Acta Sci. Math., 39 (1977), 291—301

The radical in ring varieties

A. A. ISKANDER

All rings considered here are associative and not necessarily with 1. A ring variety is a class of rings closed under subrings, homomorphic images and Cartesian products; equivalently, it is the class of all rings satisfying a set of polynomial identities. In the present paper, we study syntactic and semantic properties and also structure of varieties in which the Jacobson radical of every member is 1) nil, 2) nilpotent, or 3) a direct summand. 1) is equivalent to a one variable identity: $x^n+x^{n+1}\cdot h(x)=0$; and also the variety is locally nilpotent by finite. 2) is equivalent to an n variable identity: $x_1...x_n+f(x_1, ..., x_n)=0$, where every term in $f(x_1, ..., x_n)$ is of degree larger than n. Also, every variety satisfying 2) is generated by a finitely generated ring and is finitely based. 3) is equivalent to a finite set of two variable identities.

For an account of the variety theory, the reader may consult [1, 2, 3, 5, 12, 13, 14]. Script letters will denote classes or varieties of rings; the corresponding Latin letters will denote their T-ideals of identities. We denote the free associative ring on $\{x_1, x_2, ..., x_n, ...\}$ by F. The join of varieties will be denoted by $\vee$. Var $\mathcal{K}$ will mean the variety generated by $\mathcal{K}$. $\langle f, g, h, ...\rangle$ will mean the variety of all rings satisfying the identities $0=f=g=h= \dots$.

A residually finite ring is a ring in which every nonzero element does not belong to an ideal of finite index. A T-ideal is an ideal closed under all endomorphisms.

1. Definition 1. A ring variety is called *locally nilpotent by finite,* if every finitely generated member possesses a nilpotent ideal of finite index. A *locally finite variety* is a variety in which every finitely generated member is finite.

In [7], it is shown that locally finite ring varieties are precisely varieties satisfying $cx=0$, $x^n+x^{n+1}f(x)=0$, for some positive integers c and n and for some polynomial $f(x)$ with integral coefficients. For locally nilpotent by finite varieties, we have

Received February 12, 1976.

Theorem 1. *The following conditions on a ring variety $\mathcal{V}$ are equivalent:*
1) $\mathcal{V}$ *is locally nilpotent by finite.*
2) *every member of* $\mathcal{V}$ *generated by one element is nilpotent by finite.*
3) $\mathcal{V}$ *satisfies* $x^n+x^{n+1}f(x)=0$, *for some positive integer* n *and* $f(x)\in\mathbf{Z}[x]$.
4) *the Jacobson radical of every member of* $\mathcal{V}$ *is nil.*

2. We will first state and prove some lemmas.

Lemma 2. *Let A be an algebra over a field K. Then either A is nil, or A contains a copy of K or of $xK[x]$.*

Proof. Let $a\in A$, $a\neq 0$. The subalgebra of A generated by a is isomorphic to $xK[x]/I$, where I is an ideal of $xK[x]$ and hence principal; i.e., $I=g(x)K[x]$. If $g(x)=0$, then A contains a copy of $xK[x]$. If $g(x)\neq 0$, we can assume that $g(x)=x^s+ +x^{s+1}h(x)$. If a is not nilpotent, then $a^s=-a^{s+1}h(a)=a^{s+1}q(a)$. Hence $a^s=a^{2s}(q(a))^s$ and $(aq(a))^s$ is an idempotent element of A that generates a subalgebra of A isomorphic to K.

Lemma 3. *The following conditions on a ring variety $\mathcal{V}$ are equivalent:*
i) $x^m+x^{m+1}h(x)\in V$ *for some* $m>0$, $h(x)\in\mathbf{Z}[x]$.
ii) $\Sigma\{a_kx^k:1\leqq k\leqq s\}\in V$ *for some* $s>0$ *and* g.c.d.$(a_1,\dots,a_s)=1$.
iii) $V(x\mathbf{Z}[x])$ *the T-ideal of one variable identities of* $\mathcal{V}$ *is not contained in* $px\mathbf{Z}[x]$ *for any prime* p.

Proof. It is clear that i)$\Rightarrow$ii)$\Rightarrow$iii). Let $\mathcal{V}$ satisfy iii). By Hilbert's basis theorem, $I=V(x\mathbf{Z}[x])$ is finitely generated, say by $g_1(x), g_2(x), \dots, g_k(x)$. For any prime p one of the coefficients of $g_1(x),\dots,g_k(x)$ is not divisible by p. Let u be an integer larger than all the degrees of $g_1(x),\dots,g_k(x)$. Then

$$g(x)=\Sigma\{x^{iu}g_i(x):1\leqq i\leqq k\}\in I$$

satisfies ii).

Let $\mathcal{V}$ satisfy ii). Hence every member of $\mathcal{V}$ satisfies $rx^t=x^{t+1}f(x)$ for some positive integers r and t and some $f(x)\in\mathbf{Z}[x]$. Thus $\mathcal{V}$ satisfies

$$r^2x^t=rx^{t+1}f(x)=x^{t+2}(f(x))^2,\dots,r^tx^t=x^{2t}(f(x))^t,\quad r^{2t}x^t=x^{3t}(f(x))^{2t}.$$

