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Mean Ergodic Theorem in reflexive spaces

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The mean ergodic theorem proved by LORCH [4] states that if T is a linear operator on a reflexive Banach space X with $\|T\| \leqq 1$ then

$$\frac{1}{n}(I+T+T^2+\ldots+T^{n-1})x \to Px, \tag{1}$$

for each $x \in X$, P being a projection onto the subspace $\{x \in X: Tx = x\}$. BLUM and others in a series of papers [1, 2, 3] studied the question of the convergence

$$\frac{1}{n}(T^{k_1}+T^{k_2}+\ldots+T^{k_n})x \to Px, \tag{2}$$

where (k_n) is a given subsequence of the positive integers and X is a Hilbert space. The definitive result due to these authors is that if X is a Hilbert space and $\|T\| \leqq 1$ then (2) holds for each $x \in X$ if for each z on the unit circle it is true that

$$\frac{1}{n}(z^{k_1}+z^{k_2}+\ldots+z^{k_n})(1-z) \to 0. \tag{3}$$

This result is the best possible in the sense that if (2) holds for each contraction T then (3) must follow. The methods used to prove these results depend heavily on the Hilbert space structure and do not apply in the case where X is not a Hilbert space. We prove below a theorem which enables us to obtain a condition on the subsequence (k_n) which is sufficient for the truth of (2) where T now acts on any reflexive Banach space. Since it involves no additional effort we have stated our theorem for a sequence of polynomials more general than the one appearing in (3).

Theorem. *Let X be a reflexive Banach space, T a linear contraction on X, $(p_k)_1^\infty$ a sequence of complex polynomials and* $q(z)=(z-\lambda_1)\ldots(z-\lambda_n)$, $\lambda_1=1$, $|\lambda_i|=1$,

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$1 \leqq i \leqq n$; $\lambda_i \neq \lambda_j$ if $i \neq j$. Suppose that

$$\text{(i)} \qquad p_k(1) \to 1,\ p_k(\lambda_i) \to 0 \,(2 \leqq i \leqq n), \quad \text{as} \quad k \to \infty,$$

$$\text{(ii)} \qquad \sup_k \|p_k(T)\| < \infty,$$

$$\text{(iii)} \qquad q(T)p_k(T)x \to 0 \quad \text{as} \quad k \to \infty,\ x \in X.$$

Then for each $x \in X$, $p_k(T)x \to Px$ where P is the bounded projection onto the subspace $\{x \in X: Tx = x\}$ such that the range of $I-P$ is the closure of the range of $I-T$.

Proof. In the following, for an operator S on a reflexive space B we will denote by $R(S)$ and $N(S)$ the closure of the range of S and the null space of S, respectively. We note the well-known result that if $\|S\| \leqq 1$, then

$$B = R(I-S) \oplus N(I-S). \tag{4}$$

We now claim that the following relations hold:

$$X = N(q(T)) \oplus R(q(T)), \tag{5}$$

and

$$(Rq(T)),\ N((T-\lambda_2 I)\dots(T-\lambda_n I)) \subseteqq R(I-T). \tag{6}$$

Assuming the truth of (5) and (6), we will prove the theorem.

First, the relation (5) implies that $p_k(T)x$ converges for each $x \in X$. This is so since for $x \in R(q(T))$ and $\varepsilon > 0$, $x = q(T)y + y'$ with $\|y'\| < \varepsilon$. By (iii) and (ii) we will then have that $p_k(T)x \to 0$. If $x \in N(q(T))$ then $x = x_1 + \dots + x_n$ with $Tx_i = \lambda_i x_i$, $(1 \leqq i \leqq n)$. Thus $p_k(T)x = p_k(\lambda_1)x_1 + \dots + p_k(\lambda_n)x_n$, and by the relations in (i), the sequence $p_k(T)x$ converges to x_1.

Next, if we also have the relation (6), then noting that $N(q(T)) = N(I-T) \oplus \oplus N((T-\lambda_2 I)\dots(T-\lambda_n I))$ we have in view of the decomposition (4) that $p_k(T)x \to Px$ where P is as in the statement of the theorem.

