Complements of radicals in the class of hereditarily artinian rings

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1. Introduction

In [4], WIDIGER and WIEGANDT developed a theory of radicals for the class **K** of hereditarily artinian rings, i.e. the class **K** of all artinian rings with artinian Jacobson radical. It is remarked in their paper, that since **K** is not a variety, connections among algebraic properties are different from those in a variety. For instance, in **K** every hereditary radical class is a homomorphically closed semi-simple class, but the converse statement is not true. Other phenomena of this type are considered in this paper. It will be proved that any radical **R** in **K**, which contains **J** (the Jacobson radical) has a uniquely determined complement, which differs from the situation in a ring variety. This complement is a subidempotent radical (see [1], [2]). It is also shown that any hypernilpotent or subidempotent radical in **K** can be obtained as the upper radical, lower radical resp. of a suitable class of simple prime rings. The notation in this paper is that of [4]. For the definitions of radical class, semi-simple class etc. we refer to that paper and to [5].

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2. Hypernilpotent radicals

Let **R** be a radical class in **K**, such that **R** contains all nilpotent rings in **K**. The class of nilpotent rings in **K** coincides with the Jacobson—radical class in **K**, so $\mathbf{R} \supseteq \mathbf{J}$, where **J** is the Jacobson radical. Then any **R**-semi-simple ring is a **J**-semi-simple ring. Since any ring in **K** is artinian, an **R**-semi-simple ring is a **J**-semi-simple

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artinian ring, hence it is a finite direct sum of matrix rings over division rings by the Wedderburn—Artin theorem.

Lemma 1. *$\mathbf{R}$ is a radical class in $\mathbf{K}$, such that $\mathbf{R} \supseteqq \mathbf{J}$. Let $\mathbf{T} = \{R \in \mathbf{K} : R$ is a simple $\mathbf{R}$-semi-simple ring$\}$. Then $\mathbf{R} = \mathscr{U}\mathbf{T}$, the upper radical determined by the class $\mathbf{T}$.*

Proof. Since all rings in $\mathbf{T}$ are $\mathbf{R}$-semi-simple, it is clear that $\mathbf{R} \subseteqq \mathscr{U}\mathbf{T}$. Now if $\mathbf{R} \in \mathscr{U}\mathbf{T}$, then R has no non-zero homomorphic image in $\mathbf{T}$. We claim that R has no non-zero homomorphic image in $\mathscr{S}\mathbf{R}$, the class of $\mathbf{R}$-semi-simple rings. Indeed, if $0 \neq R/I \in \mathscr{S}\mathbf{R}$, then R/I is a finite direct sum of simple rings, which must be $\mathbf{R}$-semi-simple, since $\mathscr{S}\mathbf{R}$ is hereditary. Hence R/I, and also R, can be mapped onto a non-zero ring in $\mathbf{T}$, which is impossible. Therefore, $R \in \mathscr{U}\mathscr{S}\mathbf{R} = \mathbf{R}$ and $\mathscr{U}\mathbf{T} \subseteqq \mathbf{R}$.

Lemma 2. *Let $\mathbf{R}$ and $\mathbf{T}$ be as in Lemma 1. Let $\mathbf{D} = \{R \in \mathbf{K} : R$ is a finite direct sum of rings in $\mathbf{T}\}$. Then $\mathbf{D}$ is the class of $\mathbf{R}$-semi-simple rings. Moreover, $\mathbf{D}$ is a radical class.*