and substituting r^4x for x, we get

$$r^{6t}x^t=(r^{6t}x^t)^2g(x).$$

Let $A\in\mathcal{V}$ and denote by B the Jacobson radical of A. For every $b\in B$, $r^{6t}b^tg(b)$ is an idempotent element of B. Hence $r^{6t}b^tg(b)=0$ and B satisfies $r^{6t}x^t=0$. Let C be the ideal of B generated by all b^t, $b\in B$. Then C satisfies $r^{6t}x=0$. We will show that C satisfies $x^n=0$ where n depends only on the polynomial in ii), and consequently B satisfies $x^{nt}=0$. C is the direct sum of a finite number of rings of prime power characteristic. Hence, for our purpose, we can assume that C satisfies $p^k x = 0$ ($k < r^{6t}$). C/pC is an algebra over the field $\mathbf{Z}_p$. $C/pC \in V$ and hence satisfies $\Sigma\{a_i x^i: 1 \leqq i \leqq s\} = 0$ where g.c.d. $(a_1, \ldots, a_s) = 1$. Hence C/pC cannot contain any copies of $x\mathbf{Z}_p[x]$. If C/pC were not nil, then by Lemma 2, C/pC would contain a copy of Z_p and hence an idempotent different from 0. But pC is nilpotent $(pC)^k = 0$, and this nonzero idempotent can be lifted to an idempotent in C, contradicting the fact that B does not have any nonzero idempotents. Thus C/pC is nil by ii)

$$x^u = x^{u+1} h(x) \quad \text{where} \quad 1 \leqq u \leqq s,$$

and hence $x^u = x^{u+v}(h(x))^v$ for any $v > 0$. If $a \in C/pC$ satisfies $a^v = 0$, then $a^u = 0 = a^s$. Thus C satisfies $x^{sk} = 0$.

Now A/B is a subdirect sum of primitive rings satisfying an identity of type ii). By Kaplansky's theorem [8] every such primitive ring is of dimension $\leqq (\frac{1}{2}s)^2$ over their centers. These centers satisfy the same identity of type ii), and hence they can be only a finite number of finite fields. Thus all these fields satisfy an identity of the type $x + x^2 h(x) = 0$, and hence all these primitive rings satisfy $(x + x^2 h(x))^u = 0$ for some $u > 0$ depending only on the identity $\Sigma\{a_i x^i: 1 \leqq i \leqq s\} = 0$ [9]. Thus A satisfies $(x + x^2 h(x))^{uskt} = 0$; i.e., $\mathscr{V}$ satisfies i).

Lemma 4. *The following two conditions on a ring variety $\mathscr{V}$ are equivalent.*

i) *if $A \in \mathscr{V}$, and if A is nil, then $x^e = 0$ in A.*

ii) *$\mathscr{V}$ satisfies $x^e + x^{e+1} h(x) = 0$ for some $h(x) \in \mathbf{Z}[x]$.*

Proof. ii)$\Rightarrow$i) since $x^e + x^{e+1} h(x) = 0$ implies $x^e = -x^{e+1} h(x) = x^{e+1} H(x) = x^{e+k}(H(x))^k$ for all $k > 1$. Thus if $A (\in \mathscr{V})$ is nil, then $a^e = 0$ for all $a \in A$.

Conversely, the subvariety $\mathscr{U}$ of $\mathscr{V}$ of all rings satisfying $x^{e+1} = 0$, satisfies $x^e = 0$ by i). Hence $x_1^e \in V + (x_1^{e+1})$ where (x_1^{e+1}) is the T-ideal of identities generated by x_1^{e+1}.

Thus $x_1^e = q(x_1) + x_1^{e+1} f(x_1)$, $q(x_1) \in V$; i.e., $q(x_1) = x_1^e - x_1^{e+1} f(x_1) \in V$.

Lemma 5. *The following conditions on a ring variety $\mathscr{V}$ are equivalent:*

i) *every nilpotent member of $\mathscr{V}$ satisfies $x_1 \ldots x_e = 0$.*

ii) *$\mathscr{V}$ satisfies $x_1 \ldots x_e + f(x_1, \ldots, x_e) = 0$ for some multinomial f, all terms of which are of degree larger than e in $x_1, \ldots, x_e$.*

Proof. ii) implies that the product of any e elements of a ring $A \in \mathscr{V}$ belongs to A^{e+1}, i.e., $A^e \subseteqq A^{e+1}$. Hence $A^e = A^{e+k}$ for all $k > 0$. Thus if $A^s = 0$ for some s, $A^e = A^{e+s} = 0$.

Conversely, if $\mathscr{V}$ satisfies i) and $U = V + F^{e+1}$, then $x_1 \ldots x_e \in U$; i.e., $x_1 \ldots x_e = q + g$ where $g \in F^{e+1}$ and $q \in V$. We can assume that both q and g involve only $x_1, \ldots, x_e$; i.e., $q = x_1 \ldots x_e - g \in V$ where $g \in F^{e+1}$.

3. Proof of Theorem 1. It is obvious that 1)$\Rightarrow$2). Let $F_1(\mathscr{V})$ — the free member of $\mathscr{V}$ of rank 1 — be nilpotent by finite; i.e., there is an ideal I of $F_1(\mathscr{V})$ such that $F_1(\mathscr{V})/I$ is finite and $I^n=0$ for some $n>0$. Then $F_1(\mathscr{V})/I$ satisfies $(x+x^2h(x))^m=0$ for $m>0$, $h(x)\in \mathbf{Z}[x]$ [7, 9]. Hence $F_1(\mathscr{V})$ satisfies $(x+x^2h(x))^{mn}=0$. Thus $(x_1+x_1^2h(x))^{mn}\in V$; i.e., $\mathscr{V}$ satisfies 3).

