f, g, \varkappa)$ such that $0\mathscr{C}0$ and $(r, \varrho)\mathscr{C}(s, \sigma)$ if and only if $r=s$. The relation $\mathscr{C}$ is a 0-congruence in the sense that if $(r, \varrho)\mathscr{C}(s, \sigma)$ and $(r', \varrho')\mathscr{C}(s', \sigma')$ then $(r, \varrho)(r', \varrho')\neq 0$, $(s, \sigma)(s', \sigma')\neq 0$ imply $(r, \varrho)(r', \varrho')\mathscr{C}(s, \sigma)(s', \sigma')$.

The pair of functions $f, g: S_0^{(2)} \to C$ is said to be *trivial* if $S_0^{(2)}f = S_0^{(2)}g = 1$. In this case $\mathscr{S}^0(S, \Sigma, C, f, g, \varkappa)$ is the 0-direct product of S and Σ. Note that the semidirect product of Σ by S introduced by K. Krohn and J. Rhodes in [5] can be considered to be an extension of Σ by S over the free monoid F_S^e generated by the set S where $f, g: S\times S \to F_S^e$ are defined as follows: $(S\times S)g = 1$ while f depends on its second variable only and is a homomorphism.

The constructions used in [15] and [16] by R. J. Warne to describe the structure of $\mathscr{D}$-simple and simple regular I-semigroups, are extensions of a group and of a finite chain of groups, respectively, by the extended bicyclic semigroup if and only if they have trivial distinguished elements. Construction I applied in [1] and [2] by J. E. Ault and M. Petrich to give the structure of 0-simple ω-regular semigroups, is a 0-extension of a finite chain of groups by the 0-$\mathscr{D}$-simple ω-regular semigroup with trivial $\mathscr{H}$-equivalence if and only if the maximal idempotents belong to the same $\mathscr{D}$-class.

The aim of this paper is to investigate the properties of (0-) extensions. In section 2 we deal with functions $f, g: S_0^{(2)} \to C$ satisfying (1)—(3). The main result of this section is Theorem 2.3 characterizing these functions when S has an identity element and C is a monoid embeddable in a group. In Theorem 2.4 a necessary and sufficient condition is given for f and g enabling us to extend their definitions to S^e. Finally, applying Theorem 2.3, we describe the structure of those 0-$\mathscr{D}$-simple semigroups with identity which admit f, g of a special type. In section 3 we prove criteria for $\mathscr{S}^0(S, \Sigma, C, f, g, \varkappa)$ to be regular or inverse. We investigate Green's relations, ideals and homomorphisms of 0-extensions. We introduce a concept of equivalent 0-extensions and give conditions for 0-extensions to be equivalent. These results are essentially independent of the results of section 2. Theorem 2.3 is needed only in Theorems 3.9 and 3.11.

For brevity, if we consider functions $f, g\colon S_0^{(2)}\to C$ or a 0-extension $\mathscr{S}^0(S, \Sigma, C, f, g, \varkappa)$ we always assume conditions (1)—(3) to be satisfied. We shall write lower case Roman letters for the elements of S, in particular e for its identity element, and lower case Greek letters to denote the elements of Σ, in particular ε to denote its identity element.

S_0 together with the multiplication in S restricted to S_0 is a partial semigroup. By a right [left] ideal of S_0 we mean a non-empty subset R [L] of S_0 with the property that $r\in R$ [$l\in L$] implies $rs\in R$ [$sl\in L$] for any element s of S_0 whenever the product is defined. Analogously, a homomorphism of S_0 into a semigroup T is a mapping $\varphi\colon S_0\to T$ such that for all elements s, t in S_0 we have $(st)\varphi=s\varphi\cdot t\varphi$ provided st is defined.

For convenience, we use the expressions "if $s=o$" and "if $s\neq o$" also in the case when S has no zero element. If this is the case then $s=o$ is false, $s\neq o$ is true for every s in S.

The results and notations of [3] will be used without any comments.

2. In this section we investigate the properties of functions $f, g\colon S_0^{(2)}\to C$ satisfying conditions (1)—(3).

Lemma 2.1. *If x, s are elements of S_0 such that $sx=s$, then $f_{s,x}=1$, and if $xs=s$, then $g_{x,s}=1$.*

Proof. Assume that $sx=s$. Applying conditions (1) and (2) we get

$$f_{s,s}f_{s^2,x}=f_{s,s} \quad\text{and}\quad g_{s,s}f_{s^2,x}=f_{s,x}g_{s,s}.$$

Since C is a cancellative monoid, $f_{s^2,x}=1$ follows from the first equality and $f_{s,x}=1$ from the second one. The second half of the lemma follows by duality.

Lemma 2.2. *Let s, s', t, t' be elements of S_0 such that $s\mathscr{L}s'$, $t\mathscr{R}t'$ and $st\neq o$. Then we have*

(i) $f_{s,t}\mathscr{R}f_{s,t'}$ *and* $g_{s,t}\mathscr{L}g_{s',t}$;

(ii) *if the group of units of C is trivial, then $f_{s,t}=f_{s',t'}$ and $g_{s,t}=g_{s',t'}$.*

Proof. Suppose $t \neq t'$. Then there exist u, v in S such that $tu = t'$ and $t'v = t$. Clearly, $st' \neq o$. Condition (1) implies the following equalities:

$$f_{s,t'} = f_{s,tu} = f_{s,t} f_{st,u}, \quad f_{st,uv} = f_{st,u} f_{stu,v}.$$

For $st\,uv = st$ Lemma 2.1 shows that $f_{st,uv} = 1$. Hence $f_{st,u}$ has an inverse in C, which implies that $f_{s,t} \mathcal{R} f_{s,t'}$. If the group of units of C is trivial, then the second equality implies that $f_{st,u} = 1$ and the first one that $f_{s,t'} = f_{s,t}$. Moreover, if $s \neq s'$, then $xs = s'$, $ys' = s$ for some x and y in S. We have $yxs = s$, so it follows from Lemma 2.1 that $g_{yx,s} = 1$. (3) implies

$$g_{yx,s} = g_{x,s} g_{y,xs},$$

which gives $g_{x,s} = 1$. Analogously, one can show that $g_{x,st'} = 1$. By condition (2) we have

$$g_{x,s} f_{s',t'} = f_{s,t'} g_{x,st'},$$

that is, $f_{s',t'} = f_{s,t'} = f_{s,t}$. The proof for g is similar.

In what follows we assume that C can be embedded in a group. It is well known that if this is the case then C can be embedded in the group of right quotients which will be denoted by C^*. Let us identify C with its image under this embedding. If two functions $\chi_1, \chi_2 \colon S_0 \to C$ are given, let $\chi_1/\chi_2 \colon S_0 \to C^*$ be the mapping defined by $s\chi_1/\chi_2 = s\chi_1 (s\chi_2)^{-1}$. The next theorem characterizes the functions $f, g \colon S_0^{(2)} \to C$ by functions of one variable provided S has an identity element.

Theorem 2.3. (i) *Let S be an arbitrary semigroup and $\chi_1, \chi_2 \colon S_0 \to C$ two functions such that $R_I = \{s \in S_0 | s\chi_1 \in I\}$ is a right ideal in S_0, $L_I = \{s \in S_0 | s\chi_2 \in I\}$ is a left ideal in S_0 for every right ideal I of C, moreover, the mapping $\varphi = \chi_1/\chi_2 \colon S_0 \to C^*$ is a homomorphism. Then the functions $f, g \colon S_0^{(2)} \to C$ defined by*

$$f_{s,t} = (s\chi_1)^{-1}(st)\chi_1 \quad \text{and} \quad g_{s,t} = (t\chi_2)^{-1}(st)\chi_2 \tag{5}$$

satisfy conditions (1)—(3).

(ii) *If S has an identity element e then for all $f, g \colon S_0^{(2)} \to C$ with properties* (1)—(3) *there exists a unique pair of functions $\chi_1, \chi_2 \colon S_0 \to C$ with $e\chi_1 = e\chi_2 = 1$ such that* (5) *holds. They are*

$$s\chi_1 = f_{e,s} \quad \text{and} \quad s\chi_2 = g_{s,e}. \tag{6}$$

Furthermore, these functions satisfy the conditions required in (i).

Proof. Since the facts that $R_{s\chi_1 C}$ is a right ideal and $L_{t\chi_2 C}$ is a left ideal of S_0 ensure $f_{s,t} \in C$ and $g_{s,t} \in C$ for every $(s, t) \in S_0^{(2)}$, (i) can be checked by simple calculation.

In proving (ii) suppose S has an identity e and $e\chi_1 = e\chi_2 = 1$. Then (5) implies (6). Because of (i), it is sufficient to show that (5) holds and the conditions required in (i) are satisfied by the functions χ_1, χ_2 defined by (6). Clearly, (5) is an immediate consequence of (1) and (3). On the other hand, (2) and $st \neq o$ imply

(7) $$f_{s,t} g_{s,t}^{-1} = g_{s,e}^{-1} f_{e,t}.$$

Applying (5) and (6), this yields

$$(st)\chi_1\big((st)\chi_2\big)^{-1} = s\chi_1(s\chi_2)^{-1} t\chi_1(t\chi_2)^{-1},$$

that is, that χ_1/χ_2 is a homomorphism. Finally, if I is a right ideal of C and $s \in R_I$, then $(st)\chi_1 = s\chi_1 f_{s,t} \in I$ for every t provided that $st \neq o$. Hence R_I is a right ideal. Dually, L_I is a left ideal.

We have seen that the pair of functions f, g can be simply characterized if S has an identity element. Now it is natural to raise the problem of finding conditions under which the definition of f and g can be extended to S^e. Before treating this question we introduce some notations.

Let S be a semigroup. Denote the right and left annihilator ideals of S by Z_r and Z_l, respectively. If S does not contain a zero element, then $Z_r = Z_l = \Box$. Further, $h\colon S_0^{(2)} \to C^*$ will denote the mapping defined by $h_{s,t} = f_{s,t} g_{s,t}^{-1}$ provided $f, g\colon S_0^{(2)} \to C$ are defined and $C \subseteqq C^*$.

Theorem 2.4. *Suppose the semigroup S has the properties that $Z_r = Z_l$ (which will be denoted by Z) and for any elements s, s', t, t' in S, the relations $st, st', s't \neq o$ imply $s't' \neq o$. Let $f, g\colon S_0^{(2)} \to C$ be given, where C is a monoid embeddable in a group. The definition of f, g can be extended to S^e if and only if*

(a) *for each element q in $S \setminus Z$*

$$J_q = \bigcap_{\substack{s,t \\ st,\, sq \neq o}} (Ch_{s,t}^{-1} \cap C) h_{s,q} \cap C$$

is not empty, and for arbitrary $p, q, s, t \in S \setminus Z$

(b) $$h_{s,t} h_{p,t}^{-1} h_{p,q} h_{s,q}^{-1} = 1$$

provided $st, pt, pq, sq \neq o$.