Proof. First we show that $\mathbf{D} = \mathscr{S}\mathbf{R}$, the class of $\mathbf{R}$-semi-simple rings. Since each ring in $\mathbf{T}$ is $\mathbf{R}$-semi-simple, $\mathbf{D} \subseteqq \mathscr{S}\mathbf{R}$. Conversely, if $R \in \mathscr{S}\mathbf{R}$, then R is a finite direct sum of simple rings. Each of these simple rings is in $\mathbf{T}$, hence $R \in \mathbf{D}$, so $\mathscr{S}\mathbf{R} \subseteqq \mathbf{D}$ and $\mathbf{D} = \mathscr{S}\mathbf{R}$. Next we show that $\mathbf{D}$ is a radical class. If $R \in \mathbf{D}$, then $R = S_1 \oplus \dots \dots \oplus S_k$, S_i simple ring in $\mathbf{D}$ $(i = 1, \dots, k)$. A homomorphic image of R has the same form, hence $\mathbf{D}$ is homomorphically closed. Also, if $R/I \in \mathbf{D}$ and $I \in \mathbf{D}$ for some ideal I of R, then I has a unity, hence it is a direct summand of R, say $R = I \oplus J$. Now $R/I \cong J \in \mathbf{D}$, so $I \oplus J \in \mathbf{D}$ or $R \in \mathbf{D}$. Then $\mathbf{D}$ is closed under extensions. This shows that $\mathbf{D}$ satisfies conditions (a) and (b) of theorem 5 ([4]). Condition (c) of that theorem is vacuous, since $Z(p) \notin \mathscr{S}\mathbf{R} = \mathbf{D}$ for a prime number p ($\mathbf{R}$ contains all nilpotent rings). So $\mathbf{D}$ is a radical class ([4]).

Remark 1. Lemma 1 and the first statement of lemma 2 can be proved without any assumption about the radical class $\mathbf{R}$. However, for an arbitrary radical class $\mathbf{R}$, the class $\mathbf{D}$ may fail to be a radical class. This is a consequence of the fact that a homomorphically closed semi-simple class $\mathbf{D}$ in $\mathbf{K}$ need not be a radical class. For a counterexample, see the remark after corollary 6 in [4].

Lemma 3. *Let $\mathbf{R}$, $\mathbf{T}$ and $\mathbf{D}$ be as in Lemmas 1 and 2. Then $\mathbf{D}$ is the complement of $\mathbf{R}$.*

Proof. Let $R \in \mathbf{D} \cap \mathbf{R}$. Then, as a ring of $\mathbf{D}$, R is a finite direct sum of rings in $\mathbf{T}$. On the other hand, $R \in \mathbf{R} = \mathscr{U}\mathbf{T}$ (Lemma 1), so R has no non-zero homomorphic rings in $\mathbf{T}$. Hence $R = (0)$.

Next let $R \in \mathcal{S}\mathbf{D} \cap \mathcal{S}\mathbf{R}$. Then $R \in \mathcal{S}\mathbf{R}$ implies that $R \in \mathbf{D}$ (Lemma 2), hence $R \in \mathcal{S}\mathbf{D} \cap \mathbf{D} = (0)$. Hence $\mathbf{D}$ is a complement of $\mathbf{R}$. Also, since any ring in $\mathbf{T}$ is a simple ring with unity and $\mathbf{R} = \mathcal{U}\mathbf{T}$, it follows that $\mathbf{R}$ is hereditary. The lattice of all hereditary radicals is distributive ([3], Cor. 16, p. 212), so within this lattice a complement of $\mathbf{R}$ is uniquely determined. Since $\mathbf{D}$ is a hereditary radical, $\mathbf{D}$ is the complement of $\mathbf{R}$. Summarizing our results we get

Theorem 1. *For any radical $\mathbf{R} \supseteq \mathbf{J}$ in the category $\mathbf{K}$ there exists a uniquely determined complement $\mathbf{D}$, where $\mathbf{D}$ is the class of all finite direct sums of all simple $\mathbf{R}$-semi-simple rings. $\mathbf{D}$ is also a semi-simple class, in fact $\mathbf{D}$ is the class of all $\mathbf{R}$-semi-simple rings. Moreover, $\mathbf{R}$ is the upper radical determined by the class of all simple $\mathbf{R}$-semi-simple rings.*

Remark 2. In [1] it is shown that for any hereditary radical $\mathbf{R}$ there exists a radical $\mathbf{R}'$ which is a complement of $\mathbf{R}$ and $\mathbf{R}'$ is the upper radical determined by the class of all subdirectly irreducible rings with $\mathbf{R}$-radical heart. The $\mathbf{R}$-radical rings are the strongly $\mathbf{R}$-semi-simple rings ([1], Theorem 2).