Let $\mathscr{V}$ satisfy 3) and let $A\in\mathscr{V}$. Let B be the Jacobson radical of A. For any $b\in B$, $b^n=-b^{n+1}f(b)=b^{n+1}g(b)=b^{2n}(g(b))^n$. Thus $(bg(b))^n$ is an idempotent of B. Hence $(bg(b))^n=0$ and $b^n=b^n(bg(b))^n=0$; i.e., B is nil.

Let $\mathscr{V}$ satisfy 4), and let $A\in\mathscr{V}$ be finitely generated. $V(x\mathbf{Z}[x])$ is not contained in $px\mathbf{Z}[x]$; otherwise, $x\mathbf{Z}_p[x]$ would belong to $\mathscr{V}$, and hence $\mathscr{V}$ contains all commutative rings of characteristic p. The Jacobson radical of such rings may not be nil. Thus, by Lemma 3, $\mathscr{V}$ satisfies $x^n+x^{n+1}h(x)=0$ for some $n>0$ and $h(x)\in\mathbf{Z}[x]$. By Lemma 4 the Jacobson radical B of A satisfies $x^n=0$ since B is nil. A/B is a finitely generated semisimple ring satisfying $x^n+x^{n+1}h(x)=0$. Hence A/B is the subdirect sum of matrix rings over a finite number of finite fields. Thus A/B satisfies $cx=0$ for some positive integer c. Hence A/B satisfies $cx=0$, $x^n+x^{n+1}h(x)=0$. Thus A/B is finite by [7] as it is finitely generated. Now B is a subring of finite index in the finitely generated ring A. By a result of Lewin [10], B is a finitely generated ring. As B also satisfies $x^n=0$, by a result of Kaplansky [8], B is nilpotent; i.e., A possesses a nilpotent ideal of finite index concluding the proof of Theorem 1.

4. Let c, d, e be integers, $c\geqq 0$, $d>0$, $e>0$.

Definition 2. $\mathscr{C}(c, d, e)$ is the class of all rings A with the properties:

1) $cx=0$ for all $x\in A$.
2) Let B be a homomorphic image of a subring of A. Then
   a) if B is nilpotent, then $B^e=0$,
   b) if B is not nilpotent and B is primitive, then B is a finite simple ring of order dividing d.

For $c>0$, the class $\mathscr{C}(c, d, e)$ was defined by Kruse [9] in analogy to the corresponding definition for groups [13]. In [9] it is shown that

Proposition 6 [9]. *$\mathscr{C}(c, d, e)$, for $c>0$, is a variety generated by a finite ring.*

It is clear that the following also holds:

Proposition 7. *$\mathscr{C}(c, d, e)=\mathscr{C}(0, d, e)\cap\langle cx\rangle$. $\mathscr{C}(0, d, e)$ is closed under subrings and homomorphic images.*

Definition 4. $\mathscr{D}(c, d, e)$ is the class of all rings A with the properties:

1) $cx=0$ for all $x\in A$.
2) Let B be a homomorphic image of a subring of A. Then

a) if B is nil, B satisfies $x^e=0$,
b) if B is not nilpotent and B is primitive, then B is a finite simple ring of order dividing d.

It is clear that $\mathscr{C}(c, d, e) \leqq \mathscr{D}(c, d, e)$ for all $c \geqq 0$ and

Proposition 8. *$\mathscr{D}(c, d, e)=\mathscr{D}(0, d, e) \cap \langle cx \rangle$ and $\mathscr{D}(c, d, e)$ is closed under subrings and homomorphic images.*

In [4] EVERETT defines a ring C to be an extension of a ring A by a ring B if C possesses an ideal isomorphic to A whose factor is isomorphic to B.

Definition 5 [6, 11, 13]. Let $\mathscr{U}, \mathscr{V}$ be classes of rings. $\mathscr{U} \cdot \mathscr{V}$ is the class of all rings that are extensions of a ring of $\mathscr{U}$ by a ring of $\mathscr{V}$.

In [6, 13] it is shown that

Proposition 9. *If $\mathscr{U}, \mathscr{V}$ are varieties, then $\mathscr{U} \cdot \mathscr{V}$ is a variety satisfying $f(g_1, \ldots, g_n)=0$ for all $f(x_1, \ldots, x_n) \in U$, $g_1, \ldots, g_n \in V$.*

We will need two more results.

Proposition 10 [9]. *A finitely generated nilpotent by finite ring is residually finite.*

Proposition (HIGMAN [13]). *If a locally finite variety is generated by a family $\mathscr{K}$ of finite rings, then every finite member of the variety is a homomorphic image of a subring of a finite direct sum of members of $\mathscr{K}$.*

Higman's result was stated for groups. It also holds for rings.

We will show that all the $\mathscr{C}$ and $\mathscr{D}$ classes introduced here are actually varieties.

5. Theorem 12. *$\mathscr{D}(c, d, e)$ is a locally finite variety for all $c>0$.*

Claim 1: If $c>0$ and $A \in \mathscr{D}(c, d, e)$, then A satisfies $(x^e+x^{e+1}h(x))^e=0$ where $x+x^2h(x)$ is an identity satisfied by all finite fields of order dividing d.

