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Change of the sum of digits by multiplication

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1. Introduction

Let N be a natural number and set $\mathscr{A}_N=\{0, 1, \ldots, 2^N-1\}$. Every $n\in\mathscr{A}_N$ can be written in the form

$$(1.1)\qquad n = \sum_{i=0}^{N-1} \varepsilon_i \cdot 2^i,$$

where $\varepsilon_i=0$ or 1 ($i=0, 1, \ldots, N-1$). This representation is unique. Let $\alpha(n)$ denote the sum of the digits of n, i.e.

$$(1.2)\qquad \alpha(n) = \sum_{i=0}^{N-1} \varepsilon_i.$$

Let $M_N(x)$ denote the number of those $n\in\mathscr{A}_N$ for which

$$\frac{\alpha(n)-N/2}{\sqrt{N}/2} < x.$$

Using the central limit theorem of probability theory in the simplest form, we have that

$$2^{-N}M_N(x)\to\Phi(x)\quad(N\to\infty)$$

for every real x, where

$$(1.3)\qquad \Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,dt.$$

Furthermore, we have

$$(1.4)\qquad 2^{-N}\sum_{n\in A_N}\left(\alpha(n)-\frac{N}{2}\right)^2 = \frac{N}{4}.$$

It seems to be interesting to consider the distribution of the difference $\alpha(hn)-\alpha(n)$, $n\in\mathscr{A}_N$ for fixed h. This question is trivial for $h=2$, since $\alpha(2n)-\alpha(n)=0$.

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We shall be dealing with the case $h=3$. Let

$$\Delta(n) = \alpha(3n)-\alpha(n). \tag{1.5}$$

Our main result is the following

Theorem 1. *Let $K_N(x)$ denote the number of $n\in\mathcal{A}_N$ for which*

$$\frac{\sqrt{3}\,\Delta(n)}{\sqrt{N}} < x. \tag{1.6}$$

Then, for every real number x, we have

$$2^{-N}K_N(x) \to \Phi(x) \quad (N\to\infty). \tag{1.7}$$

We can deduce a more precise result, with a remainder term, but now we do not try to give the best one.

This and similar results may have some importance in the probabilistic treatment of rounding errors in numerical analysis.

2. The splitting of the binary representation of integers

We define the sets $\mathfrak{M}_k$ as follows. Let $\mathfrak{M}_0=\{0\}$. The sets $\mathfrak{M}_k$ contain those integers m_k for which $2^{k-1}\leqq m_k<2^k$ and the binary representation of which does not contain two consecutive zeros. Let m_k denote a general element of $\mathfrak{M}_k$, and A_k the number of its elements. It is obvious that $A_0=1$, $A_1=1$, $A_2=2$. We shall show that

$$A_k = A_{k-1}+A_{k-2} \quad (k\geqq 2). \tag{2.1}$$

Indeed, m_k can be written as

$$m_k = 2^{k-1}+m_{k-1} \quad \text{or} \quad m_k = 2^{k-1}+m_{k-2},$$

whence (2.1) immediately follows.

Let

$$F(z) = \sum_{k=0}^{\infty} A_k z^{k+2}. \tag{2.2}$$

By an easy calculation we get

$$F(z) = \frac{z^2}{1-z-z^2}. \tag{2.3}$$

Let $\mathfrak{M}=\sum_{k=0}^{\infty}\mathfrak{M}_k$. Assume that $N\geqq 2$. Then for every $n\in A_N$ there exists a unique element $m_{l_1}\in\mathfrak{M}$ for which

$$n = m_{l_1}+2^{l_1+2}u, \quad u\in A_{N-l_1-2}. \tag{2.4}$$

For $N=0$ or 1 we use the representation $n=m_{l_1}$. Repeating this, we get

$$n = m_{l_1}+2^{l_1+2}\{m_{l_2}+2^{l_2+2}\{\dots\{m_{l_{\nu(n)-1}}+2^{l_{\nu(n)-1}+2}\{m_{l_{\nu(n)}}\}\}\dots\}\}. \tag{2.5}$$