This result holds in the category of all associative rings. Our theorem 1 reveals that in the subcategory $\mathbf{K}$ a much stronger results holds. Not only is the complement $\mathbf{D}$ of $\mathbf{R}$ uniquely determined, but $\mathbf{D}$ is also a semi-simple class ($\mathbf{R} \supseteq \mathbf{J}$), i.e. the class of $\mathbf{R}$-semi-simple rings. Since the class $\mathbf{D}$ is homomorphically closed, the strongly $\mathbf{R}$-semi-simple rings are all semi-simple rings. The complement $\mathbf{R}$ of $\mathbf{D}$ is the upper radical determined by the class $\mathbf{T}$ of simple $\mathbf{R}$-semi-simple rings (=simple $\mathbf{D}$-radical rings). This class $\mathbf{T}$ of simple rings is, in general, a subclass of the class of all subdirectly irreducible rings with $\mathbf{D}$-radical hearts. However, they determine the same upper radical $\mathbf{R}$.

Examples. 1. Let $\mathbf{R} = \mathbf{J}$, the Jacobson radical. Then $\mathbf{D}$ is the class of all finite direct sums of simple $\mathbf{J}$-semi-simple rings i.e. finite direct sums of all matrix rings over division rings.

2. Let $\mathbf{R}$ be the class of all strong artinian rings, i.e. all rings $R \in \mathbf{K}$, where $(R, +)$ has d.c.c. for subgroups. It can easily be seen that $\mathbf{R}$ is a radical class, which we call $\mathbf{R}_S$. The complement of $\mathbf{R}_S$ is the class of finite direct sums of simple $\mathbf{R}_S$-semi-simple rings, i.e. finite direct sums of all matrix rings over infinite division rings.

3. Let $\mathbf{R}$ be the class of all torsion radical rings, i.e. all rings $R \in \mathbf{K}$ where $(R, +)$ is a torsion group. This is a radical class, which we call $\mathbf{R}_T$. The complement of $\mathbf{R}_T$ is the class of finite direct sums of simple $\mathbf{R}_T$-semi-simple rings, i.e. finite direct sums of simple torsion-free rings. These simple torsion-free rings are matrix rings over (infinite) torsion-free division rings.

Remark 3. Any radical $\mathbf{R} \supseteq \mathbf{J}$ is hypernilpotent i.e. $\mathbf{R}$ contains all nilpotent rings and $\mathbf{R}$ is hereditary, (Lemma 1). By corollary 6 ([4]), $\mathbf{R}$ is a homomorphically closed semi-simple class. Let $\mathbf{D}$ be the complement of $\mathbf{R}$, then $\mathbf{D}=\mathscr{S}\mathbf{R}$ (Lemma 2). One might conjecture, that $\mathbf{R}=\mathscr{S}\mathbf{D}$. This is not true in general. However, $\mathbf{R}\subseteq\mathscr{S}\mathbf{D}$. Indeed, if $R\in\mathbf{R}$ then $R\in\mathscr{U}\mathbf{T}$, so R has no non-zero homomorphic image in $\mathbf{T}$. If $\mathbf{D}(R)=S_1\oplus\ldots\oplus S_k\neq 0$, then $\mathbf{D}(R)$ is a non-zero ideal in R and $\mathbf{D}(R)$ has a unity, so it is a direct summand of R. At least one of the $S_i\neq 0$, say $S_j\neq 0$, and S_j is a direct summand of R. Now $S_j\in\mathbf{T}$ and R could be mapped homomorphically onto $0\neq S_j\in\mathbf{T}$, which is a contradiction. Hence $\mathbf{D}(R)=0$, and $R\in\mathscr{S}\mathbf{D}$. That $\mathbf{R}\neq\mathscr{S}\mathbf{D}$ may be seen by taking $\mathbf{R}=\mathbf{J}$. The ring Z_4 of integers mod 4 is not $\mathbf{J}$-radical, so $Z_4\notin\mathbf{R}$. If $\mathbf{D}$ is the complement of $\mathbf{J}$, then $Z_4\in\mathscr{S}\mathbf{D}$, however.