Let $c=p_1^{k_1} \times \ldots \times p_s^{k_s}$ be the prime factorization of c. Then $A=A_1 \times A_2 \times \ldots \times A_s$ where A_i is of prime power characteristic. It is sufficient to establish the claim for $c=p^k$. $A/pA \in \mathscr{D}(c, d, e)$. $(pA)^k=0$ and $pA \in \mathscr{D}(c, d, e)$; hence pA satisfies $x^e=0$. Let $a \in A/pA$. The subring $[a]$ of A/pA generated by a belongs to $\mathscr{D}(c, d, e)$. It is isomorphic to $x\mathbf{Z}_p[x]/I$, $I \neq 0$ since $x\mathbf{Z}_p[x] \notin \mathscr{D}(c, d, e)$. Thus $I=(x^r+x^{r+1}g(x))\mathbf{Z}_p[x]$ for some $r>0$, $g(x) \in \mathbf{Z}[x]$. $I=x^r\mathbf{Z}_p[x] \cap (x+x^2g(x))\mathbf{Z}_p[x]$. Hence $[a]$ is isomorphic to a subdirect sum of $x\mathbf{Z}_p[x]/x^r\mathbf{Z}_p[x]$ and $x\mathbf{Z}_p[x]/(x+x^2g(x))\mathbf{Z}_p[x]$; i.e., $[a]$ is isomorphic to a subdirect sum of a nil ring and a finite number of finite fields all belonging to $\mathscr{D}(c, d, e)$. Let $x+x^2h(x)$ be an identity satisfied by all fields of order dividing d. Then $[a]$ satisfies $x^e+x^{e+1}h(x)=0$ since every nil member of $\mathscr{D}(c, d, e)$ satisfies $x^e=0$. Thus A/pA satisfies $x^e+x^{e+1}h(x)=0$. Hence A satisfies $(x^r+x^{e+1}h(x))^e=0$.

Let r be the largest square free integer dividing d. All finite simple rings of order dividing d belong to $\mathscr{C}(r, d, d)$ since all are finite matrix rings over finite fields.

Claim 2: $\mathscr{D}(c, d, e) \subseteq \langle x^e \rangle \cdot \mathscr{C}(r, d, d)$ for all $c > 0$.

Let $A \in \mathscr{D}(c, d, e)$. Then by Claim 1 and Theorem 1, Rad A is nil. Hence Rad $A \in \mathscr{D}(c, d, e)$ belongs to $\langle x^e \rangle$. $A/\text{Rad}\, A$ is a subdirect sum of primitive rings belonging to $\mathscr{D}(c, d, e)$. All these primitive rings are finite simple rings of order dividing d. Hence they belong to $\mathscr{C}(r, d, d)$ which is a variety by Proposition 6.

Claim 3: $\text{Var}\,(\mathscr{D}(c, d, e)) \subseteq \mathscr{D}(c, d, ed)$.

Since by Claim 2 $\mathscr{D}(c, d, e) \subseteq \langle x^e \rangle \cdot \mathscr{C}(r, d, d)$ and by Proposition 9 $\langle x^e \rangle \cdot \mathscr{C}(r, d, d)$ is a variety, $\text{Var}\,(\mathscr{D}(c, d, e)) \subseteq \langle x^e \rangle \cdot \mathscr{C}(r, d, d)$. Let $A \in \text{Var}\,(\mathscr{D}(c, d, e))$. Then A satisfies $cx = 0$ and also $A \in \langle x^e \rangle \cdot \mathscr{C}(r, d, d)$. So, there is an ideal B of A such that B satisfies $x^e = 0$ and $A/B \in \mathscr{C}(r, d, d)$. If A is nil, then A/B is nil and hence satisfies $x^d = 0$. Thus A satisfies $x^{ed} = 0$. If A is primitive, A satisfies $(x^m + x^{m+1} f(x))^e = 0$ (by Claim 1 and Proposition 9). Hence A is finite dimensional over its center (by KAPLANSKY's theorem [8]). Thus A is a simple ring that is not nil. Hence $A \in \mathscr{C}(r, d, d)$; i.e., A is a finite simple ring of order dividing d.

Claim 4: Let $A \in \text{Var}\,(\mathscr{D}(c, d, e))$ be nil. Then A satisfies $x^e = 0$.

$\mathscr{V} = \text{Var}\,(\mathscr{D}(c, d, e))$ satisfies $cx = 0 = x^m + x^{m+1} h(x)$. Hence by [7] $\mathscr{V}$ is locally finite. Thus $\mathscr{V}$ is generated by its finite members belonging to $\mathscr{D}(c, d, e)$. It will be sufficient to establish the claim for the case A is finite. By Higman's Proposition 11, $A = T/I$ where T is a finite subdirect sum of finite rings from $\mathscr{D}(c, d, e)$. Thus T is finite, and hence its Jacobson radical R is nilpotent. T/R is generated by idempotents that can be lifted to a set of idempotents B such that T is generated by B and R. Since T/I is nil, $B \subseteq I$. Thus $T/I \cong R/R \cap I$. As $\mathscr{D}(c, d, e)$ is closed under subrings, R is a subdirect sum of members of $\mathscr{D}(c, d, e)$. Since R is nil, all these rings are nil, and hence satisfy $x^e = 0$. Thus R, and consequently A, satisfy $x^e = 0$. This argument is similar to an argument of KRUSE [9].

By Claims 3 and 4, $\mathscr{D}(c, d, e) \supseteq \text{Var}\,(\mathscr{D}(c, d, e))$. Hence $\mathscr{D}(c, d, e)$ is a variety. Since every nil member of $\mathscr{D}(c, d, e)$ satisfies $x^e = 0$, by Lemma 4 $\mathscr{D}(c, d, e)$ satisfies $x^e + x^{e+1} h(x) = 0$ for some $h \in Z[x]$. By [7], $\mathscr{D}(c, d, e)$ is a locally finite variety.

6. Theorem 13. *$\mathscr{D}(0, d, e)$ is a variety; moreover, $\mathscr{D}(0, d, e) = \bigvee \{\mathscr{D}(c, d, e) : c > 1\}$.*

Denote by $\mathscr{K}$ the join of all $\mathscr{D}(c, d, e)$, $c > 1$.