So, for every n we order the sequence of elements of $\mathfrak{M}$. It is obvious that

$$l_1+\dots+l_{\nu(n)}+2(\nu(m)-1) = N-1 \quad \text{or} \quad N. \tag{2.6}$$

Furthermore, from (2.4) we have

$$3n = 3m_{l_1}+2^{l_1+2}\cdot(3u), \quad 3m_{l_1} < 2^{l_1+2},$$

and so

$$\Delta(n) = \Delta(m_{l_1})+\Delta(n). \tag{2.7}$$

Hence we have

$$\Delta(n) = \sum_{j=1}^{\nu(n)} \Delta(m_{l_j}). \tag{2.8}$$

3. The distribution of the number of $\mathfrak{M}$-components

Now we consider the number of those integers $n\in\mathcal{A}_N$ for which $\nu(n)=H$. For the sake of brevity we use the notation

$$t_j = l_j+2. \tag{3.1}$$

So we write (2.6) in the form

$$t_1+\dots+t_H = N+2-\delta, \quad \delta = 0 \quad \text{or} \quad 1. \tag{3.2}$$

Let $2^N\beta_H(\delta)$ denote the number of $n\in\mathcal{A}_N$ for which $\nu(n)=H$ and (3.2) holds. Since

$$\int_{-1/2}^{1/2} e^{2\pi\theta}d\theta = \begin{cases} 1 & \text{if} \quad n=0 \\ 0 & \text{if} \quad n\neq 0, \quad \text{integer}, \end{cases}$$

we get

$$\beta_H(\delta) = 2^{2-\delta} \int_{-1/2}^{1/2} F\left(\frac{z}{2}\right)^H \cdot z^{-(N+2-\delta)}d\theta, \quad z = e^{2\pi i\theta}. \tag{3.3}$$

First we integrate in the neighbourhood of $\theta=0$. By taking $\omega=z-1$, $|\omega|\leqq\frac{1}{2}$, we have

$$\ln F\left(\frac{z}{2}\right) = \ln\frac{z^2}{4-2z-z^2} = \ln\frac{\omega^2+2\omega+1}{1-4\omega-\omega^2} = 6\omega+8\omega^2+O(\omega^3),$$

$$\ln z = \ln(1+\omega) = \omega-\frac{\omega^2}{2}+O(\omega^3).$$

Let

$$A_H = 6H-(N+2-\delta), \quad B_H = 8H+\frac{N+2-\delta}{2}. \tag{3.4}$$

Then

$$g(z) \stackrel{\text{def}}{=} F\left(\frac{z}{2}\right)^H z^{-(N+2-\delta)} = \exp\left(A_H\omega+B_H\omega^2+O(N\omega^3)\right).$$

Observing that $\omega=2\pi i\theta-4\pi^2\theta^2+0(\theta^3)$, $\omega^2=-4\pi^2\theta^2+O(\theta^3)$, we get

$$g(z) = \exp\left(2\pi i\cdot A_H\theta-4\pi^2(A_H+\beta_H)\theta^2\right)\exp\left(O(N\theta^3)\right). \tag{3.5}$$

Let

$$\mathscr{I}_1 = \int_{-\Lambda}^{\Lambda} g(z)d\theta, \quad \mathscr{I}_2 = \int_{\Lambda<|\theta|\leq\frac{1}{2}} g(z)d\theta, \tag{3.6}$$

where we choose Λ so that $N\Lambda^2\to\infty$, $N\Lambda^3\to 0$.

From (3.5) we get

$$\mathscr{I}_1 = \mathscr{I}_3+O(K), \tag{3.7}$$

where

$$\mathscr{I}_3 = \int_{-\Lambda}^{\Lambda} \exp\left(2\pi i A_H\theta-4\pi^2(A_H+B_H)\theta^2\right)d\theta, \tag{3.8}$$

$$K = \int_{-\Lambda}^{\Lambda} \exp\left(-4\pi^2(A_H+B_H\theta^2\right)N\theta^3 d\theta. \tag{3.9}$$

In what follows we assume that $|A_H|\ll N^{2/3}$. Then $H=\frac{N}{6}+O(N^{2/3})$, and so

$$B_H = \frac{11}{6}N+O(N^{2/3}).$$

Hence for $\mathscr{K}$ we easily get that

$$\mathscr{K}\ll\frac{N}{B_H^2}\ll\frac{1}{N}. \tag{3.10}$$

To estimate $\mathscr{I}_3$, we use the following

Lemma 1. *Let*

$$J(A, B, \Lambda) = \int_{-\Lambda}^{\Lambda} \exp(iA\varphi-B\varphi^2)d\varphi,$$