Remark. The definition of f, g can be extended to S^e if we require (a) and (b) to hold only for the elements p and q of some subsets P and Q of $S \setminus Z$, respectively, where P and Q have the following property: For each s, t, t' not contained in Z we have $sq \neq o$ for some q in Q and $pt, pt' \neq o$ for some p in P.

Proof. If f and g are defined on S^e then, applying the foregoing results, we have

$$h_{s,t} h_{p,t}^{-1} h_{p,q} h_{s,q}^{-1} = (g_{s,e}^{-1} f_{e,t})(f_{e,t}^{-1} g_{p,e})(g_{p,e}^{-1} f_{e,q})(f_{e,q}^{-1} g_{s,e}) = 1$$

for every p, q, s, t in $S\setminus Z$ with $st, pt, pq, sq\neq o$. Furthermore, if $q\in S\setminus Z$, then there exists an element s such that $sq\neq o$. If $st\neq o$, then we have

$$f_{e,q}=g_{s,e}h_{s,q}=g_{s,e}h_{s,t}h_{p,t}^{-1}h_{p,q}=f_{e,t}h_{p,t}^{-1}h_{p,q}=f_{e,t}h_{s,t}^{-1}h_{s,q}.$$

Hence $f_{e,q}\in J_q$, and the proof of necessity is complete.

As for sufficiency we prove the stronger statement formulated in the Remark. Suppose that (a) and (b) hold for some subsets P and Q of $S\setminus Z$. We define a relation $\sim$ on S by writing $s\sim s'$ if and only if $s=s'$ or ts and $ts'\neq o$ for some t in S. Clearly, this relation is reflexive and symmetric. If $s\sim s'$ and $s'\sim s''$, then ts, ts', $t's', t's''\neq o$ for some t and t' in S. But then $t's\neq o$, that is, $s\sim s''$. Hence $\sim$ is an equivalence relation. We restrict this relation to Q and choose an element q^0 from each equivalence class of Q and an element c_{q^0} from J_{q^0}. If $q\sim q^0$ and $q\neq q^0$,then, by the definition of P, we have $pq^0, pq\neq o$ for some p in P. Now we define c_q by the equality $c_q=c_{q^0}h_{p,q^0}^{-1}h_{p,q}$. Since $c_{q^0}\in J_{q^0}$ and C is cancellative, there exists a unique element c in C such that $c_{q^0}=ch_{p,q}^{-1}h_{p,q^0}$ and $ch_{p,q}^{-1}=c_{q^0}h_{p,q^0}^{-1}\in C$. Clearly, $c=c_q$ and hence $c_q\in C$. Let s be an element of $S\setminus Z$ such that $sq^0\neq o$. Since $pq, pq^0\neq o$, we have $sq\neq o$ and (b) implies $h_{p,q^0}^{-1}h_{p,q}h_{s,q}^{-1}=h_{s,q^0}^{-1}$. Thus we have

$$c_qh_{s,q}^{-1}=c_{q^0}h_{p,q^0}^{-1}h_{p,q}h_{s,q}^{-1}=c_{q^0}h_{s,q^0}^{-1}.$$

Hence $c_q\in J_q$. Relation (b) ensures that c_q is welldefined. Let s, t be elements of S not contained in Z. Then, on the one hand, there exists an element q in Q such that $sq\neq o$ and, on the other hand, there is an element p in P such that $pt\neq o$ and hence an element q' such that $pq'\neq o$. Let us define $f_{e,t}, g_{s,e}$ to be the uniquely determined elements of C such that

$$f_{e,t}h_{p,t}^{-1}h_{p,q'}=c_{q'}\quad\text{and}\quad g_{s,e}h_{s,q}=c_q.$$

(b) implies that $f_{e,t}$ and $g_{s,e}$ are well defined. If $z\in Z$, then $f_{e,z}$ and $g_{z,e}$ can be arbitrarily defined. By Theorem 2.3 it suffices to check (5) for the mappings defined by (6) and to check (7). Let s, t be elements of S with $st\neq o$. Then $pst\neq o$ for some p in P and $pq'\neq o$ for some q' in Q. Clearly, $ps\neq o$ and we have

$$\begin{aligned}f_{e,s}^{-1}f_{e,st}&=h_{p,s}^{-1}h_{p,q'}c_{q'}^{-1}c_{q'}h_{p,q'}^{-1}h_{p,st}=g_{p,s}f_{p,s}^{-1}f_{p,st}g_{p,st}^{-1}=\\&=g_{p,s}f_{ps,t}g_{p,st}^{-1}=f_{s,t}.\end{aligned}$$

In the last two equalities conditions (1) and (2) are applied. Analogously, we have $g_{t,e}^{-1}g_{st,e}=g_{s,t}$ if $st\neq o$. Finally, if $st\neq o$, then $sq\neq o$ for a q in Q and hence $pt, pq\neq o$ for some p in P. Applying (b), we have

$$g_{s,e}^{-1}f_{e,t}=h_{s,q}c_q^{-1}c_qh_{p,q}^{-1}h_{p,t}=h_{s,t},$$

as was to be proved.

It is easy to see that for any elements p, q, s, t of S and x, y, u, v of S^e

$$h_{s,t}h_{p,t}^{-1}h_{p,q}h_{s,q}^{-1} = h_{us,tv}h_{xp,tv}^{-1}h_{xp,qy}h_{us,qy}^{-1}$$

provided $ustv, xptv, xpqy, usqy \neq o$. One has only to observe that

$$h_{s,t} = f_{s,t}g_{s,t}^{-1} = f_{s,tv}f_{st,v}^{-1}f_{st,v}g_{s,tv}^{-1}f_{t,v}^{-1} = h_{s,tv}f_{t,v}^{-1}$$

and dually

$$h_{s,t} = g_{u,s}h_{us,t}.$$

A subset M of $S \setminus Z_r$ will be called *left* 0*-reversible* if for any pair s, s' of elements of S the existence of elements m in M and t in S with $st, sm, s't, s'm \neq o$ implies the existence of an element x in $tS \cap mS$ such that sx and $s'x \neq o$. It follows by straightforward calculation that in this case

$$h_{s,t}h_{s',t}^{-1}h_{s',m}h_{s,m} = h_{s,x}h_{s',x}^{-1}h_{s',x}h_{s,x} = 1.$$

Hence Theorem 2.4 implies the following

Corollary 2.5. *Suppose $Z = Z_r = Z_l$ holds in the semigroup S. Assume, furthermore, that S has the property that for any elements s, s', t, t' the relations $st, st', s't \neq o$ imply $s't' \neq o$ and S contains a left* 0*-reversible subset M such that for every element s of S not belonging to Z the set M has an element m with $sm \neq o$. Then the definition of $f, g\colon S_0^{(2)} \to C$ can be extended to S^e if and only if for each element m of M*

$$J_m = \bigcap_{\substack{s,t \\ st, sm \neq o}} (Ch_{s,t}^{-1} \cap C)h_{s,m} \cap C$$

is not empty.

The assumption of Corollary 2.5 is satisfied for example if S is an inverse semigroup in which the semilattice of idempotents is an orthogonal sum of semilattices. M can be chosen to be the set of idempotents. If S has no zero element then in Theorem 2.4 P and Q can be chosen to be singletons. In this case the assumption of Corollary 2.5 means that S contains a left reversible element m.

Evidently, condition (a) of Theorem 2.4 is satisfied if C is a group.

The following example shows that there exist functions $f, g\colon S_0^{(2)} \to C$ which cannot be extended to S^e, while considered as functions $f, g\colon S_0^{(2)} \to C^*$ they can be extended. Let S be the extended bicyclic semigroup defined by R. J. Warne in [14]. We denote the set of integers by I. S is the set $I \times I$ equipped with the multiplication

$$(i, j)(k, l) = \bigl(i+k-\min(j, k), l+j-\min(j, k)\bigr).$$

Clearly, $f, g\colon S \times S \to N^0$ defined by $f_{(i,j),(k,l)} = k - \min(j, k)$, $g_{(i,j),(k,l)} = j - \min(j,k)$ satisfy (1)—(3), while $J_{(k_0, l_0)}$ is empty for every (k_0, l_0). On the other hand, S is an inverse semigroup without zero, hence the definition of f, g can be extended to S^e if negative integers are allowed to be used.

Now we determine all the pairs of functions $f, g\colon F_X\times F_X\to C$ which can be defined on the free semigroup freely generated by its subset X.

Theorem 2.6. *Let C be a monoid embeddable in a group and let $\check{\chi}_1, \check{\chi}_2\colon F_X\to C$ be two functions such that $X\check{\chi}_1=X\check{\chi}_2=1$, $R_I=\{s\in F_X|s\check{\chi}_1\in I\}$ is a right ideal, $L_I=\{s\in F_X|s\check{\chi}_2\in I\}$ is a left ideal of F_X for every right ideal I of C and the mapping $\check{\varphi}=\check{\chi}_1/\check{\chi}_2$ satisfies the following condition: for all $s=x_1\dots x_n\in F_X\setminus X$, where $x_i\in X$ $(i=1,\dots,n)$, we have*

$$s\check{\varphi}=(x_1,x_2)\check{\varphi}(x_2x_3)\check{\varphi}\dots(x_{n-1}x_n)\check{\varphi}. \tag{8}$$

Then $f, g\colon F_X\times F_X\to C$ defined by

$$f_{s,t}=(s\check{\chi}_1)^{-1}(st)\check{\chi}_1,\quad g_{s,t}=(t\check{\chi}_2)^{-1}(st)\check{\chi}_2 \tag{9}$$

have the properties (1)—(3). *Conversely, for any $f, g\colon F_X\times F_X\to C$ satisfying* (1)—(3) *there exists a unique pair of functions $\check{\chi}_1, \check{\chi}_2\colon F_X\to C$ with the above properties. These functions are defined on $F_X\setminus X$ by*

$$s\check{\chi}_1=f_{x_1,x_2\dots x_n},\quad s\check{\chi}_2=g_{x_1\dots x_{n-1},x_n} \tag{10}$$

where $x_1,\dots,x_n\in X$ and $s=x_1\dots x_n$.

Proof. Since $R_{s\check{\chi}_1C}$ is a right ideal and $L_{t\check{\chi}_2C}$ is a left ideal of F_X, we have $f_{s,t}$ and $g_{s,t}\in C$ for all s, t in F_X. The first statement of the theorem can be verified by calculation.