Claim 5: $\mathscr{K}$ satisfies $(x^e + x^{e+1} h(x))^e = 0$ where $x + x^2 h(x) = 0$ is an identity satisfied by all finite fields of order dividing d.

This is immediate from Claim 1.

Claim 6: $\mathscr{D}(0, d, e) \subseteq (\mathscr{K} \cdot \langle x^e \rangle) \cdot \mathscr{K}$.

Let $A\in\mathscr{D}(0, d, e)$, $B=\bigcap\{cA: c>1\}$, and let C be the torsion ideal of B. It is clear that A/B, $C\in\mathscr{K}$ and $B/C\in\mathscr{D}(0, d, e)$ is an algebra over the field Q of rational numbers. By Lemma 2, if B/C is not nil, it contains a copy of Q or a copy of $xQ[x]$ in contradiction to the statement $B/C\in\mathscr{D}(0, d, e)$. Thus B/C is nil and hence satisfies $x^e=0$.

Claim 7: $\mathscr{D}(0, d, e)\subseteq\mathscr{K}$.

From Claims 5 and 6 and by Proposition 9, $\mathscr{D}(0, d, e)$ satisfies an identity of the type $x^{e^3}+x^{e^3+1}g(x)=0$. By Theorem 1 $\operatorname{Var}(\mathscr{D}(0, d, e))$ is locally nilpotent by finite. Hence every finitely generated member of $\mathscr{D}(0, d, e)$ is nilpotent by finite, and by Proposition 10, is residually finite. Thus every finitely generated member of $\mathscr{D}(0, d, e)$ is a subdirect sum of finite members of $\mathscr{D}(0, d, e)$ and hence is also a subdirect sum of finite members of $\mathscr{K}$. Thus $\mathscr{D}(0, d, e)\subseteq\mathscr{K}$.

Claim 8: $\operatorname{Var}(\mathscr{D}(0, d, e))=\mathscr{K}\subseteq\langle x^e\rangle\mathscr{C}(r, d, d)$.

It is clear that $\mathscr{D}(c, d, e)\subseteq\mathscr{D}(0, d, e)$ for all $c>1$. Hence $\mathscr{K}\subseteq\operatorname{Var}(\mathscr{D}(0, d, e))\subseteq\mathscr{K}$ (by Claim 7). Also by Claim 2, $\mathscr{D}(c, d, e)\subseteq\langle x^e\rangle\cdot\mathscr{C}(r, d, d)$.

Claim 9: $\operatorname{Var}(\mathscr{D}(0, d, e))\subseteq\mathscr{D}(0, d, ed)$.

The proof is the same as in Claim 3.

Claim 10: $\mathscr{K}$ satisfies $(x+x^2f(x))^e=0$ where $x+x^2f(x)$ is an identity satisfied by all fields of order dividing d.

Consider A — the free member of rank 1 of $\mathscr{D}(c, d, e)$. If R is the Jacobson radical of A, R is nilpotent and A/R is a finite semisimple commutative ring. Thus A/R is the direct sum of finite fields belonging to $\mathscr{D}(c, d, e)$. Hence A/R satisfies $x+x^2f(x)=0$ where f depends only on d and R satisfies $x^e=0$. Hence $\mathscr{D}(c, d, e)$ satisfies $(x+x^2f(x))^e=0$, and so does $\mathscr{K}$.

From Claims 8, 9 and 10 and by Lemma 4, $\operatorname{Var}(\mathscr{D}(0, d, e))\subseteq\mathscr{D}(0, d, e)$, concluding the proof of Theorem 13.

7. Theorem 14. *A variety is locally finite iff it is contained in $\mathscr{D}(c, d, e)$ for some positive integers c, d, e.*

By Theorem 12, $\mathscr{D}(c, d, e)$ $(c>0)$ is a locally finite variety. Hence every subvariety of $\mathscr{D}(c, d, e)$ is locally finite. Conversely, if $\mathscr{V}$ is a locally finite variety, it satisfies $cx=0=x^e+x^{e+1}h(x)$ [7]. If d is the least common multiple of the orders of all nonnilpotent finite simple rings satisfying $x^e+x^{e+1}h(x)=0$, then $\mathscr{V}\subseteq\mathscr{D}(c, d, e)$.

Theorem 15. *A variety is locally nilpotent by finite iff it is contained in $\mathscr{D}(0, d, e)$ for some positive integers d, e.*

A subvariety of $\mathscr{D}(0, d, e)$ satisfies $x^e+x^{e+1}g(x)=0$ and hence is locally nilpotent by finite (by Theorem 1). Conversely, if a variety is locally nilpotent by finite, it satisfies $x^e+x^{e+1}h(x)=0$, and hence it is contained in $\mathscr{D}(0, d, e)$ where d is the least common multiple of the orders of all finite simple rings satisfying $x^e+x^{e+1}h(x)=0$.

8. Theorem 16. *$\mathscr{C}(0, d, e)$ is a variety; moreover, $\mathscr{C}(0, d, e)=\bigvee\{\mathscr{C}(c, d, e): c>1\}$.*

Claim 11: $\mathscr{C}(0, d, e)\leqq\langle x_1\dots x_e\rangle\cdot\mathscr{C}(r, d, d)$.

