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A bound for the nilstufe of a group

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In this paper we are concerned solely with torsion-free abelian groups of finite rank. Such a group is said to have an (associative) multiplication defined on it if there is an (associative) ring with additive structure isomorphic to G. There may be many non-isomorphic rings all having isomorphic additive structure and most significantly a group may have associative and non-associative multiplications defined on it.

T. SZELE [5] defined $\nu(G)$, the nilstufe of G, to be the positive integer n such that there is an associative multiplication on G having a non-zero product of n group elements but there being no associative multiplication on G allowing a non-zero product of more than n group elements. If no such n exists then $\nu(G)=\infty$. Following FEIGELSTOCK [2] we define the strong nilstufe of G, $N(G)$, similarly but also considering non-associative multiplications on G. It will be seen later that the two invariants $\nu(G)$ and $N(G)$ are not necessarily equal.

The case where the rank of G, $r(G)$, is one is trivial, for all torsion-free rank one groups can be considered as subgroups of the rational numbers Q. As such they are either associative subrings of the rationals or do not admit non-trivial multiplication. Hence if $r(G)=1$ then $\nu(G)=N(G)=1$ or ∞. Other results concerning rank one groups are given in [2]. In the remainder of this paper we obtain useful bounds for $\nu(G)$ and $N(G)$ using well-known results on algebras of finite dimension.

Theorem. *If G is a torsion-free abelian group of finite rank $r(G)$, then*

(a) $\nu(G) \leqq r(G)$ *or* $\nu(G) = \infty$, (b) $N(G) \leqq 2^{r(G)-1}$ *or* $N(G) = \infty$.

To prove this result we require two lemmas concerning finite dimensional algeb-

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ras and a transition from torsion-free groups of finite rank to algebras of finite dimension. The following is a standard result and so no proof is given here.

Lemma 1. *If A is an associative algebra of finite dimension d over some field K such that, for some positive integer n, $A^n \neq 0$ and $A^{n+1}=0$ then n is at most d.*

The next lemma concerns non-associative algebras and we may no longer use A^n without ambiguity. Thus we define $A^{(n)}$ to be the subalgebra of A generated by all products of n elements of A. Clearly if A is associative then $A^{(n)}=A^n$.

For any algebra we define the associative subalgebra $E(A)$ of the endomorphism algebra of the K-module A^+ as being generated by all endomorphisms L_a, R_a over all a in A, where

$$L_a(x) = ax, \quad R_a(x) = xa \quad \text{for all} \quad x \text{ in } A.$$

Then we have the following sequence of submodules;

$$A \supseteqq AE(A) \supseteqq AE(A)^2 \supseteqq \cdots \supseteqq AE(A)^r \supseteqq \cdots.$$

But if we know that all products of $n+1$ elements of A are zero then $E(A)^n=0$ and the sequence above becomes;

$$\text{(I)} \qquad A \supseteqq AE(A) \supseteqq AE(A)^2 \supseteqq \cdots \supseteqq AE(A)^{n-1} \supset 0$$

if we suppose that $E(A)^{n-1} \neq 0$. If further we suppose that for some integer k, $1 \leqq k \leqq n-1$, we have $AE(A)^k = AE(A)^{k+1}$ then;

$$AE(A)^{k+1} = AE(A)^k E(A) = AE(A)^{k+1} E(A) = AE(A)^{k+2}.$$

Thus ultimately we get that $AE(A)^k = AE(A)^n = 0$, which is a contradiction since $k<n$. So the sequence (I) strictly decreases to zero. By considering the dimension of A we obtain that the length of the sequence, n, is at most the dimension of A, d.

Lemma 2. *For any finite dimensional algebra A over the field K we have $A^{(k)} \subseteqq AE(A)^n$ for all integers $k > 2^{n-1}$.*

Proof. If $n=1$ then $2^{n-1}=1$ and trivially $A^{(k)} \subseteqq AE(A)$ for all $k>1$. Let $n>1$ and proceed by induction on n. Take x in A to be a product of $k>2^{n-1}$ elements of A. Then $x=u\cdot v$ where at least one of u or v is a product of at least $2^{n-2}+1$ elements, u say. Then by hypothesis u is in $AE(A)^{n-1}$ and $u\cdot v$ is in $AE(A)^n$, proving the lemma.

Corollary. *If $E(A)^n=0$ for some integer n, then $A^{(k)}=0$ for all $k>2^{n-1}$.*

Recall that we saw above that if an integer n exists such that $E(A)^n=0$ then n is at most the dimension of A.

We now perform the promised transition from groups to algebras. This is done by noting that any (associative) multiplication on the group G induces an (associative) algebra structure on $A=Q^{+}\otimes G$ over Q. It is easy to verify that

(1) $A^{(n)}=0$ if and only if $G^{(n)}=0$.

(2) The dimension of A over Q is equal to the rank of G.

Proof of Theorem.

(a) We are dealing only with associative multiplications on G hence $A=Q\otimes G$ is an associative algebra of dimension $r(G)$ over Q and so if $\nu(G)=n$ is finite, Lemma 1 applies to give that $n\leqq r(G)$.

(b) We now admit non-associative multiplications and if $N(G)$ is finite then $E(A)^{n}=0$. We conclude firstly that $n\leqq r(G)$ and secondly that, applying the Corollary to Lemma 2, $A^{(k)}=0$ for all integers k such that $k>2^{n-1}$ which combined with (1) above gives $N(G)\leqq 2^{r(G)-1}$.

Thus the proof of the theorem is complete. It should be noted that the special case for G of rank two was obtained by FEIGELSTOCK [3] who seems to have overlooked that Lemma 1 of [3] drawn from BEAUMONT and WISNER [1] requires the ring to be associative, which in Theorem 1 of [3] it need not be.

Finally an example of a group G is given where $\nu(G)$ and $N(G)$ are not equal. Let $R<Q$ have type $(1, 0, 1, 1, 0, 1, \ldots)$ (for the definition of type see [4] from which the notation is borrowed), $S<Q$ have type $(2, 0, 2, 2, 0, 2, \ldots)$ and $T<Q$ have type $(\infty, 1, 4, \infty, 1, 4 \ldots)$. We recall that the type of a product is at least the product of the types. So if $\mathrm{t}(a)=(n_p)$, $\mathrm{t}(b)=(m_p)$ then $\mathrm{t}(a\cdot b)\geqq(n_p+m_p)$. Hence for any multiplication on $G=\mathrm{Ra}\oplus\mathrm{Sb}\oplus\mathrm{Tc}$ where a, b, c are linearly independent the type of each summand demands that;

$$a\cdot x\in Sb\oplus Tc \quad \text{for any} \quad x \text{ in } G, \quad b\cdot b\in Tc, \quad b\cdot c=c\cdot b=c\cdot c=0.$$

Hence both $\nu(G)$ and $N(G)$ are finite. Thus $\nu(G)\leqq 3$, $N(G)\leqq 4$ from the Theorem. The following table defines a non-associative multiplication on G in which the product $(a\cdot a)\cdot(a\cdot a)\neq 0$.

	a	b	c
a	b	c	c
b	0	c	0
c	c	0	0

Thus it can be seen that the bounds given by the Theorem are attained by at least one group.

References

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