3. Subidempotent radicals

Definition. A ring R will be called *hereditarily idempotent* if every ideal of R is idempotent. A hereditary radical $\mathbf{R}$ will be called a subidempotent radical if $\mathbf{R}$-radical rings are hereditarily idempotent rings (cf. [1]).

Examples. In the category $\mathbf{K}$ the complements of hypernilpotent radicals are subidempotent.

Lemma 4. *$\mathbf{E}$ is a subidempotent radical. Let $\mathbf{P}=\{R\in\mathbf{K}:R$ is a simple $\mathbf{E}$-radical ring$\}$. Then $\mathbf{E}=\mathscr{L}\mathbf{P}$ the lower radical determined by the class $\mathbf{P}$.*

Proof. Since every ring in $\mathbf{P}$ is $\mathbf{E}$-radical, it is clear that $\mathscr{L}\mathbf{P}\subseteq\mathbf{E}$. Next let $R\in\mathbf{E}$. Then R is a hereditarily idempotent ring. Hence any ideal of R is idempotent. However $\mathbf{J}(R)$ is nilpotent, so $\mathbf{J}(R)=(0)$. Then R is a finite direct sum of matrix rings over division rings. Each of the direct summands is a simple ring and, since $\mathbf{E}$ is hereditary, a simple $\mathbf{E}$-radical ring. A non-zero homomorphic image of R is in $\mathbf{E}$ since $\mathbf{E}$ is homomorphically closed. Such an image is again a finite direct sum of simple $\mathbf{E}$-radical rings, hence it has a non-zero ideal in $\mathbf{P}$. Then $R\in\mathbf{P}_2$. Since $\mathbf{P}$ is a homomorphically closed class of idempotent rings, $\mathscr{L}\mathbf{P}=\mathbf{P}_2$ ([5], Corollary 12.6), so $R\in\mathscr{L}\mathbf{P}$. Therefore $\mathbf{E}\subseteq\mathscr{L}\mathbf{P}$.

Lemma 5. *Let $\mathbf{E}$ and $\mathbf{P}$ be as in Lemma 4. Let $\mathbf{Q}=\{R\in\mathbf{K}:R$ is a finite direct sum of rings in $\mathbf{P}\}$. Then $\mathbf{Q}$ is the class of $\mathbf{E}$-radical rings. Moreover, $\mathbf{Q}=\mathbf{E}$ is a semi-simple class, in fact, $\mathbf{Q}$ is the class of $\mathscr{U}\mathbf{P}$-semi-simple rings.*

Proof. Since every ring in $\mathbf{P}$ is $\mathbf{E}$-radical, it is clear that any ring in $\mathbf{Q}$ is $\mathbf{E}$-radical. From the proof of Lemma 4 it follows that if $R\in\mathbf{E}$, R is a finite direct sum of simple $\mathbf{E}$-radical rings, i.e. $R\in\mathbf{Q}$. This shows that $\mathbf{Q}=\mathbf{E}$.

Since $\mathbf{Q}=\mathbf{E}$ is hereditary and closed under extensions, it follows that $\mathbf{Q}$ is a semi-simple class ([4], Theorem 1). Now we show that $\mathbf{Q}$ is the class of $\mathscr{U}\mathbf{P}$-semi-simple rings. Let $R \in \mathbf{Q}$ then R is a finite direct sum of rings in $\mathbf{P}$, each of which is $\mathscr{U}\mathbf{P}$-semi-simple, hence R is $\mathscr{U}\mathbf{P}$-semi-simple. Conversely, assume that R is $\mathscr{U}\mathbf{P}$-semi-simple. Any ring in $\mathbf{P}$ is a simple prime ring, hence a $\mathbf{J}$-semi-simple ring, so $\mathbf{P} \subseteqq \mathscr{S}\mathbf{J}$, which implies $\mathscr{U}\mathscr{S}\mathbf{J} \subseteqq \mathscr{U}\mathbf{P}$ or $\mathbf{J} \subseteqq \mathscr{U}\mathbf{P}$. Then R is a $\mathbf{J}$-semi-simple ring and a finite direct sum of simple $\mathbf{J}$-semi-simple rings, i.e. simple $\mathscr{U}\mathbf{P}$-semi-simple rings. But a simple $\mathscr{U}\mathbf{P}$-semi-simple ring is a simple ring in $\mathbf{P}$. Hence R is a finite direct sum of rings in $\mathbf{P}$ or $R \in \mathbf{Q}$. Therefore $\mathbf{Q}$ is the class of $\mathscr{U}\mathbf{P}$-semi-simple rings.