Since $\mathscr{C}(0, d, e)\leqq\mathscr{D}(0, d, e)$, by Theorem 15, $\mathscr{C}(0, d, e)$ is locally nilpotent by finite. Hence, if $A\in\mathscr{C}(0, d, e)$, then the Jacobson radical R of A is nil satisfying $x^e=0$. Thus by KAPLANSKY's theorem [8], R is locally nilpotent. Hence R satisfies $x_1\dots x_e=0$ since $R\in\mathscr{C}(0, d, e)$. A/R is a subdirect sum of primitive rings belonging to $\mathscr{D}(0, d, e)$ and hence to $\mathscr{C}(r, d, d)$.

Claim 12: $\mathscr{C}(0, d, e)\leqq\bigvee\{\mathscr{C}(c, d, e): c>1\}$.

Let $A\in\mathscr{C}(0, d, e)$ be finitely generated. Hence A is nilpotent by finite, and by Proposition 10, A is residually finite. Hence A is a subdirect sum of finite rings belonging to $\mathscr{C}(0, d, e)$. These finite rings belong to $\bigcup\{\mathscr{C}(c, d, e): c>1\}$. Thus $A\in\bigvee\{\mathscr{C}(c, d, e): c>1\}$. Hence $\mathscr{C}(0, d, e)\leqq\bigvee\{\mathscr{C}(c, d, e): c>1\}$.

Claim 13: $\mathscr{C}(c, d, e)$ satisfies $r^e x_1\dots x_e=0$ for all $c\geqq 0$.

By Claim 11 and by Proposition 9, $\mathscr{C}(0, d, e)$ satisfies $(rx_1)(rx_2)\dots(rx_e)=0$. Also $\mathscr{C}(0, d, e)\leqq\bigvee\{\mathscr{C}(c, d, e): c>1\}$.

Claim 14: If $\mathscr{C}(r^e, d, e)$ satisfies an identity in $x_1, \dots, x_n$, for some $n\geqq e$, where every term in the identity involves precisely all $x_1, \dots, x_n$, then the same identity holds in $\mathscr{C}(c, d, e)$ for all $c>1$.

Denote by V_c the T-ideal of the variety $\mathscr{C}(c, d, e)$. It is clear that $\mathscr{C}(c_1, d, e)\leqq\leqq\mathscr{C}(c_2, d, e)$ if $c_1|c_2$. Thus, if $s=$g.c.d. (c, r^e), then

$$V_s = V_c+sF = V_c+cF+sF = V_c+cF+r^eF = V_c+r^eF.$$

If $n\geqq e$ and $g(x_1, \dots, x_n)\in V_{r^e}$ is as described in Claim 14, then $g\in V_{r^e}\leqq V_s=V_c+r^eF$. Thus there are $v\in V_c$ and $f\in F$ such that $g=v+r^ef$. By substituting 0 for all variables outside $Y\subseteqq\{x_1, x_2, \dots\}$ we get equality between the sum of terms involving variables from Y only in g and $v+r^ef$. Hence we can assume that every term in v and f involves precisely $x_1, \dots, x_n$. Thus $f\in F^n\subseteqq F^e$. But $V_c\geqq r^eF^e$. Thus $r^ef\in V_c$, and $g=v+r^ef\in V_c$.

Claim 15: $\mathscr{C}(0, d, e)$ satisfies $x_1\dots x_e+f(x_1, \dots, x_e)=0$ where $f\in F^{e+1}$.

By Lemma 5 $\mathscr{C}(r^e, d, e)$ satisfies $x_1\dots x_e+f(x_1, \dots, x_e)=0$ since every nilpotent member of $\mathscr{C}(r^e, d, e)$ satisfies $x_1\dots x_e=0$. By Claim 14, $\mathscr{C}(c, d, e)$ satisfies $x_1\dots x_e+ +f(x_1, \dots, x_e)=0$. Hence Claim 15 follows from Claim 12.

Var $(\mathscr{C}(0, d, e))$ satisfies $x_1\dots x_e+f=0$, $f\in F^{e+1}$. Hence every nilpotent member of Var $(\mathscr{C}(0, d, e))$ satisfies $x_1\dots x_e=0$. If A is primitive and not nilpotent, and $A\in$Var$(\mathscr{C}(0, d, e))$, then $A\in\mathscr{D}(0, d, e)$. Hence A is a finite simple ring of order dividing d. Thus

$$\mathrm{Var}\,(\mathscr{C}(0, d, e)) \leqq \mathscr{C}(0, d, e)).$$

But $\mathrm{Var}\,(\mathscr{C}(0, d, e)) = \bigvee\{\mathscr{C}(c, d, e) : c > 1\}$. This concludes the proof of Theorem 16.

9. The following is a generalization of KRUSE's theorem [9] that the identities of a finite ring are finitely based.

Theorem 17. *If $\mathscr{V} \leqq \mathscr{C}(0, d, e)$ then the identities of $\mathscr{V}$ are finitely based.*

Denote by F_k the free associative ring on $\{x_1, \ldots, x_k\}$. $\mathscr{V}^{(k)}$ the variety defined by the k-variable identities of V is finitely based for any k. Since $F_k \cap V$ — the T-ideal of V in F_k — determines the variety $\mathscr{V}^{(k)}$ $F_k/F_k \cap V = F_k(V)$ is nilpotent by finite. Hence $V \cap F_k \leqq I$ for some ideal I of F_k of finite index and $I/V \cap F_k$ is nilpotent. By LEWIN's result [10], I is finitely generated, and so $F_k \cap V$ is finitely generated.