A, B, Λ real numbers, $B>0$, $\Lambda>0$. Then

$$J(A, B, \Lambda) = \exp\left(-\frac{A^2}{4B}\right)\cdot\sqrt{\frac{\pi}{B}}+O\left(B^{-3/2}|A|\exp(-\Lambda^2 B)\right)+$$

$$+O\left(B^{-1/2}\exp\left(-\frac{A^2}{4B}-\Lambda^2 B\right)\right).$$

Proof.

$$J(A, B, \Lambda) = B^{-1/2} \int_{-\Lambda\sqrt{B}}^{\Lambda\sqrt{B}} \exp\left(i\frac{A}{\sqrt{B}}\tau - \tau^2\right) d\tau =$$

$$= \frac{\exp\left(-\frac{A^2}{4B}\right)}{i\sqrt{B}} \int_{-\Lambda\sqrt{B}}^{\Lambda\sqrt{B}} \exp\left(\left(\frac{A}{2\sqrt{B}} + i\tau\right)^2\right) d\left(\tau i + \frac{A}{2\sqrt{B}}\right) = \frac{\exp\left(-\frac{A^2}{4B}\right)}{i\sqrt{B}} \int_L e^{z^2} dz,$$

where L denotes the segment $\left[\frac{A}{2\sqrt{B}} - \Lambda\sqrt{B}, \frac{A}{2\sqrt{B}} + \Lambda\sqrt{B}\right]$. Transforming the integral to the imaginary axis, by an easy estimation we get the desired result.

Applying this Lemma by choosing $A = 2\pi A_H$, $B = 4\pi(A_H + B_H)$, $\Lambda = N^{-1/3-\varepsilon}$ $(\varepsilon > 0)$, we get

$$\mathscr{I}_1 = \mathscr{I}_3 + O(\mathscr{K}) = \frac{\exp\left(-\frac{A_H^2}{4(A_H+B_H)}\right)}{2\sqrt{\pi}\sqrt{A_H+B_H}} + O(1/N). \tag{3.11}$$

Now we estimate $\mathscr{I}_2$. By taking $Y = \cos 2\pi\theta$, $Y = 1 - t$, we get

$$|4 - 2z - z^2|^2 = 1 + 44t - 16t^2.$$

So in $\Lambda \leqq |\theta| \leqq \frac{1}{2}$ we get

$$|4 - 2z - z^2|^{-H} \leqq \left(1 + 44\pi^2\Lambda^2(1-\varepsilon)\right)^{-H/2} \leqq \exp(-c_1 H\Lambda^2),$$

$c_1 > 0$ constant. Consequently

$$\mathscr{I}_2 \ll \exp(-c_1 N\Lambda^2). \tag{3.12}$$

Finally, taking into account (3.3), (3.6), (3.7), (3.10), (3.11), (3.12), we get

$$\beta_H \stackrel{\text{def}}{=} \beta_H(0) + \beta_H(1) = \frac{1}{\sigma\sqrt{2\pi N}} \exp\left(-\frac{(H - N/6)^2}{2\sigma^2 N}\right) + O(1/N),$$

where $\sigma = \frac{1}{6} \cdot \sqrt{\frac{11}{3}}$.

So we have proved:

Theorem 2. *Let $2^N \beta_H$ denote the number of those $n \in \mathscr{A}_N$ for which $v(n) = H$. Then*

$$\beta_H = \frac{1}{\sigma\sqrt{2\pi N}} \exp\left(-\frac{(H - N/6)^2}{2\sigma^2 N}\right) + O(1/N)$$

where $\sigma = \frac{1}{6} \cdot \sqrt{\frac{11}{3}}$, uniformly in H.

4. The function $H(\tau)$

Let

$$S_l(\tau) = \sum_{m_l \in \mathfrak{M}_l} e^{i\tau\Delta(m_l)}, \quad H(\tau) + \sum_{l=0}^{\infty} 2^{-l-2} S_l(\tau). \tag{4.1}$$

It is obvious that $S_0(\tau)=1$, $S_1(\tau)=e^{i\tau}$, $S_2(\tau)=1+e^{i\tau}$. Let $\mathfrak{M}_k=\mathfrak{M}_k^1 \cup \mathfrak{M}_k^0$, where $\mathfrak{M}_k^0$ contains the even numbers of $\mathfrak{M}_k$, and $\mathfrak{M}_k^1$ the odd numbers. Let

$$S_k^{(i)}(\tau) = \sum_{m_k \in \mathfrak{M}_k^i} e^{i\tau\Delta(m_k)}.$$

First of all we observe that

$$S_k^{(0)}(\tau) = S_{k-1}^{(1)}(\tau)$$

for $k \geqq 2$. Then

$$H(\tau) = \frac{1}{4} + \frac{3e^{i\tau}}{16} + \frac{3}{8} \cdot \sum_{l=2}^{\infty} \frac{S_l^{(1)}(\tau)}{2^l}. \tag{4.2}$$