Lemma 6. *Let* $\mathbf{E}$, $\mathbf{P}$ *and* $\mathbf{Q}$ *be as in Lemmas 4 and 5. Then* $\mathscr{U}\mathbf{P}$ *is the complement of* $\mathbf{E}$.

Proof. Let $R \in \mathbf{E} \cap \mathscr{U}\mathbf{P}$. Then $R \in \mathbf{Q}$ (Lemma 5), so R is a finite direct sum of rings in $\mathbf{P}$. But $R \in \mathscr{U}\mathbf{P}$ implies that R cannot be mapped homomorphically onto a non-zero ring in $\mathbf{P}$. Hence $R=(0)$. Next, let $R \in \mathscr{S}\mathbf{E} \cap \mathscr{S}\mathscr{U}\mathbf{P}$. Since $\mathbf{Q}=\mathscr{S}\mathscr{U}\mathbf{P}$ (Lemma 5), it follows that $R \in \mathbf{Q}$. Also $\mathbf{E}=\mathbf{Q}$, so $R \in \mathscr{S}\mathbf{Q}$. Then $R \in \mathbf{Q} \cap \mathscr{S}\mathbf{Q}$ implies $R=(0)$.

This shows that $\mathscr{U}\mathbf{P}$ is a complement of $\mathbf{E}$. Each ring in $\mathbf{P}$ is a simple $\mathbf{E}$-radical ring and a simple $\mathbf{J}$-semi-simple ring (proof of Lemma 4). So such a ring is a simple ring with unity and $\mathscr{U}\mathbf{P}$ is a hereditary radical. It follows that $\mathscr{U}\mathbf{P}$ is the complement of $\mathbf{E}$. In the proof of Lemma 5 we have seen that $\mathbf{J} \subseteqq \mathscr{U}\mathbf{P}$, so $\mathscr{U}\mathbf{P}$ is a hypernilpotent radical. Summarizing the results we get

Theorem 2. *Let* $\mathbf{E}$ *be an arbitrary subidempotent radical in the category* $\mathbf{K}$. *Then* $\mathbf{E}=\mathscr{L}\mathbf{P}$, *where* $\mathbf{P}$ *is the class of simple* $\mathbf{E}$*-radical rings. Any ring in* $\mathbf{E}$ *is a finite direct sum of rings in* $\mathbf{P}$. *Also* $\mathbf{E}$ *is a semi-simple class, i.e. the class of* $\mathscr{U}\mathbf{P}$*-semi-simple rings. The radical* $\mathscr{U}\mathbf{P}$ *is hypernilpotent and the complement of* $\mathbf{E}$.

Remark 4. It can easily be seen that using the notation of Lemmas 1, 2 and 3, the complement $\mathbf{D}$ of $\mathbf{R}$ equals $\mathscr{L}\mathbf{T}$, the lower radical determined by $\mathbf{T}$. Indeed, $\mathbf{D}$ is a subidempotent radical and $\mathbf{T}$ is the class of simple $\mathbf{R}$-semi-simple rings i.e. simple $\mathbf{D}$-radical rings (Lemma 2). Now apply Lemma 4.

By theorem 2 of [1] the class $\mathbf{D}$ can also be characterized as the upper radical determined by the class of all subdirectly irreducible rings with $\mathbf{R}$-radical hearts.