$\mathscr{W} = \mathscr{V} \cap \langle r^e x \rangle$ is a subvariety of $\mathscr{C}(r^e, d, e)$. Hence by [7] $\mathscr{W}$ is generated by a finite ring and by KRUSE's theorem [9], $\mathscr{W}$ is finitely based; i.e., $W = V + r^e F$ is finitely generated as a T-ideal. Thus all identities of $\mathscr{W}$ are consequences of $r^e x, v_1, \ldots, v_n$ where $v_1, \ldots, v_n$ can be chosen in V. Let $v \in V$ involve precisely $x_1, \ldots, x_m$, $m \geqq e$. Then $v \equiv r^e f + w$ where $f \in F$ and w is a consequence of $v_1, \ldots, v_n$. Comparing the terms involving the same set of variables, we get

$$v \equiv r^e f' + w', \quad 0 \equiv r^e f'' + w''$$

where f', w' are the sums of all terms of f and w involving precisely $x_1, \ldots, x_m$, $f'' \equiv f - f'$, $w'' \equiv w - w'$. But w', w'' are also consequences of $v_1, \ldots, v_n$ and $r^e f' \in V$ since $f' \in F^e$. $r^e f'' \equiv -w''$. Hence $v \equiv r^e f' - w'' + w$. Thus v is a consequence of $r^e x_1 \ldots x_e$, $v_1, \ldots, v_n$. Thus $V \cap F_{e-1} \cup \{r^e x_1 \ldots x_e, v_1, \ldots, v_n\}$ is a basis for V. Hence $\mathscr{V}$ is finitely based.

Theorem 18. *The following conditions on a variety $\mathscr{V}$ are equivalent:*

1) *$\mathscr{V} \leqq \mathscr{C}(0, d, e)$ for some positive integers d, e.*

2) *$\mathscr{V}$ satisfies $x_1 \ldots x_n + f(x_1, \ldots, x_n) = 0$ for some $f \in F^{n+1}$ and some $n > 0$.*

3) *the Jacobson radical of every member is nilpotent.*

We have established that 1)$\Rightarrow$2)$\Rightarrow$3). If $\mathscr{V}$ satisfies 3), then by Theorem 1, $\mathscr{V}$ is locally nilpotent by finite. Thus by Theorem 15, $\mathscr{V} \leqq \mathscr{D}(0, d, e)$ for some positive integers d and e. By Claim 8, $\mathscr{V} \leqq \langle x^e \rangle \cdot \mathscr{C}(r, d, d)$. Let A be the free ring of $\mathscr{V}$ of rank ω. Then $A/\mathrm{Rad}\, A \in \mathscr{C}(r, d, d)$ and $\mathrm{Rad}\, A$ is nilpotent say $(\mathrm{Rad}\, A)^m = 0$. Thus $A \in \langle x_1 \ldots x_m \rangle \cdot \mathscr{C}(r, d, d)$. Since A generates $\mathscr{V}$

$$\mathscr{V} \leqq \langle x_1 \ldots x_m \rangle \cdot \mathscr{C}(r, d, d) \leqq \mathscr{C}(0, d, md).$$

Corollary 1. *If a variety satisfies $x_1 \ldots x_n + f(x_1, \ldots, x_n) = 0$, where $f \in F^{n+1}$ and $n > 0$, then it is finitely based.*

Corollary 2. *A variety is generated by a finite ring iff it satisfies* $cx=0$ *and* $x_1 \dots x_n + f(x_1, \dots, x_n) = 0$ *for some* $n>0$, $c>0$, *and* $f \in F^{n+1}$.

This is immediate from Theorem 18 and the observation that a finite ring belongs to $\mathscr{C}(c, d, e)$ for some $c, d, e>0$.

10. We now come to the condition that the Jacobson radical be a direct summand.

Theorem 19. *The following conditions on a ring variety are equivalent:*
1) *the Jacobson radical of every finitely generated member is a direct summand;*
2) *the Jacobson radical of every member generated by two elements is a direct summand.*

Proof. It is obvious that 1)⇒2).

If $\mathscr{V}$ is a variety such that $V(x\mathbf{Z}[x]) \leqq px\mathbf{Z}[x]$, for some prime p, then $\mathscr{V}$ contains $x\mathbf{Z}_p[x]$ and hence all commutative rings of characteristic p. The ring $\{(x, y): x, y \in \mathbf{Z}_p\}$ with component-wise addition and $(x, y)(z, t) = (xt+yz, yt)$ is commutative of characteristic p; its Jacobson radical is $\{(x, 0): x \in \mathbf{Z}_p\}$, and the radical is not a direct summand. However, this ring is generated by (1, 1). Thus, if $\mathscr{V}$ satisfies 2), $V(x\mathbf{Z}[x])$ is not contained in $px\mathbf{Z}[x]$ for any prime p. By Lemma 3, $\mathscr{V}$ satisfies $x^n + x^{n+1}f(x) = 0$, and hence by Theorem 1, the radical of every member of $\mathscr{V}$ is nil; moreover, it satisfies $x^n=0$. Let A be a finitely generated member of $\mathscr{V}$, and let R be the Jacobson radical of A. Hence A/R is a finite semisimple ring. So A/R has 1. As R is nil, 1 can be lifted to an idempotent $c \in A$. Hence $A = cA + R$. Let $b \in R$. Then the subring B generated by b and c belongs to $\mathscr{V}$, and hence its radical C is a direct summand. But the radical C of B contains $R \cap B$. The projection of B onto C sends idempotents to 0. Thus $cb = bc = 0$. Hence $A(cA) = (cA+R)(cA) = (cA)(cA) + R(cA) = c(AcA) \subseteqq cA$. Hence cA is a two sided ideal of A. $R \cap cA = 0$ and $R + cA = A$. Hence $A = R \oplus cA$.