Now we compute $S_k^{(1)}(\tau)$. The general form of the binary representation of n_A is the following one:

$$n = \overbrace{\boxed{}}^{A}\;\underbrace{\boxed{0\,|\,1\,|\;\dots\;|\,0\,|\,1}}_{2h \text{ places}},$$

where A is one of the following types:

1) $A = \boxed{1}$

2) $A = \boxed{1\,|\,1}$

3) $A = \boxed{1\,|\qquad\qquad|\,1\,|\,1}$

Case 1) holds for odd k only. If $k=2t+1$ and $m_k=101\dots01$, then, obviously

$$\Delta(101\dots01) = t+1. \tag{4.3}$$

In the other cases we say that $m_k \in \mathfrak{M}_k^1$ is of $\mathscr{B}_{h,r}$ type if in A there exist exactly r zeros. In case 2) $k-2h=2$ (k even, $r=0$). In case 3) $k-2h \geqq 3$. It is easy to see that the number of elements of type $\mathscr{B}_{h,r}$ is

$$\binom{k-2h-r-2}{r}.$$

We observe that for $m_k \in \mathscr{B}_{h,r}$, $\Delta(m_k)=h-r$. We consider $3m_k$ as the sum of $2m_k$ and m_k. See the following figure.

m_k				1	0...	...	0	... 1 1	0 1 0 1 ... 0 1
$2m_k$			1				1	... 1 0	1 0 1 0 ... 1 0
$3m_k$								1	1 .... 1 1
$\Delta(m_k)$			0		−1 ...	...		−1	$+h$

Now we take

$$\sum_{l=2}^{\infty} 2^{-l} S_l^{(1)}(\tau) = \sum_{t=1}^{\infty} 2^{-2t} S_{2t}^{(1)}(\tau) + \sum_{t=1}^{\infty} 2^{-2t-1} S_{2t+1}^{(1)}(\tau) = \Sigma_A + \Sigma_B .$$

We have

$$\Sigma_A =$$

$$= \sum_{t=1}^{\infty} 2^{-2t} \cdot \sum_{h,r} \binom{2t-2h-r-2}{r} e^{i\tau(h-r)} = \sum_{r=0}^{\infty} \sum_{h=0}^{\infty} e^{i\tau(h-r)} \sum_{v=1}^{\infty} 2^{-2(h+r+v)} \cdot \binom{r+2v-2}{r} =$$

$$= \left(\sum_{h=0}^{\infty} 2^{-2h} e^{i\tau h}\right) \cdot \frac{1}{4} \left\{ \sum_{r=0}^{\infty} 2^{-2r} e^{-i\tau r} \cdot \sum_{v=0}^{\infty} \binom{r+2v}{r} \cdot 2^{-2v} \right\} = \Sigma_0 \cdot \frac{1}{4} \cdot \{\Sigma_1\}.$$

We observe that

$$\Sigma_1 = \sum_{s=0}^{\infty} \sum_{2v \leqq r} \binom{s}{2v} \left(\frac{1}{2}\right)^{2v} \cdot \left(\frac{e^{-i\tau}}{2^2}\right)^{r} = \sum_{s=0}^{\infty} \frac{1}{2} \left\{ \left(\frac{1}{2} + \frac{e^{-i\tau}}{4}\right)^{s} + \left(-\frac{1}{2} + \frac{e^{-i\tau}}{4}\right)^{s} \right\} =$$

$$= \frac{2}{2-e^{-i\tau}} + \frac{2}{3-e^{-i\tau}} .$$

Furthermore,

$$\Sigma_0 = \frac{1}{1-e^{i\tau}/4} .$$

So we have

$$\Sigma_A = \frac{1}{4-e^{i\tau}} \left\{ \frac{2}{2-e^{-i\tau}} + \frac{2}{3-e^{-i\tau}} \right\} .$$

In the sum Σ_B the extraordinary case (4.3) occurs. We get

$$\Sigma_B = \Sigma_E + \Sigma_C ,$$

where

$$\Sigma_E = \sum_{t=0}^{\infty} 2^{-2t-1} \cdot e^{i\tau(t+1)} = \frac{e^{2i\tau}}{2(4-e^{i\tau})} , \quad \Sigma_C = \sum_{h,r} e^{i\tau(h-r)}$$