Now let f, g be given with properties (1)—(3). Relations (9) ensure that the only functions $\check{\chi}_1$, $\check{\chi}_2$ with $X\check{\chi}_1=X\check{\chi}_2=1$ are the ones defined by (10). All we need to prove is that these functions have the required properties. (9) is implied immediately by (1) and (3). If I is a right ideal of C, $s\in R_I$, $t\in F_X$, then $(st)\check{\chi}_1=s\check{\chi}_1f_{s,t}\in I$, that is, R_I is a right ideal of F_X. Dually, L_I is a left ideal of F_X. Let $x_1,\dots,x_n\in X$, where $n\geqq 3$. Applying (9) and (10), relation (2) implies that

$$f_{x_1,x_2\dots x_n}g^{-1}_{x_1\dots x_{n-1},x_n}=f_{x_1,x_2}g^{-1}_{x_1,x_2}f_{x_2,x_3\dots x_n}g^{-1}_{x_2\dots x_{n-1},x_n},$$

that is, we have

$$(x_1\dots x_n)\check{\varphi}=(x_1x_2)\check{\varphi}(x_2\dots x_{n-1})\check{\varphi}.$$

By induction on n one can show that (8) holds, which completes the proof.

Now it is easy to construct a pair of functions on a free semigroup such that its definition cannot be extended to the free monoid generated by the same set. Let X be the two-element set $\{x, y\}$, C the cancellative monoid of non-negative integers with the usual addition. Define $\check{\chi}_1$ in the following way: let $x^2\check{\chi}_1=y^2\check{\chi}_1=0$, $(xy)\check{\chi}_1=(yx)\check{\chi}_1=1$ and $(x_1\dots x_n)\check{\chi}_1=(x_1x_2)\check{\chi}_1+\dots+(x_{n-1}x_n)\check{\chi}_1$ if $n\geqq 3$ and $x_1,\dots,x_n\in\{x, y\}$. Let $\check{\chi}_2$ be identically 0. Obviously, these functions have the required properties enabling us to define $f, g\colon F_X\times F_X\to C$ by (5). However, $h_{y,y}-h_{x,y}+h_{x,x}-h_{y,x}=-2$.

In what follows we prove a structure theorem for 0-$\mathscr{D}$-simple semigroups with identity on which a nontrivial pair of functions $f, g\colon S_0^{(2)} \to N^0$ is defined where N^0 denotes the additive semigroup of nonnegative integers. The operation in N^0 will be denoted by $+$. Clearly, N^{0*} is the infinite cyclic group.

Lemma 2.7. *Let the semigroup S have an identity e and two elements a, b such that $ba=e$. If $f_{e,a}=n$ $(n\in N^0)$, then for all nonnegative integers k and m*

$$f_{e,a^kb^m} = kn, \quad g_{a^kb^m,e} = mn.$$

Proof. Since $ba=e$, we have $b^ka^k=e$ for all k in N^0. Hence $a^k\mathscr{L}e$ and $b^k\mathscr{R}e$. This implies by Lemmas 2.2 (ii) and 2.1 that

$$f_{e,b^k} = f_{e,e} = 0 \quad \text{and} \quad g_{a^k,e} = g_{e,e} = 0.$$

Using the homomorphism φ defined in Theorem 2.3 we have

$$f_{e,a^k} = f_{e,a^k} - g_{a^k,e} = a_k\varphi = k(a\varphi) = k(f_{e,a} - g_{a,e}) = kf_{e,a} = kn.$$

On the other hand, we have

$$0 = e\varphi = (ba)\varphi = b\varphi + a\varphi = -g_{b,e} + f_{e,a}$$

whence $g_{b,e}=n$. In the same way as above one can prove that $g_{b^k,e}=kn$. Since for all k, m in N^0 we have $a^k\mathscr{R}a^kb^m\mathscr{L}b^m$, Lemma 2.2 (ii) ensures that $f_{e,a^kb^m}= =f_{e,a^k}=kn$ and $g_{a^kb^m,e}=g_{b^m,e}=mn$.

An immediate consequence of this lemma is

Corollary 2.8. *The functions $f, g\colon B\times B\to N^0$ definable on the bicyclic semigroup B are exactly the constant multiples of f^* and g^** (see § **1.**).

Let S be a semigroup with identity e and zero element 0 on which a nontrivial pair of functions $f, g\colon S_0^{(2)}\to N^0$ is given. Let

$$F_i = \{s\in S_0 | f_{e,s} = i\}, \quad G_i = \{s\in S_0 | g_{s,e} = i\}$$

for all i in N^0. $\{F_i | i\in N^0\}$ and $\{G_i | i\in N^0\}$ are partitions of S_0. The equivalence relations induced by them will be denoted by $\mathscr{F}$ and $\mathscr{G}$, respectively. Let $\mathscr{K}=\mathscr{F}\cap\mathscr{G}$. Clearly, its equivalence classes are the sets $K_{i,j}=F_i\cap G_j$.

We remark that $R_{k+N^0}=\bigcup_{i=k}^{\infty} F_i$ and $L_{k+N^0}=\bigcup_{i=k}^{\infty} G_i$ where R_{k+N^0} and L_{k+N^0} denote the right and left ideals of S_0 respectively used in Theorem 2.3. Since φ defined in the same theorem is a homomorphism, $\bigcup_{i=k}^{\infty} K_{i,i}\cup 0$ is a subsemigroup of S for every k in N^0. Lemma 2.2 (ii) implies that $\mathscr{R}\subseteq\mathscr{F}$ and $\mathscr{L}\subseteq\mathscr{G}$. Hence if S is 0-$\mathscr{D}$-simple, then the following holds: $hr=r\,[rh=r]$ for all r in $F_i[G_j]$, whenever $hk=k[kh=k]$ for all k in $K_{i,j}$. These facts will be used without reference. To prove Theorem 2.10 we need

Lemma 2.9. *Assume that nontrivial functions $f, g\colon S_0^{(2)} \to N^0$ are given on a 0-$\mathscr{D}$-simple semigroup S with identity e such that the subsemigroup $\bigcup_{i=1}^{\infty} K_{i,i} \cup 0$ has an identity element e_1. Let $e_1 \in K_{n,n}$. If $e_1 = ab$ with $ba = e$, then $a^m b^m$ is the identity element of $\bigcup_{i=mn}^{\infty} K_{i,i} \cup 0$.*

Proof. We prove by induction on m that $r \in \bigcup_{i=mn}^{(m+1)n-1} K_{i,0}$ implies $a^m b^m r = r$. Clearly, this holds for $m=0$ and $r \in K_{n,0}$ implies $abr = r$. Suppose that $r \in \bigcup_{i=mn}^{(m+1)n-1} K_{i,0}$ implies $a^m b^m r = r$ for all m smaller than m' $(m' \geqq 1)$ and $r \in K_{m'n,0}$ implies $a^{m'} b^{m'} r = r$. Now let $r \in K_{j,0}$, where $m'n < j \leqq (m'+1)n$. Since ab is an identity element of $\bigcup_{i=1}^{\infty} K_{i,i}$, we have $abr = r$. Hence $br \mathscr{L} r$, that is, $br \in G_0$. On the other hand, we have

$$(br)\varphi = b\varphi + r\varphi = -n + j,$$

whence $br \in K_{-n+j,0}$. By assumption, $(a^{m'-1} b^{m'-1}) br = br$, that is,

$$r = abr = a\,a^{m'-1} b^{m'-1} br = a^{m'} b^{m'} r.$$

Moreover, if $j = (m'+1)n$, i.e. $-n+j = m'n$, then $br = (a^{m'} b^{m'}) br$ and

$$r = abr = a^{m'+1} b^{m'+1} r.$$

This completes the proof of the fact that $a^m b^m$ is a left identity element in $\bigcup_{i=mn}^{\infty} K_{i,i} \cup 0$. Dually, it follows that it is also a right identity.

Theorem 2.10. *Let a nontrivial pair of functions $f, g\colon S_0^{(2)} \to N^0$ be given on the 0-$\mathscr{D}$-simple semigroup S with identity e such that the subsemigroup $\bigcup_{i=1}^{\infty} K_{i,i} \cup 0$ has an identity element e_1. Assume that $e_1 \in K_{n,n}$. Then*

(i) *the ranges of f and g are the set of multiples of n.*

(ii) *If, moreover, $K_{0,0}^0 = K_{0,0} \cup 0$ is a subsemigroup of S and $e_1 K_{0,0} \subseteq K_{n,n}$, then $S \cong \mathscr{S}^0(B, K_{0,0}^0, N^0, f^*, g^*, \varkappa)$, where B denotes the bicyclic semigroup and the endomorphism $\pi = 1\varkappa$ preserves e.*

Proof. Since $e_1 \in F_n$, the number n is the least positive integer with $F_n \neq \square$. The semigroup S is 0-$\mathscr{D}$-simple, hence $e \mathscr{L} a \mathscr{R} e_1$ for some a and since S is regular, there exists an inverse b of a such that $ba = e$, $ab = e_1$. Suppose that in contrast to (i), $F_p \neq \square$ for some p, where $n \nmid p$. Let d be the greatest common divisor of n and p. Then $up - vn = d$ for some positive integers u, v. Let c be an element of F_p such that $e \mathscr{L} c$. By Lemma 2.7 we have $c^u \in K_{up,0}$, and since $up > vn$, Lemma 2.9 implies $a^v b^v c^u = c^u$. Hence $b^v c^u \mathscr{L} c^u \mathscr{L} e$. However, we have

$$(b^v c^u)\varphi = v(b\varphi) + u(c\varphi) = -vn + up = d,$$

whence it follows that $b^v c^u \in K_{d,0}$ with $d<n$, a contradiction. On the other hand, $a^m \in F_{mn}$ by Lemma 2.7, which proves (i) for f. Dually, one can show (i) for g.

