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Every subring R of N with $A(\bar{D})\subset R$ is not adequate

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Let N (Nevanlinna class) be the ring of all functions of bounded characteristic and $A(\bar{D})$ (disc algebra) the subring of all holomorphic functions in the open unit disc D, which are continuously extendible to $\bar{D}$. (For details see [1], p. 16 ff and [3].)

In answering a question raised by Szűcs ([5], p. 201) E. A. Nordgren [2] showed that the ring N^+ is not adequate in showing (by an example) that there exists a finitely generated ideal, which is not principal. In his review [7] N. Yanagihara remarks that the same is true in the ring $F^+\cap N$ by the same construction. We remark that for other rings R (e.g. $R=N$) the construction does not carry over.

The purpose of this note is to give an example which works in any subring R of N with $R\supset A(\bar{D})$ and is for some reason simpler than the example in [2]. Also the construction works after appropriate modifications in other rings of holomorphic functions.

Theorem. *In every subring R of N with $A(\bar{D})\subset R$ there exists a finitely generated ideal which is not principal.*

Proof. Take $f_i(z)=(1-z)B_i(z)$ $(i=1, 2)$ with the Blaschke products

$$B_1(z) = \prod_{n=1}^{\infty} \frac{a_n-z}{1-a_n z} \quad \text{and} \quad B_2(z) = \prod_{n=1}^{\infty} \frac{b_n-z}{1-b_n z}$$

where $a_n=1-n^{-2}$, $b_n=a_n+\varepsilon_n$ and $\varepsilon_n>0$ is tending very rapidly to zero, e.g. $\varepsilon_n=n^{-2}\exp[-(1-a_n)^{-2}]$.

Clearly $f_1, f_2\in R$, since $f_1, f_2\in A(\bar{D})\subset R$. We claim that the ideal (f_1, f_2) is not principal. Assume the contrary, i.e. that there exist $d, g_1, g_2\in R$ such that $d=f_1g_1+f_2g_2$ and $(d)=(f_1, f_2)$. Then there exist $h_1, h_2\in R$ with $f_1=h_1d, f_2=h_2d$. Since B_1 and B_2 have no common zero and in view of the factorization theorem in N ([1], p. 25) there exist $d_1, d_2\in N$ such that $h_1=d_1B_1$, $h_2=d_2B_2$. This yields $d=d_1B_1dg_1+d_2B_2dg_2$ or $1=d_1B_1g_1+d_2B_2g_2$. For $z=a_m$ it follows $1=(d_2B_2g_2)(a_m)$

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or $|(g_2 d_2)(a_m)| = |B_2(a_m)|^{-1}$.

$$|B_2(a_m)| = \frac{b_m - a_m}{1 - b_m a_m}\left|\prod_{n\neq m} \frac{b_n - a_m}{1 - b_n a_m}\right| < \frac{\varepsilon_m}{1 - a_m} = m^2 \varepsilon_m = \exp[-(1 - a_m)^{-2}].$$

It follows $|(g_2 d_2)(a_m)| > \exp[(1 - a_m)^{-2}]$. But this is a contradiction to the fact that every function $f \in N$ (here $f = g_2 d_2$) fulfills $|f(z)| \leqq \exp[C(1 - |z|)^{-1}]$ for some constant $C > 0$ (see [3], p. 57).

Remark. The idea behind the proof is not new (it seems that WHITTAKER [6], p. 256 was the originator) and has the advantage to carry over to other rings of holomorphic functions restricted by a growth condition and with some type of canonical factorization, for example the Hadamard-Weierstrass factorization in the ring of all entire functions of exponential type (see [4], p. 10 for an analogous construction).

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