We hawe, similarly as for Σ_A,

$$\Sigma_C = \frac{1}{4-e^{i\tau}} \left\{ \frac{2}{2-e^{-i\tau}} - \frac{2}{3-e^{-i\tau}} \right\} .$$

Summing up, we have

$$\sum_{l=2}^{\infty} 2^{-l} S_l^{(1)}(\tau) = \frac{1}{4-e^{i\tau}}\left\{\frac{e^{2i\tau}}{2}+\frac{4}{2-e^{-i\tau}}\right\}.$$

So we have from (4.2) that

$$H(\tau) = \frac{1}{4}+\frac{3}{16}\cdot e^{i\tau}+\frac{3}{16}\cdot\frac{e^{2i\tau}}{4-e^{i\tau}}+\frac{3}{2}\cdot\frac{1}{(2-e^{-i\tau})(4-e^{i\tau})}.$$

Differentiating two times we can deduce that $H(0)=1$, $H'(0)=0$, $H''(0)=-2$.

5. Proof of Theorem 1

From (2.1) we have that

(5.1) $$A_k = c_1\theta_1^k + c_2\theta_2^k,$$

where

(5.2) $$\theta_1 = \frac{1+\sqrt{5}}{2},\quad \theta_2 = \frac{1-\sqrt{5}}{2},\quad c_1 = \frac{5+\sqrt{5}}{10},\quad c_2 = \frac{5-\sqrt{5}}{10}.$$

Lemma 2. *Let $C(N, l_0)$ be the number of those $n \in A_N$ the longest component of which is greater than l_0. Then*

(5.3) $$2^{-N} C(N, l_0) \ll N\cdot\left(\frac{\theta_1}{2}\right)^{l_0}.$$

Proof. Assume that the longest $\mathfrak{M}$ component of n is $l\,(\geqq l_0)$. Then for a suitable integer t we have $n = h+2^{t+2}u+2^{t+l+4}v$, where

(5.4) $$h < 2^t,\quad v < 2^{N-t-l-4},\quad u \in \mathfrak{M}_l.$$

The number of n satisfying (5.4) is $\ll A_l\cdot 2^t\cdot 2^{N-t-l-4}$. Summing up for t, and l, we have

$$2^{-N}C(N, l_0) \leqq N\sum_{l\geqq l_0} 2^{-l-2}A_l \ll N\left(\frac{\theta_1}{2}\right)^{l_0}.$$

Lemma 3. *Let $H_1 = \frac{N}{6} - \varrho(N)\sqrt{N}$, where $\varrho(N)\to\infty\ (N\to\infty)$, and*

(5.5) $$S = \sum_{t_1+\dots+t_{H_1}\geqq N}\ \prod_{j=1}^{H_1}\frac{A_{t_j-2}}{2^{t_j}}.$$

Then

(5.6) $$s \ll e^{-\sqrt{\varrho(N)}}.$$

Proof. First we observe that for $c>1$

$$S \leqq c^{-N} \sum_{t_1,\dots,t_{H_1}} \prod_{j=1}^{H_1} A_{t_j-2}\cdot\left(\frac{c}{2}\right)^{t_j} = c^{-N} F\left(\frac{c}{2}\right)^{H_1}.$$

We take $c=1+\delta$, $\delta\to 0$. Then, by repeating the estimation that we used for the deduction of (3.5), we get

$$c^{-N}F^{H_1}\left(\frac{c}{2}\right) = \exp\left[(6H_1-N)\delta+(8H_1+N/2)\delta^2+O(N\delta^3)\right].$$

By choosing $\delta=(N\varrho(N))^{-1/2}$, we get (5.6).