Turning to (ii), we first show that all the elements of S_0 can be uniquely represented in the form $a^k h b^m$, where $h \in K_{0,0}$. Let $s \in K_{kn,mn}$. Since S is 0-$\mathscr{D}$-simple, Green's lemma ensures that $s = a_k h' b_m$ for some a_k, b_m and h' such that $e \mathscr{L} a_k \mathscr{R} s$, $e \mathscr{R} b_m \mathscr{L} s$ and $h' \mathscr{H} e$. Applying Lemma 2.9 we obtain

$$s = a^k b^k s a^m b^m = a^k (b^k a_k) h' (b_m a^m) b^m.$$

Since $a_k \in K_{kn,0}$, the equality $a^k b^k a_k = a_k$ holds. Hence $b^k a_k \mathscr{L} a_k$, that is, $b^k a_k \in G_0$. Moreover, $(b^k a_k)\varphi = -kn + kn = 0$, whence $b^k a_k \in K_{0,0}$. The fact that $b_m a^m \in K_{0,0}$ can be proved similarly. By assumption $K^0_{0,0}$ is a subsemigroup of S. This implies that $h = (b^k a_k) h' (b_m a^m) \in K^0_{0,0}$ and $h \neq 0$ because $s \neq 0$. We have obtained that $s = a^k h b^m$. To prove uniqueness suppose that we have

$$a^k h b^m = a^{k'} h' b^{m'}$$

for some h, h' in $K_{0,0}$. Since

$$(a^k h b^m)\varphi = (k-m)n, \quad (a^{k'} h' b^{m'})\varphi = (k'-m')n,$$

we have $k-m = k'-m'$, that is, $k-k' = m-m' = r$. Without loss of generality we can assume that r is nonnegative. Multiplying the equality above by $b^{k'}$ on the left and by $a^{m'}$ on the right it yields $a^r h b^r = h'$. Hence $h' a^r b^r = h'$. Should $r>0$ hold, then $a^r b^r$ would belong to L_{1+N^0} implying $h' \in L_{1+N^0}$. Since $h' \in K_{0,0}$, we have $r=0$ and $h=h'$, as was to be proved.

Let h be any element of $K_{0,0}$. Since $(bh)\varphi = b\varphi + h\varphi = -n$ we have either $bh \in K_{0,n}$ or $bh \in L_{n+1+N^0}$. The latter would imply $abh = e_1 h \in L_{n+1+N^0}$, contrary to the assumption $e_1 h \in K_{n,n}$. Hence $bh \in K_{0,n}$ and by the foregoing $bh = \hat{h} b$ for the unique element $\hat{h} = bha$ of $K_{0,0}$. Similarly, we have $ha \in K_{n,0}$ and hence $ha = (ab)ha = a\hat{h}$. If $h=0$, then $\hat{h}=0$ is the unique element such that $bh = \hat{h} b$ and $ha = a\hat{h}$. The mapping π sending h into $\hat{h}$ is an endomorphism of $K^0_{0,0}$ as we have

$$[(gh)\pi]b = b(gh) = (bg)h = (g\pi)bh = (g\pi)(h\pi)b,$$

whence $(gh)\pi = (g\pi)(h\pi)$. Obviously, $e\pi = e$. Using these results we obtain the product of any two elements of S_0 in the form

$$(a^m g b^n)(a^p h b^q) = a^m g b^{n-r} a^{p-r} h b^q = a^m g a^{p-r} b^{n-r} h b^q = \\ = a^{m+p-r} (g\pi^{p-r} h\pi^{n-r}) b^{n+q-r},$$

where $r = \min(n, p)$. In the second step we made use of the equality $b^{n-r} a^{p-r} = a^{p-r} b^{n-r}$ implied by the fact that at least one of the exponents equals 0. This implies that the mapping Φ defined by

$$s\Phi = \begin{cases} (m, h, n) & \text{if} \quad s \neq 0 \quad \text{and} \quad s = a^m h b^n, \quad h \in K_{0,0} \\ 0 & \text{if} \quad s = 0 \end{cases}$$

is an isomorphism of S onto $\mathscr{S}^0(B, K^0_{0,0}, C, f^*, g^*, \varkappa)$, where $\varkappa$ is the homomorphism of N^0 into the endomorphism monoid of $K^0_{0,0}$ mapping k into π^k.

Corollary 2.11. *In a semigroup S satisfying the conditions of Theorem* 2.10 *the relations $\mathscr{F}, \mathscr{G}$ and $\mathscr{K}$ are* 0*-congruences.*

Now we construct a $\mathscr{D}$-simple semigroup S with identity and a pair of functions $f, g\colon S\times S\to N^0$ with range N^0 which fail to have the property that the subsemigroup $\bigcup_{i=1}^{\infty} K_{i,i}$ has an identity element. We shall use the notions and results of W. D. Munn's paper [8].

The descending ω-chain as a meet semilattice is isomorphic to the semilattice $N^0_\wedge$ with underlying set N^0 and operation defined by

$$m\wedge n = \max(m, n).$$

Let E be the direct product $N^0_\wedge \times N^0_\wedge$. The semilattice E is uniform and has a greatest element (0, 0). The set T_E of all isomorphisms of a principal ideal of E onto another one considered as partial mappings of E together with the usual multiplication is a $\mathscr{D}$-simple inverse semigroup with the semilattice of idempotents isomorphic to E. The principal ideal of E generated by (m, n) will be denoted by $[m, n]$. For each pair of elements (m, n), (p, q) of E there exist two isomorphisms $\alpha^+_{(m,n),(p,q)}$ and $\alpha^-_{(m,n),(p,q)}$ of $[m, n]$ onto $[p, q]$ defined by

$$(m+i, n+j)\alpha^+_{(m,n)(p,q)} = (p+i, q+j), \quad (m+i, n+j)\alpha^-_{(m,n)(p,q)} = (p+j, q+i),$$

where $i, j\geqq 0$. Let us define the functions χ_1 and $\chi_2\colon T_E\to N^0$ as follows:

$$\alpha^\eta_{(m,n),(p,q)}\chi_1 = m+n, \quad \alpha^\eta_{(m,n),(p,q)}\chi_2 = p+q,$$

where η may be $+$ as well as $-$. Since $\alpha^\eta_{(m,n),(p,q)}T_E \subseteq \alpha^{\eta'}_{(m',n'),(p',q')}T_E$ if and only if $(m,n)\leqq(m',n')$ in E, the set $R_{k+N^0}=\{\beta\in T_E | \beta\chi_1\in k+N^0\}$ is a right ideal of T_E. Dually, $L_{k+N^0}=\{\beta\in T_E | \beta\chi_2\in k+N^0\}$ is a left ideal of T_E. Furthermore, denoting χ_1/χ_2 by φ, we have

$$\begin{gathered}
(\alpha^\eta_{(m,n),(p,q)}\alpha^{\eta'}_{(m',n'),(p',q')})\varphi = \\
= \alpha^{\eta\eta'}_{(p\wedge m', q\wedge n')\alpha^\eta_{(p,q),(m,n)},(p\wedge m', q\wedge n')\alpha^{\eta'}_{(m',n'),(p',q')}}\varphi = \\
= (m+n+(p\wedge m'-p)+(q\wedge n'-q))-(p'+q'+(p\wedge m'-m')+(q\wedge n'-n')) = \\
= (m+n)-(p+q)+(m'+n')-(p'+q') = \\
= \alpha^\eta_{(m,n),(p,q)}\varphi + \alpha^{\eta'}_{(m',n'),(p',q')}\varphi.
\end{gathered}$$

Hence φ is a homomorphism. Theorem 2.3 implies that $f, g\colon T_E\times T_E\to N^0$ defined by (5) have the desired properties (1)—(3). There are two idempotents in $K_{1,1}$ which are dual atoms in the semilattice of idempotents: $\alpha^+_{(0,1),(0,1)}$, $\alpha^+_{(1,0),(1,0)}$. Consequently, the subsemigroup $\bigcup_{i=1}^{\infty} K_{i,i}$ has no identity element.

3. In the present section we deal with the properties of 0-extensions $\mathcal{S}^0(S, \Sigma, C, f, g, \varkappa)$ of a semigroup Σ by S. We state a proposition on the 0-congruence induced by the 0-extension. Necessary and sufficient conditions are given for $\mathcal{S}^0(S, \Sigma, C, f, g, \varkappa)$ to have an identity, to be a regular or an inverse semigroup. We investigate its Green's relations and ideals, too. The homomorphisms of a semigroup $\mathcal{S}^0(S, \Sigma, C, f, g, \varkappa)$ into $\mathcal{S}^0(\bar{S}, \bar{\Sigma}, \bar{C}, \bar{f}, \bar{g}, \bar{\varkappa})$ are investigated in some special cases. We introduce a concept of equivalence between the 0-extensions of a semigroup Σ by another one denoted by S and deal with the equivalent 0-extensions. This section is mostly independent of section 2, the main result of which, Theorem 2.3, is used in Theorems 3.9 and 3.11 only.

Let the semigroups S and Σ be given. Consider a 0-extension $\mathcal{S}^\circ(S, \Sigma, C, f, g, \varkappa)$ of Σ by S over the cancellative monoid C. For brevity, denote $\mathcal{S}^\circ(S, \Sigma, C, f, g, \varkappa)$ by $\mathbf{S}$.

The 0-congruence induced by the 0-extension $\mathbf{S}$ will be denoted by $\mathscr{C}$. Its congruence classes are $C_r = \{(r, \sigma) | \sigma \in \Sigma_\omega\}$ and $C_0 = \{0\}$. Denote $C_r \cup 0$ by C_r^0.

Proposition 3.1. (i) *All congruence classes C_i^0 with 0 adjointed corresponding to nonzero idempotents of S are subsemigroups of* $\mathbf{S}$ *isomorphic to Σ^ω.*

(ii) *If Σ has an idempotent element ι preserved by all the endomorphisms in $\{f_{s,t}, g_{s,t} | s, t \in S, st \neq o\}\varkappa$ then*

$$\{(s, \iota) | s \in S_0\} \cup 0$$

is a subsemigroup of $\mathbf{S}$ *isomorphic to S^0.*

Proof. Since, by Lemma 2.1, $f_{i,i} = g_{i,i} = 1$ we have

$$(i, \varrho)(i, \sigma) = \begin{cases} (i, \varrho\sigma) & \text{if} \quad \varrho\sigma \neq \omega \\ 0 & \text{otherwise} \end{cases}$$

and

$$(i, \varrho)0 = 0(i, \varrho) = 0 \cdot 0 = 0.$$

Hence C_i^0 is isomorphic to Σ^ω. As for (ii) if ι has the required property then

$$(s, \iota)(t, \iota) = \begin{cases} (st, \iota) & \text{if} \quad st \neq o \\ 0 & \text{otherwise} \end{cases}$$

and

$$(s, \iota)0 = 0(s, \iota) = 0 \cdot 0 = 0.$$

Hence $\{(s, \iota) | s \neq o\} \cup 0$ is a subsemigroup ismorphic to S^0.

Proposition 3.2. (i) *An element* (i, ι) *of* $\mathbf{S}$ *is idempotent if and only if i and ι are idempotents in S and Σ, respectively.*

(ii) *The element* (e, ε) *of* $\mathbf{S}$ *is the identity of* $\mathbf{S}$ *if and only if* e, ε *are the identities of* S *and* Σ, *respectively, and the endomorphisms of* Σ *contained in* $\{f_{s,t}, g_{s,t} | s, t \in S, st \neq o\}\varkappa$ *preserve* ε.