Comparing our results with those of theorem 4 of [1] it turns out that, contrary to the general situation in the category of associative rings, any radical $\mathbf{R} \supseteqq \mathbf{J}$ is a dual radical, i.e. the complement of $\mathbf{D}$ is $\mathbf{R}$, if $\mathbf{D}$ is the complement of $\mathbf{R}$. Here $\mathbf{R}$ is a dual hypernilpotent radical, while $\mathbf{D}$ is a dual subidempotent radical.

The radical $\mathbf{R}$ resp. $\mathbf{D}$ is the upper radical resp. lower radical determined by the same class $\mathbf{T}$, i.e. the class $\mathbf{T}$ of simple $\mathbf{R}$-semi-simple rings or simple $\mathbf{D}$-radical rings. In the next section we investigate radicals, determined by a class of simple prime rings.

4. Simple prime rings in K

Let $\mathbf{M}$ be an arbitrary non-empty class of simple prime rings in $\mathbf{K}$. Then $\mathbf{M}$ is a class of simple rings with unity. Let $\mathbf{Q}=\{R\in\mathbf{K}: R$ is a finite direct sum of rings from $\mathbf{M}\}$. Then $\mathbf{Q}$ is homomorphically closed, closed under extensions and has no non-zero nilpotent rings. Hence $\mathbf{Q}$ is a radical class in $\mathbf{K}$ ([4] Theorem 5). All-rings in $\mathbf{M}$ are $\mathbf{Q}$-radical, hence $\mathscr{L}\mathbf{M}\subseteqq\mathbf{Q}$. But if $R\in\mathbf{Q}$, then $0\neq R/I\in\mathbf{Q}$ for any ideal I in R, so R/I has a non-zero ideal in $\mathbf{M}$ which implies $R\in\mathbf{M}_2$. Since $\mathbf{M}$ is a class of idempotent rings $\mathscr{L}\mathbf{M}=\mathbf{M}_2$, hence $R\in\mathscr{L}\mathbf{M}$. Therefore $\mathbf{Q}=\mathscr{L}\mathbf{M}$.

Since $\mathbf{Q}$ is hereditary and closed under extensions, $\mathbf{Q}$ is a semi-simple class ([4], Theorem 1). From the proof of Lemma 5 it follows that $\mathbf{Q}=\mathscr{S}\mathscr{U}\mathbf{M}$. Also both $\mathscr{U}\mathbf{M}$ and $\mathscr{L}\mathbf{M}$ are hereditary radicals, since $\mathbf{M}$ is a hereditary class. From $\mathbf{Q}=\mathscr{L}\mathbf{M}=\mathscr{S}\mathscr{U}\mathbf{M}$ it follows directly that $\mathscr{U}\mathbf{M}$ and $\mathscr{L}\mathbf{M}$ are complements.
This shows:

Theorem 3. *Let $\mathbf{M}$ be an arbitrary non-empty class of simple prime rings in $\mathbf{K}$. Then both $\mathscr{U}\mathbf{M}$ and $\mathscr{L}\mathbf{M}$ are hereditary radicals, where $\mathscr{U}\mathbf{M}$ is hypernilpotent and $\mathscr{L}\mathbf{M}$ is subidempotent. In addition, $\mathscr{L}\mathbf{M}=\mathscr{S}\mathscr{U}\mathbf{M}$ and $\mathscr{L}\mathbf{M}$ and $\mathscr{U}\mathbf{M}$ are complementary radicals.*

From Lemmas 1 and 4 it follows that any hypernilpotent (subidempotent) radical $\mathbf{R}(\mathbf{E})$ is the upper radical (lower radical), determined by a class $\mathbf{T}(\mathbf{P})$ of simple prime rings.

Remark 5. Finally we want to compare our results with theorem 10 of [1], the so-called duality theorem for radicals. It is said there that all dual hypernilpotent and dual subidempotent radicals can be obtained both as upper radicals determined by certain classes of subdirectly irreducible rings with idempotent hearts. In our case any hypernilpotent or subidempotent radical is dual and the hypernilpotent radicals are upper radicals, while the subidempotent ones are lower radicals. Both are determined by classes of simple prime rings, which are matrix rings over division rings.

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