Let

$$\varphi(\tau) = 2^{-N}\sum_{n\in\mathscr{A}_N} e^{i\tau\Delta(n)}, \tag{5.7}$$

$$A(n, H_1) = \sum_{j=1}^{H_1}\Delta(m_{l_j}), \tag{5.8}$$

$$B(n, H_1) = \Delta(n)-A(n, H_1). \tag{5.9}$$

We take $A(n, H_1)=A(n, \nu(n))$, when $\nu(n)<H_1$. Let

$$\varphi_0(\tau) = 2^{-N}\sum_{n\in\mathscr{A}_N} e^{i\tau A(n,H_1)}. \tag{5.10}$$

First we consider $\varphi_0(\tau)$. It is obvious that

$$\varphi_0(\tau) = \sum_{t_1+\dots+t_{H_1}\leqq N}\prod_{j=1}^{H_1}\frac{S_{l_j}(\tau)}{2^{t_j}}. \tag{5.11}$$

From Lemma 3 we get

$$\varphi_0(\tau) = H(\tau)^{H_1}+O(S) = H(\tau)^{H_1}+O\left(e^{-\sqrt{\varrho(N)}}\right). \tag{5.12}$$

Now we estimate the difference $\varphi(\tau)-\varphi_0(\tau)$. Let $\mathscr{A}$ denote the set of those integers $n\in\mathscr{A}_N$ for which

$$t_1+t_2+\dots+t_{H_1}\leqq N-\varrho(N)(\log N)^2\sqrt{N}.$$

Let $\mathscr{B}=\mathscr{A}_N\setminus\mathscr{A}$. We show that $\mathscr{A}$ has at most $O(2^N/N)$ elements. From Theorem 2 it follows easily that the number of those $n\in\mathscr{A}_N$ for which

$$\nu(n)\geqq\frac{N}{6}+\varrho(N)\sqrt{N}$$

is $O(2^N/N)$. For the remaining elements of $\mathscr{A}$ we get

$$\varrho(N)(\log N)^2\sqrt{N}\leqq t_{H_1+1}+\dots+t_{\nu(n)}\leqq$$
$$\leqq \max(l_j+2)\cdot(\nu(n)-H_1)\leqq 4(\max l_j)\cdot\varrho(N)\sqrt{N},$$

i.e.

$$\max l_j \geqq \frac{1}{4}(\log N)^2. \tag{5.13}$$

From Lemma 2 we have that the number of $n\in\mathcal{A}_N$ satisfying (5.13) is smaller than $O(2^N/N)$. We have

$$|\varphi_0(\tau)-\varphi(\tau)| \ll 2^{-N}\sum_{n\in\mathcal{A}}1+2^{-N}|\tau|\sum_{n\in\mathcal{B}}|B(n,H_1)| = \tag{5.14}$$

$$= 2^{-N}(\Sigma_1+|\tau|\cdot\Sigma_2).$$

It is obvious that

$$\Sigma_1 \ll 2^N/N.$$

We can write $n\in\mathcal{B}$ in the following form:

$$n = M+2^s l,\quad t_1+\ldots+t_{H_1}=s,\quad l\in\mathcal{A}_{N-s-2},$$

where M has the components $m_{l_1},\ldots,m_{l_{H_1}}$.

Let $\mathcal{D}_M$ denote the set of these elements. Then, by (1.4), applying the Cauchy inequality, we have

$$\sum_{n\in\mathcal{D}_M}|B(n,H_1)| \ll \sum_{j<2^{N-s}}|\Delta(j)| \ll 2^{N-s}\sqrt{N-s}.$$

Observing that

$$N-s \leqq \varrho(N)(\log^2 N)\cdot\sqrt{N}$$

for $n\in\mathcal{B}$, we get that

$$\Sigma_2 \ll 2^N\cdot N^{1/4}\sqrt{\varrho(N)}\cdot\log N.$$

So we get that

$$|\varphi_0(\tau)-\varphi(\tau)| \ll \frac{1}{N}+|\tau|N^{1/4}\sqrt{\varrho(N)}\cdot\log N. \tag{5.15}$$

Consequently,

$$\varphi(\tau) = H(\tau)^{H_1}+O\left(e^{-\sqrt{\varrho(N)}}\right)+O(1/N)+O\,|\tau|\,N^{1/4}\sqrt{\varrho(N)}\cdot\log N).$$

Observing that $H(0)=1$, $H'(1)=0$, $H''(0)=-2$, we get

$$H(\tau) = 1-\tau^2+O(\tau^3) = \exp\left(-\tau^2+O(\tau^3)\right).$$

By taking $\tau=\varkappa/N$, we have

$$\varphi(\varkappa/\sqrt{N}) = \exp(-\varkappa^2/6)+o(1)\quad (N\to\infty)$$

uniformly for every $\varkappa$ in an arbitrary bounded interval. But $\exp(-\varkappa^2/6)$ is the characteristic function of the normal distribution function with zero mean, and variance $1/\sqrt{3}$. Using the well-known theorem of probability theory on the convergence of characteristic functions, we get Theorem 1 immediately.

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