Proof. Using the definition of $\mathbf{S}$, the element (i, ι) is idempotent if and only if

$$i^2 = i \quad \text{and} \quad \iota(f_{i,i}\varkappa)\iota(g_{i,i}\varkappa) = \iota.$$

By Lemma 2.1, $i^2=i$ implies $f_{i,i}=g_{i,i}=1$. Thus the above condition is equivalent to the following one: $i^2=i$ and $\iota^2=\iota$. Similarly, (e, ε) is an identity if and only if for any s, σ

$$se = es = s \quad \text{and} \quad \sigma(f_{s,e}\varkappa)\varepsilon(g_{s,e}\varkappa) = \varepsilon(f_{e,s}\varkappa)\sigma(g_{e,s}\varkappa) = \sigma.$$

Lemma 2.1 ensures $f_{s,e}=g_{e,s}=1$, so that the latter equality says that

$$\sigma\varepsilon(g_{s,e}\varkappa) = \varepsilon(f_{e,s}\varkappa)\sigma = \sigma$$

for any s, σ. Taking $s=e$ this yields that ε is the identity of Σ. But then $\varepsilon(f_{e,s}\varkappa) = \varepsilon(g_{s,e}) = \varepsilon$ for all $s \neq o$. Let s, t be any elements of S such that $st \neq o$. Applying (1) and the fact that $\varkappa$ is a homomorphism we have

$$\varepsilon = \varepsilon(f_{e,st}\varkappa) = \varepsilon(f_{e,s}\varkappa)(f_{s,t}\varkappa) = \varepsilon(f_{s,t}\varkappa),$$

and similarly, by (3), we have $\varepsilon = \varepsilon(g_{s,t}\varkappa)$. Conversely, if e and ε are identities of S and Σ and $\varkappa$ has the desired property, then (e, ε) is clearly an identity.

Theorem 3.3. (i) *The semigroup* $\mathbf{S}$ *is regular if and only if both* S *and* Σ *are regular.*

(ii) $\mathbf{S}$ *is an inverse semigroup if and only if both* S *and* Σ *are inverse semigroups.*

Proof. We show that two elements (r, ϱ) and (s, σ) of $\mathbf{S}$ are inverses of each other if and only if r, s and $\varrho, \sigma(f_{s,r}g_{r,sr}\varkappa)$ are inverses of each other in S and Σ, respectively, where $f_{s,r}g_{r,sr}\varkappa$ is an automorphism of Σ. This proves the theorem. By definition (r, ϱ) and (s, σ) are inverses of each other if and only if

$$rsr = r, \quad srs = s, \tag{11}$$

$$\varrho(f_{r,sr}\varkappa)\sigma(f_{s,r}g_{r,sr}\varkappa)\varrho(g_{rs,r}\varkappa) = \varrho, \quad \sigma(f_{s,rs}\varkappa)\varrho(f_{r,s}g_{s,rs}\varkappa)\sigma(g_{sr,s}\varkappa) = \sigma. \tag{12}$$

Using (2), (1) and (3), (11) implies that

$$f_{r,s}g_{s,rs}f_{s,r}g_{r,sr} = f_{r,s}f_{rs,r}g_{s,r}g_{r,sr} = f_{r,sr}g_{rs,r}.$$

It follows from Lemma 2.1 that $f_{r,sr}=g_{rs,r}=f_{s,rs}=g_{sr,s}=1$. Hence $f_{r,sr}\varkappa = g_{rs,r}\varkappa = f_{s,rs}\varkappa = g_{sr,s}\varkappa$, the identity automorphisms of Σ and $f_{r,s}g_{s,rs}\varkappa$, $f_{s,r}g_{r,sr}\varkappa$ are automorphisms of Σ, inverses of each other. Thus conditions (11), (12) are equivalent to (11) and

$$(13) \qquad \varrho\sigma(f_{s,r}g_{r,sr}\varkappa)\varrho = \varrho, \quad \sigma(f_{s,r}g_{r,sr}\varkappa)\varrho\sigma(f_{s,r}g_{r,sr}\varkappa) = \sigma(f_{s,r}g_{r,sr}\varkappa),$$

as was to be proved.

As for Green's relations and ideals of the semigroup $\mathrm{S}=\mathcal{S}^0(S, \Sigma, C, f, g, \varkappa)$ in general one cannot say more than the definitions of them. Therefore we deal with the case when S is regular.

Lemma 3.4. *Let S be a regular semigroup. The principal left [right] ideal of* S *generated by (s, σ) is contained in the one generated by (r, ϱ) if and only if $(r, \varrho)=$ $=(s, \sigma)$ or the following conditions are satisfied:*

(a) $s\in Sr[s\in rS]$,

(b) $\sigma\in\Sigma\varrho\pi[\sigma\in g\pi\Sigma]$ *where* $\pi = g_{x,r}\varkappa(xr = s)\,[\pi = f_{r,x}(rx = s)]$

is an endomorphism of Σ depending only on r and s.

Proof. First we note that $xr=s$ and $x'r=s$ imply $g_{x,r}=g_{x',r}$. Indeed, if i is an idempotent in the $\mathcal{L}$-class containing r, then $ri=r$ and $si=s$. Thus by (3) we have

$$g_{r,i}g_{x,r} = g_{s,i} \quad \text{and} \quad g_{r,i}g_{x',r} = g_{s,i}.$$

Since C is cancellative, $g_{x,r}=g_{x',r}$.

By definition, $(s, \sigma)\in\mathrm{S}(r, \varrho)$ means that there exist elements x and ξ in S and Σ, respectively, such that

$$(14) \qquad xr = s, \quad \xi(f_{x,r}\varkappa)\varrho(g_{x,r}\varkappa) = \sigma.$$

Hence the necessity of (a), (b) is proved. Conversely, assume that (a), (b) hold, that is, there exist x and ξ' in S and Σ, respectively, such that

$$xr = s, \quad \xi'\varrho\pi = \sigma.$$

Since S is regular, x can be chosen to satisfy $x\mathcal{R}s$. If j is an idempotent belonging to the $\mathcal{R}$-class of s, then $j=sw$ for some w and the equality

$$f_{j,x}f_{x,r}f_{s,w} = f_{j,s}f_{s,w} = f_{j,j} = 1$$

follows from (1) and Lemma 2.1. Hence $f_{x,r}$ is in the group of units of C and $f_{x,r}\varkappa$ is an automorphism. Thus $\xi'=\xi(f_{x,r}\varkappa)$ for some ξ, that is, by (14), $(s, \sigma)\in\mathrm{S}(r, \varrho)$. The statement for right ideals can be proved dually.

The next theorem deals with Green's relations of S.

Theorem 3.5. *Let S be a regular semigroup. Two distinct elements (r, ϱ) and (s, σ) of* S *are $\mathcal{L}[\mathcal{R}]$-equivalent if and only if*

(a) $r\mathcal{L}s[r\mathcal{R}s]$ *in S and*

(b) $\sigma\in\Sigma\varrho\alpha,\ \varrho\alpha\in\Sigma\sigma[\sigma\in\varrho\alpha\Sigma,\ \varrho\alpha\in\sigma\Sigma]$ *where* $\alpha=g_{x,r}\varkappa\,(xr=s)\ [\alpha=f_{r,x}\varkappa(rx=s)]$ *is an automorphism of Σ depending only on r and s.*

Note that if the group of units of C is trivial, then $g_{x,r}=1[f_{r,x}=1]$, whence α is the identity automorphism.

Proof. By Lemma 3.4. all we need to show is that if $xr=s, ys=r$ then $g_{x,r}\varkappa$ and $g_{y,s}\varkappa$ are automorphisms of Σ being inverses of each other. To prove this one has only to observe that we have

$$g_{x,r}g_{y,s}=g_{x,r}g_{y,xr}=g_{yx,r}=1$$

by (3) and Lemma 2.1.

An immediate consequence of this theorem is

Corollary 3.6. *Let S be regular and Σ have the property that $\sigma\in\sigma\Sigma\cap\Sigma\sigma$ for all elements σ in Σ.*

(i) *The distinct elements (r, ϱ) and (s, σ) of* $\mathbf{S}$ *are $\mathcal{D}$-equivalent if and only if $r\mathcal{D}s$ and there exists an element t in S such that $r\mathcal{L}t\mathcal{R}s$ and $\varrho\alpha\mathcal{D}\sigma\beta$, where $\alpha=g_{x,r}\varkappa$, $\beta=f_{s,y}\varkappa\,(xr=sy=t)$ are automorphisms of Σ depending only on r, s and t.*

(ii) *If both S and Σ are (0-)$\mathcal{D}$-simple, then* $\mathbf{S}$ *is also 0-$\mathcal{D}$-simple.*

(iii) *If the group of units of C is trivial, then* $\mathbf{S}$ *is 0-$\mathcal{D}$-simple if and only if S and Σ are (0-)$\mathcal{D}$-simple.*

To make the formulation of the theorem on the ideals of $\mathbf{S}$ easier we introduce the following notations. If $\mathbf{C}$ is a subset of $\mathbf{S}$ containing the 0 element let

$$C=\{s\in S|\exists\sigma\in\Sigma\ (s,\sigma)\in\mathbf{C}\}$$

provided S has no zero element and adjoin o to C if $o\in S$.

For all c in C different from o define Γ_c as

$$\Gamma_c=\{\sigma\in\Sigma|(c,\sigma)\in\mathbf{C}\}$$

if Σ has no zero element and adjoin ω to Γ_c if $\omega\in\Sigma$.

In particular, the subsets corresponding to the subsets of $\mathbf{S}$ denoted by $\mathbf{L}$, $\mathbf{R}$ and $\mathbf{D}$ are denoted by L, $\Lambda_l(l\in L, l\neq o)$, R, $P_r\,(r\in R, r\neq o)$ and D, $\Delta_d\,(d\in D, d\neq o)$, respectively.

Theorem 3.7. *Let S be a regular semigroup. A subset* $\mathbf{L}[\mathbf{R}]$ *of the semigroup* $\mathbf{S}$ *containing the 0 element is a left [right] ideal if and only if*

(a) *$L[R]$ is a left [right] ideal of S and*

(b) *for all elements l of L[r of R], l' of $Sl\,[r'$ of $rS]$ different from o and λ of $\Lambda_l[\varrho$ of $P_r]$ $\Sigma\lambda\pi\subseteq\Lambda_{l'}[\varrho\pi\Sigma\subseteq P_{r'}]$ holds where $\pi=g_{x,l}\varkappa(xl=l')[\pi=f_{r,x}\varkappa(rx=r')]$ is an endomorphism of Σ depending only on l and l'[r and r'].*

The proof of this theorem is easy thanks to Lemma 3.4, therefore it is left to the reader.