From Theorem 19, the condition 1), equivalent to 2), is equivalent to $\mathscr{V}^{(2)}$, the variety of all rings satisfying the two variable identities of $\mathscr{V}$, satisfies condition 1) of Theorem 19. But the identities of $\mathscr{V}^{(2)}$ are finitely based. Thus condition 2) is equivalent to a finite set of two variable identities. The following shows that this cannot be improved.

Theorem 20. *Let* $\mathscr{V}$ *be a ring variety for which the condition that every finitely generated member of* $\mathscr{V}$ *has the radical as a direct summand is equivalent to a set of one variable identities. Then* $\mathscr{V}$ *satisfies* $x^e=0$ *or* $x + x^2h(x) = 0$.

In the varieties $\langle x^e \rangle$, every ring is radical. In the varieties $\langle x + x^2h(x) \rangle$, the radical is 0.

If $\mathscr{V}$ is a variety in which the radical of every ring generated by one element is a direct summand, then $V(x\mathbf{Z}[x])$ is not containedi n $px\mathbf{Z}[x]$ for any prime p; otherwise, all commutative rings of characteristic p belong to $\mathscr{V}$. In the proof of Theorem 19, we have shown that there is a ring of characteristic p generated by one element and its radical is not a direct summand. Thus, by Lemma 3, $\mathscr{V}$ satisfies $x^e+x^{e+1}h(x)=0$. For some prime p, $\mathscr{V}$ contains the minimal variety $\langle px, xy\rangle$ [15]; otherwise, $\mathscr{V}$ is a variety not containing $\langle px, xy\rangle$ for any prime p; and by [6], this is equivalent to the validity of $x+x^2f(x)=0$ in $\mathscr{V}$ for some $f(x)\neq 0$. Also for some prime q, $\mathscr{V}$ contains the prime field of q elements $\mathbf{Z}_q$. This is true since by Theorem 15, $\mathscr{V}\leqq\mathscr{D}(0, d, e)$, where d is the least common multiple of all orders of finite simple nonnilpotent rings belonging to $\mathscr{V}$. If $\mathscr{V}$ does not contain any nonnilpotent simple finite ring, then $\mathscr{V}\leqq\mathscr{D}(0, 1, e)=\langle x^e\rangle$. Thus, if $\mathscr{V}$ does not satisfy $x^e=0$, then $\mathscr{V}$ contains a nonnilpotent finite simple ring whose center is $\mathbf{Z}_q$ for some prime q. The variety $\mathscr{U}=\langle qx, xy-x^qy, xy-xy^q\rangle$ is contained in $\langle qx, xy\rangle\vee\langle qx, x-x^q\rangle\leqq\mathscr{V}$. The one variable identities of $\mathscr{U}$ are all consequences of $qx=x^2-x^{q+1}=0$. The ring $\{(a, b)\colon a, b\in\mathbf{Z}_q\}$ with $(a, b)+(c, d)=(a+c, b+d)$, $(a, b)(c, d)=(ac, ad)$ satisfies $qx=0=x^2-x^{q+1}$. Its radical is $\{(0, b)\colon b\in\mathbf{Z}_p\}$, and the radical is not a direct summand. Thus, if $\mathscr{V}$ is a variety not satisfying $x^e=0$ or $x+x^2h(x)=0$ for any $e>0$ or $h(x)\neq 0$, then the condition that the radical of every finitely generated member is a direct summand is not equivalent to any set of one variable identities.

References

[1] G. Birkhoff, *Lattice Theory* (3rd ed), AMS Colloq. Publ. (Providence, 1967).
[2] P. M. Cohn, *Universal Algebra*, Harper & Row (New York, 1965).
[3] P. M. Cohn, *Free Rings and their Relations*, Academic Press (London and New York, 1971).
[4] C. J. Everett, An extension theory for rings, *Amer. J. Math.*, **64** (1942), 363—370.
[5] G. Grätzer, *Universal Algebra*, Van Nostrand (Princeton, N. J., 1968).
[6] A. A. Iskander, Product of ring varieties and attainability, *Trans. Amer. Math. Soc.*, **193** (1974), 231—238.
[7] A. A. Iskander, Locally finite ring varieties, *Proc. Amer. Math. Soc.*, to appear.
[8] I. Kaplansky, Rings with a polynomial identity, *Bull. Amer. Math. Soc.*, **54** (1948), 575—580.
[9] R. L. Kruse, Identities satisfied by a finite ring, *J. Algebra*, **26** (1973), 298—318.
[10] J. Lewin, Subrings of finite index in finitely generated rings, *J. Algebra*, **5** (1967), 84—88.
[11] A. I. Mal'cev, On class multiplication of algebraic systems, *Sibirsk. Math. Ž.*, **8**: 2 (1967), 346—355., (Russian).
[12] A. I. Mal'cev, *Algebraic Systems*, Nauka (Moscow, 1970) (Russian); English transl., Die Grundlehren der math. Wissenschaften, Band 192, Springer-Verlag (Berlin and New York, 1973).
[13] Hanna Neumann, *Varieties of Groups*, Ergebnisse der Math. und ihrer Grenzgebiete, vol. 37, Springer-Verlag (New York, 1967).
[14] J. M. Osborn, Varieties of algebras, *Advances in Math.*, **8** (1972), 163—369.
[15] A. Tarski, Equationally complete rings and relation algebras, *Nederl. Akad. Wetensch. Proc.* Ser. A. **59** = *Indag. Math.*, **18** (1956), 39—46.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SOUTHWESTERN LOUISIANA
LAFAYETTE, LOUISIANA 70501, USA