We will now prove by induction on n that

$$X = N(I-T) \oplus \dots \oplus N(I-\bar{\lambda}_n T) \oplus Y, \tag{7}$$

where $\overline{(I-T)Y} = \dots = \overline{(I-\bar{\lambda}_n T)Y} = Y$. This surely implies (5) and (6).

Let us suppose that for $n-1$ there exists such a $Y = Y_{n-1}$. This Y_{n-1} is necessarily invariant under T, and by (4), we have

$$Y_{n-1} = R(I-\bar{\lambda}_n T | Y_{n-1}) \oplus N(I-\bar{\lambda}_n T | Y_{n-1}).$$

Now it is immediate that $N(I-\bar{\lambda}_n T) \subseteqq Y_{n-1}$, thus $N-(I-\bar{\lambda}_n T | Y_{n-1}) = N(I-\bar{\lambda}_n T)$ and we only have to show that for $Y_n = R(I-\bar{\lambda}_n T | Y_{n-1})$ we have $\overline{(I-T)Y_n} = \dots = \overline{(I-\bar{\lambda}_n T)Y_n} = Y_n$. The last equality is immediate, the others follow from the corresponding equalities for Y_{n-1}, from the fact that $N(I-\bar{\lambda}_n T)$ is invariant under

T and from the boundedness of the projections defined by the decomposition of Y_{n-1}. Thus the proof of (7) and therefore that of the theorem are complete.

The following corollaries now follow directly from the theorem. These corollaries are stated in such a way that the conditions on the operator T and the sequence (p_k) are independent of each other.

For $p(z)=\sum_0^N a_n z^n$, set $\|p\|_A=\sum_0^N |a_n|$ and $\|p\|_\infty=\sup\{|p(z)|: |z|\leqq 1\}$.

Corollary 1. *Let X be a reflexive Banach space and T a linear contraction on X. Let (p_k), q be as in the theorem and suppose that the relations* (i) *of the theorem hold. Suppose further that*

(ii)′ $$\sup_k \|p_k\|_A < \infty,$$

(iii)′ $$\|qp_k\|_A \to 0 \quad as \quad k \to \infty.$$

Then $p_k(T)x \to Px$ $(x\in X)$ where P is as in the theorem.

Corollary 2. *Let X be a reflexive Banach space and T a linear operator on X such that for every polynomial p, $\|p(T)\|\leqq\|p\|_\infty$. Let (p_k), q be as in the theorem and suppose that the relations* (i) *of the theorem hold. Suppose further that*

(ii)″ $$\sup \|p_k\|_\infty < \infty,$$

(iii)″ $$\|qp_k\|_\infty \to 0 \quad as \quad k \to \infty.$$

Then $p_k(T)x \to Px$ $(x\in X)$ where P is as in the theorem.

We now return to the problem discussed in the introduction. Let (k_n) be a subsequence of the positive integers satisfying (3) and take $p_n(z)=\frac{1}{n}(z^{k_1}+\ldots+z^{k_n})$, $q(z)=z^\nu-1$, ν a positive integer. Then all the conditions except (iii)′ of Corollary 1 are satisfied. The condition (iii)′ will also be fulfilled if

(8) $$\lim_{n\to\infty}\frac{1}{N}\operatorname{card}\big(E_N\cap(E_N+\nu)\big)=1,$$

where $E_N=\{k_1, \ldots, k_N\}$ and $E_N+\nu$ is the translate of E_N by ν. We can therefore conclude that for a linear contraction T on a reflexive space X if a sequence (k_n) satisfies (3) then the condition (8) is sufficient for the convergence of (2). The example in [2], p. 428 is of a sequence (k_n) satisfying (3) and (8) with $\nu=2$.

We note that any linear contraction T on a Hilbert space satisfies the hypothesis (on T) of Corollary 2. However, as shown in [3], the conclusion of Corollary 2 holds under weaker hypothesis on (p_n). Thus the Corollary 2 has significance only when the reflexive space X is not a Hilbert space.

References

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