Corollary 3.8. *Suppose the semigroup S is regular. The semigroup* $\mathbf{S}$ *is* 0*-simple if and only if S is* (0-)*simple and*

$$\Sigma = \bigcup_{\pi \in E_t^s} \Sigma \sigma \pi \Sigma \tag{16}$$

for all s, t and σ different from o and ω, respectively, where

$$E_t^s = \{(g_{x,s} f_{xs,y})\varkappa \mid x \mathcal{R} t \mathcal{L} y \quad \text{and} \quad xsy = t\}.$$

Proof. $\mathbf{S}$ is 0-simple if and only if for every element (s, σ) in it the ideal $\mathbf{D} = \mathbf{S}(s, \sigma)\mathbf{S}$ is equal to S itself. By the last theorem this means that S is (0-)simple and $\Delta_t = \Sigma$ for every $t \neq o$. So it is sufficient to prove that the right side of the equality (16) is equal to Δ_t. Theorem 3.7 (b) ensures that $\bigcup_{\pi \in E_t^s} \Sigma \sigma \pi \Sigma \subseteqq \Delta_t$. Conversely, since the nonzero elements of $\mathbf{D}$ have the form $(x', \xi)(s, \sigma)(y', \eta)$, Δ_t is contained in the ideal $\bigcup_{\substack{x', y' \\ x' s y' = t}} \Sigma \sigma (g_{x',s} f_{x's,y'})\varkappa \Sigma$. Let i_1 and i_2 be idempotents such that $i_1 \mathcal{R} t \mathcal{L} i_2$ and let $x = i_1 x'$, $y = y' i_2$. Obviously, $x \mathcal{R} t \mathcal{L} y$ and $xsy = t$. Applying identities (1)—(3) and Lemma 2.1, we see that

$$g_{x',s} f_{x's,y'} = g_{x',s} f_{x's,y'} g_{i_1,x'sy'} = g_{x',s} g_{i_1,x's} f_{xs,y'} = g_{x,s} f_{xs,y'} =$$
$$= g_{x,s} f_{xs,y'} f_{xsy',i_2} = g_{x,s} f_{xs,y}.$$

Hence $(g_{x',s} f_{x's,y'})\varkappa \in E_t^s$ and $\Delta_t \subseteqq \bigcup_{\pi \in E_t^s} \Sigma \sigma \pi \Sigma$, which completes the proof.

In what follows we deal with homomorphisms of a semigroup $\mathbf{S} = \mathcal{S}^0(S, \Sigma, C, f, g, \varkappa)$ into another one $\bar{\mathbf{S}} = \mathcal{S}^0(\bar{S}, \bar{\Sigma}, \bar{C}, \bar{f}, \bar{g}, \bar{\varkappa})$ in two special cases. In the first case we assume that $C\varkappa$ and $\bar{C}\bar{\varkappa}$ are contained in the group of automorphisms of Σ and $\bar{\Sigma}$, respectively. Then without loss of generality we can assume C and $\bar{C}$ to be groups. In the alternative case we suppose that S and $\bar{S}$ are inverse semigroups.

Theorem 3.9. *Suppose that C and $\bar{C}$ are groups and the definition of f, g and $\bar{f}$, $\bar{g}$ can be extended to S^e and $\bar{S}^{\bar{e}}$, respectively. Denote the suitable homomorphisms used in Theorem 2.3 by $\varphi\colon S_0^e \to C$ and $\bar{\varphi}\colon \bar{S}_{\bar{0}}^{\bar{e}} \to \bar{C}$. Suppose four mappings $m_1\colon S^0 \to \bar{S}^{\bar{0}}$, $\mu_1\colon \Sigma^\omega \to \bar{S}^{\bar{0}}$, $m_2\colon S^0 \to \bar{\Sigma}^{\bar{\omega}}$ and $\mu_2\colon \Sigma^\omega \to \bar{\Sigma}^{\bar{\omega}}$ are given with the following properties:*

(a) $om_1 = \omega\mu_1, \quad om_2 = \omega\mu_2, \quad \bar{o}m_1^{-1} = \bar{\omega}m_2^{-1}, \quad \bar{o}\mu_1^{-1} = \bar{\omega}\mu_2^{-1}.$

(b) *For any r, s, ϱ, σ in S and Σ, respectively,*

$$(rs)m_1 = rm_1 \cdot sm_1, \tag{17}$$

$$(\varrho\sigma)\mu_1 = \varrho\mu_1 \cdot \sigma\mu_1 \tag{18}$$

if $(rs)m_1 \neq \bar{o}$ *and* $(\varrho\sigma)\mu_1 \neq \bar{o}$, *furthermore*

$$(rs)m_2 = rm_2 \cdot sm_2(rm_1\bar{\varphi}\bar{\varkappa})^{-1}, \tag{19}$$

$$(\varrho\sigma)\mu_2 = \varrho\mu_2 \cdot \sigma\mu_2(\varrho\mu_1\bar{\varphi}\bar{\varkappa})^{-1} \tag{20}$$

whenever $(rs)m_1 \neq \bar{o}$ *or* $rm_1 \cdot sm_1 \neq \bar{o}$ *and* $(\varrho\sigma)\mu_1 \neq \bar{o}$ *or* $\varrho\mu_1 \cdot \sigma\mu_1 \neq \bar{o}$.

(c) *For any* r, ϱ *in* S *and* Σ, *respectively,*

$$rm_1 \cdot \varrho\mu_1 = \varrho(r\varphi\varkappa)^{-1}\mu_1 \cdot rm_1, \tag{21}$$

$$rm_2 \cdot \varrho\mu_2(rm_1\bar{\varphi}\bar{\varkappa})^{-1} = \varrho(r\varphi\varkappa)^{-1}\mu_2 \cdot rm_2\big(\varrho(r\varphi\varkappa)^{-1}\mu_1\bar{\varphi}\bar{\varkappa}\big)^{-1}$$

if all the four elements are different from $\bar{o}$ *and* $\bar{\omega}$, *respectively. In addition, the left hand sides differ from zero if and only if the right hand sides do so.*

Define a mapping $\Phi\colon \mathbf{S} \to \bar{\mathbf{S}}$ *in the following way. Let*

$$(r, \varrho)\varphi = \big(\varrho'\mu_1 \cdot rm_1, (\varrho'\mu_2 \cdot rm_2(\varrho'\mu_1\bar{\varphi}\bar{\varkappa})^{-1})(f_{\bar{e},\varrho'\mu_1 \cdot rm_1}\bar{\varkappa})\big),$$

where $\varrho' = \varrho(f_{e,r}\varkappa)^{-1}$, *when both components on the right are nonzero and* $(r,\varrho)\Phi = \bar{0}$ *otherwise. Further, let* $0\Phi = (om_1, om_2)$ *if* $om_1 \neq \bar{o}$ *and* $0\Phi = \bar{0}$ *otherwise. Then*

(i) *the mapping* Φ *is a homomorphism.*

(ii) Φ *is an isomorphism if and only if* $0\Phi = \bar{0}$ *and for all nonzero elements* $\bar{r}$, $\bar{\varrho}$ *of* $\bar{S}$ *and* $\bar{\Sigma}$, *respectively, there exist uniquely determined elements* r *and* ϱ *in* S *and* Σ *such that*

$$\bar{r} = \varrho\mu_1 \cdot rm_1 \quad \text{and} \quad \bar{\varrho} = \varrho\mu_2 \cdot rm_2(\varrho\mu_1\bar{\varphi}\bar{\varkappa})^{-1}.$$

(iii) *If the semigroups* S, $\bar{S}$, Σ *and* $\bar{\Sigma}$ *have identity elements, then all the homomorphisms of* $\mathbf{S}$ *into* $\bar{\mathbf{S}}$ *are of this form.*

Proof. It is not difficult to check statement (i) by computation. (ii) is implied immediately by the definition of Φ and the fact that the elements of $C\varkappa$ and $\bar{C}\bar{\varkappa}$ are automorphisms. Turning to (iii), consider the semigroups S, $\bar{S}$, Σ, $\bar{\Sigma}$ with identity. Since $f_{e,r}\varkappa$ is an automorphism, for any nonzero r, ϱ we have

$$(r, \varrho) = \big(e, \varrho(f_{e,r}\varkappa)^{-1}\big)(r, \varepsilon).$$

Hence all the nonzero elements of $\mathbf{S}$ can be uniquely written in the form $(e, \varrho)(r, \varepsilon)$, where $\{(e, \varrho) | \varrho \in \Sigma_\omega\} \cup 0$ and $\{(r, \varepsilon) | r \in S_0\} \cup 0$ are subsemigroups isomorphic to Σ^ω and S^0, respectively. Let $\Phi\colon \mathbf{S} \to \bar{\mathbf{S}}$ be a homomorphism. Define the mappings $m_1\colon S^0 \to \bar{S}^{\bar{o}}$, $\mu_1\colon \Sigma^\omega \to \bar{S}^{\bar{o}}$, $m_2\colon S^0 \to \bar{\Sigma}^{\bar{\omega}}$, $\mu_2\colon \Sigma^\omega \to \bar{\Sigma}^{\bar{\omega}}$ as follows. Let $om_1 = \omega\mu_1 = \bar{o}$, $om_2 = \omega\mu_2 = \bar{\omega}$ if $0\Phi = \bar{0}$ and $om_1 = \omega\mu_1 = \bar{r}$, $om_2 = \omega\mu_2 = \bar{\varrho}$ if $0\Phi = (\bar{e}, \bar{\varrho})(\bar{r}, \bar{\varepsilon})$. Let $rm_1 = \bar{o}$, $rm_2 = \bar{\omega}$ if $(r, \varepsilon)\Phi = \bar{0}$ and $\varrho\mu_1 = \bar{o}$, $\varrho\mu_2 = \bar{\omega}$ if $(e, \varrho)\Phi = \bar{0}$, respectively. In the opposite case, let

$$(r, \varepsilon)\Phi = (\bar{e}, rm_2)(rm_1, \bar{\varepsilon}) = \big(rm_1, rm_2(\bar{f}_{\bar{e},rm_1}\bar{\varkappa})\big),$$

$$(e, \varrho)\Phi = (\bar{e}, \varrho\mu_2)(\varrho\mu_1, \bar{\varepsilon}) = \big(\varrho\mu_1, \varrho\mu_2(\bar{f}_{\bar{e},\varrho\mu_1}\bar{\varkappa})\big).$$

Clearly, $om_1=\omega\mu_1$, $om_2=\omega\mu_2$, $\bar{o}m_1^{-1}=\bar{\omega}m_2^{-1}$ and $\bar{o}\mu_1^{-1}=\bar{\omega}\mu_2^{-1}$. Relations (1), (3) and the fact that $\bar{\varphi}$ is a homomorphism yield that for any $\bar{r}$, $\bar{s}$ with $\bar{r}\bar{s}\neq\bar{o}$ we have

$$\bar{f}_{\bar{e},\bar{r}}\bar{f}_{\bar{r},\bar{s}}=\bar{f}_{\bar{e},\bar{r}\bar{s}},$$
(22)
$$\bar{f}_{\bar{e},\bar{s}}\bar{g}_{\bar{r},\bar{s}}=\bar{f}_{\bar{e},\bar{s}}\bar{g}_{\bar{s},\bar{e}}^{-1}\bar{g}_{\bar{r}\bar{s},\bar{e}}=\bar{s}\bar{\varphi}(\bar{r}\bar{s}\bar{\varphi})^{-1}\bar{f}_{\bar{e},\bar{r}\bar{s}}=(\bar{r}\bar{\varphi})^{-1}\bar{f}_{\bar{e},\bar{r}\bar{s}}.$$

Since $\varkappa$ is a homomorphism, we have

$$(r,\varepsilon)\Phi\cdot(s,\varepsilon)\Phi=\big(rm_1\cdot sm_1,\, rm_2(\bar{f}_{\bar{e},rm_1}\bar{\varkappa})(\bar{f}_{rm_1,sm_1}\bar{\varkappa})\cdot sm_2(\bar{f}_{\bar{e},sm_1}\bar{\varkappa})(\bar{g}_{rm_1,sm_1}\bar{\varkappa})\big)=$$
$$=\big(rm_1\cdot sm_1,\big(rm_2\cdot sm_2(rm_1\bar{\varphi}\bar{\varkappa})^{-1}\big)\bar{f}_{\bar{e},rm_1\cdot sm_1}\bar{\varkappa}\big)$$

for every pair r, s, whenever both components on the right side are nonzero and $(r,\varepsilon)\Phi\cdot(s,\varepsilon)\Phi=\bar{0}$ in the opposite case. The same equality holds if (r,ε) or (s,ε) is replaced by 0, that is, if $r=o$ or $s=o$. But Φ is a homomorphism, $(r,\varepsilon)\Phi\cdot(s,\varepsilon)\Phi=$ $=(rs,\varepsilon)\Phi$, which proves that (17) and (19) hold under the conditions mentioned in the theorem. Investigating Φ restricted to the subsemigroup $\{(e,\varrho)|\varrho\in\Sigma_\omega\}\cup 0$ one can verify (18) and (20). Observe that for any r and ϱ

$$(r,\varepsilon)(e,\varrho)=\big(e,\varrho(r\varphi\varkappa)^{-1}\big)(r,\varepsilon).$$

Hence $(r,\varepsilon)\Phi(e,\varrho)\Phi=\big(e,\varrho(r\varphi\varkappa)^{-1}\big)\Phi\cdot(r,\varepsilon)\Phi$, that is, denoting $\varrho(r\varphi\varkappa)^{-1}\mu_1$ by $\bar{s}$ and $\varrho(r\varphi\varkappa)^{-1}\mu_2$ by $\bar{\sigma}$, we have

$$\big(rm_1\cdot\varrho\mu_1,\big(rm_2\cdot\varrho\mu_2(rm_1\bar{\varphi}\bar{\varkappa})^{-1}\big)(\bar{f}_{\bar{e},rm_1\cdot\varrho\mu_1}\bar{\varkappa})\big)=\big(\bar{s}\cdot rm_1,\big(\bar{\sigma}\cdot rm_2(\bar{s}\bar{\varphi}\bar{\varkappa})^{-1}(\bar{f}_{\bar{e},\bar{s}\cdot rm_1})\big)$$

if all the components are nonzero. Moreover, if a component is zero on one side, then so is one on the other side. This is equivalent to condition (21), which completes the proof.

In the next theorem we use the notation $f_{r,s}^{-1}$ only if $f_{r,s}$ is contained in the group of units of C. If r is an element of an inverse semigroup the unique idempotent belonging to the $\mathscr{R}$-class containing r will be denoted by $[r]$.

Theorem 3.10. *Let S and $\bar{S}$ be inverse semigroups, E and $\bar{E}$ the semilattices of their idempotents, respectively. Assume that mappings $m_1\colon S^0\to\bar{S}^{\bar{0}}$, $m_2\colon S^0\to\bar{\Sigma}^{\bar{\omega}}$, $\mu_1^i\colon\Sigma^\omega\to\bar{S}^{\bar{0}}$, $\mu_2^i\colon\Sigma^\omega\to\bar{\Sigma}^{\bar{\omega}}$ $(i\in E_0)$ are given such that they have the following properties.*

(a) *For each i in E_0 we have $om_1=\omega\mu_1^i$, $om_2=\omega\mu_2^i$, $\bar{o}m_1^{-1}=\bar{\omega}m_2^{-1}$ and $\bar{o}(\mu_1^i)^{-1}=$ $=\bar{\omega}(\mu_2^i)^{-1}$.*

(b) *For any r, s, ϱ, σ in S and Σ, respectively, and i in E_0 we have*

$$(rs)m_1=rm_1\cdot sm_1,$$

$$(\varrho\sigma)\mu_1^i=\varrho\mu_1^i\cdot\sigma\mu_1^i.$$

if $(rs)m_1 \neq \bar{o}$ *and* $(\varrho\sigma)\mu_1^i \neq \bar{o}$. *Further we have*

$$(rs)m_2 = rm_2(\bar{f}_{[rm_1],[rm_1\cdot sm_1]}\bar{\varkappa}) \cdot sm_2((\bar{g}_{rm_1,[sm_1]}\bar{f}^{-1}_{[rm_1\cdot sm_1],rm_1[sm_1]})\bar{\varkappa}),$$

$$(\varrho\sigma)\mu_2^i = \varrho\mu_2^i(\bar{f}_{[\varrho\mu_1^i],[\varrho\mu_1^i\cdot\sigma\mu_1^i]}\bar{\varkappa}) \cdot \sigma\mu_2((\bar{g}_{\varrho\mu_1^i,[\sigma\mu_1^i]}\bar{f}^{-1}_{[\varrho\mu_1^i\cdot\sigma\mu_1^i],\varrho\mu_1^i[\sigma\mu_1^i]})\bar{\varkappa})$$

if $(rs)m_1 \neq \bar{o}$ *or* $rm_1 \cdot sm_1 \neq \bar{o}$ *and* $(\varrho\sigma)\mu_1^i \neq \bar{o}$ *or* $\varrho\mu_1^i \cdot \sigma\mu_1^i \neq \bar{o}$.

(c) *For any* r, ϱ *in* S *and* Σ *and* i *in* E_0

$$rm_1 \cdot \varrho\mu_1^i = \varrho((g_{r,i}f^{-1}_{[ri],ri})\varkappa)\mu_1^{[ri]} \cdot (ri)m_1,$$

$$rm_2(\bar{f}_{[rm_1],[rm_1\cdot\varrho\mu_1^i]}\bar{\varkappa}) \cdot \varrho\mu_2^i((\bar{g}_{rm_1,[\varrho\mu_1^i]}\bar{f}^{-1}_{[rm_1\cdot\varrho\mu_1^i],rm_1[\varrho\mu_1^i]})\bar{\varkappa}) =$$

$$= \varrho'\mu_2^{[ri]}(\bar{f}_{[\varrho'\mu_1^{[ri]}],[\varrho'\mu_1^{[ri]}\cdot(ri)m_1]}\bar{\varkappa}) \cdot (ri)m_2((\bar{g}_{\varrho'\mu_1^{[ri]},[(ri)m_1]}\bar{f}^{-1}_{[\varrho'\mu_1^{[ri]}\cdot(ri)m_1],\varrho'\mu_1^{[ri]}[(ri)m_1]})\bar{\varkappa})$$

if all the four elements are different from zero. Moreover, the left hand sides of the equalities differ from zero if and only if the right hand sides do so, too. In the second equality $\varrho' = \varrho((g_{r,i}f^{-1}_{[ri],ri})\varkappa)$. *Define a mapping* $\Phi\colon \mathbf{S} \to \bar{\mathbf{S}}$ *in the following way. Let*

$$(r, \varrho)\Phi = (\sigma\mu_1^{[r]} \cdot rm_1, (\sigma\mu_2^{[r]}(\bar{f}_{[\sigma\mu_1^{[r]}],[\sigma\mu_1^{[r]}\cdot rm_1]}\bar{\varkappa}) \cdot$$

$$\cdot rm_2((\bar{g}_{\sigma\mu_1^{[r]},[rm_1]}\bar{f}^{-1}_{[\sigma\mu_1^{[r]}\cdot rm_1],\sigma\mu_1^{[r]}[rm_1]})\bar{\varkappa}))(\bar{f}_{[\sigma\mu_1^{[r]}\cdot rm_1],\sigma\mu_1^{[r]}\cdot rm_1}\bar{\varkappa})),$$

where $\sigma = \varrho(f^{-1}_{[r],r}\varkappa)$, *whenever both components are different from zero,* $(r, \varrho)\Phi = \bar{0}$ *otherwise. Further, put* $0\Phi = (om_1, om_2)$ *if* $om_1 \neq \bar{o}$ *and* $0\varphi = \bar{0}$ *otherwise. Then*

(i) *the mapping* Φ *is a homomorphism,*

(ii) *if the semigroups* Σ *and* $\bar{\Sigma}$ *have identity elements preserved by all endomorphisms* $f_{r,s}\varkappa$, $g_{r,s}\varkappa$ *and* $\bar{f}_{\bar{r},\bar{s}}\bar{\varkappa}$, $\bar{g}_{\bar{r},\bar{s}}\bar{\varkappa}$, *respectively, then all homomorphisms of* $\mathbf{S}$ *into* $\bar{\mathbf{S}}$ *are of this form.*

Proof. (i) can be verified by computation. If S is an inverse semigroup and Σ has an identity preserved by the endomorphisms $f_{r,s}\varkappa$ and $g_{r,s}\varkappa$, then all the nonzero elements (r, ϱ) of S can be uniquely written in the form $([r], \varrho')(r, \varepsilon)$ with $\varrho' = \varrho(f^{-1}_{[r],r}\varkappa)$ because $f_{[r],r}$ is in the group of units of C by Lemmas 2.2 and 2.1 and hence $f^{-1}_{[r],r}\varkappa$ is an automorphism. Since the proof of (ii) is similar to that of Theorem 3.9 (iii), it is left to the reader. We note only that by (1)—(3) we have

$$f_{[r],r}f_{r,s} = f_{[r],rs} = f_{[r],[rs]}f_{[rs],rs}.$$

Since $[rs] = rss^{-1}r^{-1}$, we have $[rs]r[s] = r[s]$ and

$$f_{[rs],r[s]}f_{[rs]r[s],s}f_{rs,s^{-1}r^{-1}} = f_{[rs],rs}f_{rs,s^{-1}r^{-1}} = f_{[rs],[rs]} = 1.$$

Hence $f_{[rs],r[s]}$ is in the group of units of C and we have

$$f_{[s],s}g_{r,s} = g_{r,[s]}f_{r[s],s} = g_{r,[s]}f^{-1}_{[rs],r[s]}f_{[rs],rs}.$$

Let S and Σ be two semigroups and consider two 0-extensions $\mathbf{S} = \mathscr{S}^0(S, \Sigma, C, f, g, \varkappa)$ and $\mathbf{S}' = \mathscr{S}^0(S, \Sigma, \bar{C}, \bar{f}, \bar{g}, \bar{\varkappa})$ of Σ by S.

Definition. The 0-extension $\mathbf{S}$ is said to be equivalent to $\bar{\mathbf{S}}'$ if for every s in S_0 there exists an automorphism ψ_s of Σ such that the mapping $\Psi: \mathbf{S} \to \mathbf{S}'$ defined by $0\Psi = \bar{0}$, $(s, \sigma)\Psi = (s, \sigma\psi_s)$ is an isomorphism.

This definition clearly determines an equivalence relation on the class of 0-extensions of Σ by S.

In the next theorems we investigate the equivalent 0-extensions.

Before formulating the first one we note that if the images of the functions $f\varkappa$ and $g\varkappa$ are contained in the group of automorphisms of Σ, then Theorem 2.3 applies to them provided S has an identity. The homomorphism used in this theorem will be denoted by $\varphi^\varkappa$.

Theorem 3.11. *Let S be a semigroup with identity and Σ a reductive semigroup. Assume that the images of $f\varkappa$ and $g\varkappa$ are in the group of automorphisms of Σ. The 0-extension $\mathbf{S}$ is equivalent to $\mathbf{S}'$ if and only if the images $\bar{f}\bar{\varkappa}$ and $\bar{g}\bar{\varkappa}$ are included in the group of automorphisms of Σ and $\bar{\varphi}^{\bar{\varkappa}} = \varphi^\varkappa \mathfrak{A}$ for some inner automorphism $\mathfrak{A}$ of the group of automorphisms of Σ.*

Proof. Suppose $\mathbf{S}$ and $\mathbf{S}'$ are equivalent. This means that for any r, s in S with $rs \neq o$ and ϱ, σ in Σ we have

$$(r, \varrho\psi_r)(s, \sigma\psi_s) = (rs, (\varrho(f_{r,s}\varkappa)\sigma(g_{r,s}\varkappa))\psi_{rs})$$

provided they are nonzero or else both of them are zero. In both cases we have

$$\varrho\psi_r(\bar{f}_{r,s}\bar{\varkappa})\sigma\psi_s(\bar{g}_{r,s}\bar{\varkappa}) = \varrho(f_{r,s}\varkappa)\psi_{rs}\sigma(g_{r,s}\varkappa)\psi_{rs}.$$

For $r=e$ this yields

$$\varrho\psi_e(\bar{f}_{e,s}\bar{\varkappa})\sigma\psi_s = \varrho(f_{e,s}\varkappa)\psi_s\sigma\psi_s.$$

Σ is reductive and ψ_s is an automorphism. Hence for any $s \neq o$ we have

$$\psi_e(\bar{f}_{e,s}\bar{\varkappa}) = (f_{e,s}\varkappa)\psi_s.$$

Dually, one can see that

$$g_{s,}\psi_{e,}(\bar{g}_{se}\bar{\varkappa}) = ({}_e\varkappa)\psi_s$$

for $s \neq o$. From these equalities it follows that $\bar{f}_{e,s}\bar{\varkappa}$ and $\bar{g}_{s,e}\bar{\varkappa}$ are automorphisms for all $s \neq o$, which implies by (1) and (3) that so are $\bar{f}_{r,s}\bar{\varkappa}$ and $\bar{g}_{r,s}\bar{\varkappa}$, where $rs \neq o$. Moreover, we have

$$s\bar{\varphi}^{\bar{\varkappa}} = \psi_e^{-1}(s\varphi^\varkappa)\psi_e,$$

which completes the proof of the only if part. Conversely, suppose that the conditions of the theorem are satisfied. Denote the automorphism of Σ inducing $\mathfrak{A}$ by ψ. Define ψ_s by

$$\psi_s = (f_{e,s}\varkappa)^{-1}\psi(\bar{f}_{e,s}\bar{\varkappa}).$$

Making use of the equalities (22) and the fact that $s\bar{\varphi}^{\bar{\varkappa}} = \psi^{-1}(s\varphi^\varkappa)\psi$ holds for every

$s \neq o$, one can obtain by computation

$$\psi_r(\bar{f}_{r,s}\bar{\varkappa}) = (f_{r,s}\varkappa)\psi_{rs}, \quad \psi_s(g_{r,s}\bar{\varkappa}) = (g_{r,s}\varkappa)\psi_{rs},$$

whenever $rs \neq o$.

Theorem 3.12. *Consider a regular semigroup S and a reductive semigroup Σ. The 0-extensions* S *and* S′ *are equivalent if and only if there exist automorphisms ψ_i of Σ indexed by the idempotents of S_0 such that for every pair i, i' of idempotents $i'i=i'$ and $i'i=i'$ imply*

$$(23) \qquad (f_{i,i'}\varkappa)\psi_{i'} = \psi_i(\bar{f}_{i,i'}\bar{\varkappa}) \quad \text{and} \quad (g_{i',i}\varkappa)\psi_{i'} = \psi_i(\bar{g}_{i',i}\bar{\varkappa}),$$

respectively, and $i\mathscr{D}i'$ implies that

$$(24) \qquad (f_{i,r}^{-1}\varkappa)\psi_i(\bar{f}_{i,r}\bar{\varkappa}) = (g_{r,i'}^{-1}\varkappa)\psi_{i'}(\bar{g}_{r,i'}\bar{\varkappa})$$

for any r such that $i\mathscr{R}r\mathscr{L}i'$.

Proof. In the proof of the last theorem we saw that the 0-extensions are equivalent if and only if the equality

$$\varrho\psi_r(\bar{f}_{r,s}\bar{\varkappa})\sigma\psi_s(\bar{g}_{r,s}\bar{\varkappa}) = \varrho(f_{r,s}\varkappa)\psi_{rs}\sigma(g_{r,s}\varkappa)\psi_{rs}$$

holds for any r, s such that $rs \neq o$ and for arbitrary ϱ, σ in Σ. If $ir=r$, this implies by Lemma 2.1 that

$$\varrho\psi_i(\bar{f}_{i,r}\bar{\varkappa}) \cdot \sigma\psi_r = \varrho(f_{i,r}\varkappa)\psi_r \cdot \sigma\psi_r.$$

Since Σ is reductive we have

$$(f_{i,r}\varkappa)\psi_r = \psi_i(\bar{f}_{i,r}\bar{\varkappa}).$$

Dually, we can obtain that

$$(g_{r,i'}\varkappa)\psi_r = \psi_{i'}(\bar{g}_{r,i}\bar{\varkappa})$$

if $ri'=r$. In particular, this yields (23) if r is an idempotent. If $i\mathscr{R}r\mathscr{L}i'$, then, as it has been verified above, $f_{i,r}$ and $g_{r,i'}$ belong to the group of units of C. Hence $f_{i,r}\varkappa$ and $g_{r,i'}\varkappa$ are automorphisms and it follows from the foregoing that

$$(25) \qquad \psi_r = (f_{i,r}^{-1}\varkappa)\psi_i(\bar{f}_{i,r}\bar{\varkappa}) = (g_{r,i'}^{-1}\varkappa)\psi_{i'}(\bar{g}_{r,i'}\bar{\varkappa}).$$

Conversely, assume that the conditions of the theorem hold for some automorphisms ψ_i. Let r be an element of S_0 and i, i' idempotents such that $i\mathscr{R}r\mathscr{L}i'$. Define ψ_r by (25). Obviously, ψ_r is well defined. If j is an idempotent such that $ji=i$, then applying (1) and (23) we have

$$(f_{j,r}\varkappa)\psi_r = ((f_{j,r}f_{i,r}^{-1})\varkappa)\psi_i(\bar{f}_{i,r}\bar{\varkappa}) = (f_{j,i}\varkappa)\psi_i(\bar{f}_{i,r}\bar{\varkappa}) =$$
$$= \psi_j((\bar{f}_{j,i}\bar{f}_{i,r})\bar{\varkappa}) = \psi_j(\bar{f}_{j,r}\bar{\varkappa}).$$

Hence if r, s are elements of S such that $rs \neq o$, then denoting an idempotent in the $\mathscr{R}$-class of r by i, we have

$$\psi_r(\bar{f}_{r,s}\bar{\varkappa}) = (f_{i,r}^{-1}\varkappa)\psi_i((\bar{f}_{i,r}\bar{f}_{r,s})\bar{\varkappa}) = (f_{i,r}^{-1}\varkappa)\psi_i(\bar{f}_{i,rs}\bar{\varkappa}) =$$
$$= ((f_{i,r}^{-1}f_{i,rs})\varkappa)\psi_{rs} = (f_{r,s}\varkappa)\psi_{rs}.$$

Similarly, one can show that $\psi_s(\bar{g}_{r,s}\bar{\varkappa})=(g_{r,s}\varkappa)\psi_{rs}$. This completes the proof.

Note that if S is (0-)$\mathcal{D}$-simple and C has a trivial group of units, then (24) implies that all ψ_i are equal. Conversely, if all ψ_i coincide, then, denoting ψ_i ($i\in E_0$) by ψ, (23) implies

$$\bar{f}_{i,r}\bar{\varkappa}=\psi^{-1}(f_{i,r}\varkappa)\psi \quad \text{and} \quad g_{r,j}\bar{\varkappa}=\psi^{-1}(g_{r,j}\varkappa)\psi,$$

whenever $ir=r$ and $rj=r$, respectively. Hence (24) holds trivially. Thus we have proved the following

Corollary 3.13. *If S is a (0-)$\mathcal{D}$-simple regular semigroup and Σ is reductive, then the* 0*-extensions* $\mathbf{S}$ *and* $\mathbf{S}'$ *are equivalent if and only if there exists an automorphism ψ of Σ such that for all idempotents i, i' the equalities*

$$\bar{f}_{i,i'}\bar{\varkappa}=\psi^{-1}(f_{i,i'}\varkappa)\psi \quad \text{and} \quad \bar{g}_{i',i}\bar{\varkappa}=\psi^{-1}(g_{i',i}\varkappa)\psi$$

are implied by $ii'=i'$ and $i'i=i'$, respectively.

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