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Differentiability for Rademacher series on groups

C. W. ONNEWEER

In the paper [1] P. L. BUTZER and H. J. WAGNER defined the derivative of real-valued functions f defined on the dyadic group, both in the pointwise sense and in the strong sense, that is, with respect to the norm of the space to which f belongs. They proved that this derivative has many properties similar to properties of the ordinary derivative of functions on the circle group. In the present paper we shall extend this definition to functions defined on groups G that are the direct product of countably many groups of prime order. Furthermore, we shall give some applications to functions that are defined as the sum of a Rademacher series on G.

1. Introduction

Let $\{p_n\}$ be a sequence of prime numbers and let G be the direct product of groups of order p_n, that is, $G=\prod_{n=1}^{\infty} Z(p_n)$. Thus the elements of G are of the form $x=(x_1, x_2, \ldots)$, with $0 \leqq x_n < p_n$ for each $n \geqq 1$ and for x, y in G the n-th coordinate of their sum $x+y$ is obtained by adding the n-th coordinates of x and y modulo p_n. Furthermore, if we define the subgroups G_n of G by $G_0=G$ and for $n \geqq 1$

$$G_n = \{x \in G;\ x_1 = \ldots = x_n = 0\},$$

then the G_n's form a basis for the neighborhoods of $0=(0, 0, \ldots)$ in G. Finally, for $n \geqq 1$ we define the elements e_n of G by $(e_n)_i=0$ if $i \neq n$ and $(e_n)_n=1$.

Next, let $\hat{G}$ denote the character group of G. We enumerate the elements of $\hat{G}$ as follows. For each $k \geqq 0$ and each x in G let $\varphi_k(x)$ be defined by

$$\varphi_k(x) = \exp(2\pi i x_{k+1}/p_{k+1}).$$

Thus, $\varphi_k(e_j)=1$ if $j \neq k+1$ and $\varphi_k(e_{k+1})=\exp(2\pi i/p_{k+1})=\omega_k$. We observe here

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that for each $j \geqq 0$

$$\sum_{l=0}^{p_{j+1}-1} (\omega_j)^{lk} = \begin{cases} 0 & \text{if } 0 < k < p_{j+1}, \\ p_{j+1} & \text{if } k = 0. \end{cases} \tag{1}$$

Let the sequence $\{m_n\}$ be defined by $m_0=1$ and $m_n=p_n \cdot m_{n-1}$ for $n \geqq 1$. Next, if $n \geqq 0$ is represented as $n=a_0m_0+\ldots+a_km_k$, with $0 \leqq a_i < p_{i+1}$ for each $i \geqq 0$, then we define χ_n by

$$\chi_n(x) = \varphi_0^{a_0}(x) \cdot \ldots \cdot \varphi_k^{a_k}(x) = \prod_{v=0}^{k} \exp(2\pi i a_v x_{v+1}/p_{v+1}). \tag{2}$$

The χ_n's are precisely the elements of $\hat{G}$. The functions φ_n are called the Rademacher functions on G and the χ_n are called the generalized Walsh functions on G.

Remark 1. If $p_n=2$ for all n, then G is the so-called dyadic group. The elements of the character group $\hat{G}$, when ordered as indicated here, are the Walsh (-Paley) functions, see [3].

Let dx or m denote normalized Haar measure on G. For f in $L_1(G)$ we define its generalized Walsh-Fourier series by

$$\sum_{k=0}^{\infty} \hat{f}(k)\chi_k(x), \quad \text{where} \quad \hat{f}(k) = \int_G f(t)\overline{\chi_k(t)}\,dt.$$

In a number of previous papers, [4] and [5], we have studied several properties of such generalized Walsh-Fourier series. Among other things we defined the concept of r-generalized bounded fluctuation. We recall the definition here. For each subgroup G_n of G we denote the m_n cosets of G_n in G by $z_{q,n}+G_n$, $q=0, 1, \ldots, m_n-1$, with $z_{0,n}+G_n=G_n$. If f is a function on G and if $H \subset G$ then

$$\operatorname{osc}(f;\, H) = \sup\{|f(x)-f(y)|\,;\, x, y \in H\}.$$

Definition 1. Let f be a function on G, r a real number with $r \geqq 1$, and

$$V_r(f) = \sup\left\{\left\{\sum_{q=0}^{m_n-1} (\operatorname{osc}(f;\, z_{q,n}+G_n))^r\right\}^{1/r};\ n = 0, 1, \ldots\right\}.$$

The function f is of *r-generalized bounded fluctuation* ($f \in r$-GBF) if $V_r(f) < \infty$.

In [6] and [5] it was shown that functions in r-GBF have many properties similar to properties of functions of r-bounded variation (r-BV) on the circle group T. However, we shall show that the differentiability properties of functions in GBF, that is, in 1-GBF, are unlike those of functions in BV.

2. Differentiation of functions on G

Definition 2. If for a complex-valued function f on G and for x in G

$$\lim_{m\to\infty} \sum_{j=0}^{m} m_j \sum_{k=0}^{p_{j+1}-1} kp_{j+1}^{-1} \sum_{l=0}^{p_{j+1}-1} (\omega_j)^{-lk} f(x+le_{j+1})$$

exists, then we call this limit the *pointwise derivative* of f at x, denoted by $f^{[1]}(x)$.

Definition 3. Let $X(G)$ denote either $C(G)$ or $L_p(G)$, $1 \leqq p < \infty$, with the usual norm. If for f in $X(G)$ there exists a g in $X(G)$ such that

$$\lim_{m\to\infty} \left\| \sum_{j=0}^{m} m_j \sum_{k=0}^{p_{j+1}-1} kp_{j+1}^{-1} \sum_{l=0}^{p_{j+1}-1} (\omega_j)^{-lk} f(\cdot+le_{j+1}) - g(\cdot) \right\|_{X(G)} = 0,$$

then g is the *strong derivative* of f, denoted by $D^{[1]}f$.

Higher order derivatives are defined recursively. If $p_n=2$ for all n then these definitions agree with the definitions of Butzer and Wagner [1]. These authors showed that the Walsh functions $\chi_n(x)$ have the property that $D^{[1]}\chi_n = n\chi_n$ in each space $X(G)$ and $\chi_n^{[1]}(x) = n\chi_n(x)$ for all x in G. Further results in [1] are largely based on these identities. Therefore we shall prove that the derivatives as presently defined for functions on G satisfy the same identities, after which it is easy to extend most of the results in [1] to functions on G.

Remark. We would like to thank the referee for bringing the paper by Gibbs and Ireland [4] to our attention. In it the authors define the derivative for functions on groups G which are the direct product of finitely many cyclic groups. Their definition closely resembles our Definition 2, see [4, Section VI].

Theorem 1. *For each $n \geqq 0$ and each x in G we have $\chi_n^{[1]}(x) = n\chi_n(x)$.*

Proof. Since $\chi_0(x) \equiv 1$, the theorem is clearly true for $\chi_0(x)$.

Assume $n = a_0m_0 + \ldots + a_rm_r$, with $0 \leqq a_i < p_{i+1}$ for each $i \geqq 0$ and $a_r \neq 0$. Take a fixed j with $0 \leqq j \leqq r$. Then

$$m_j \sum_{k=0}^{p_{j+1}-1} kp_{j+1}^{-1} \sum_{l=0}^{p_{j+1}-1} (\omega_j)^{-lk} \chi_n(x+le_{j+1}) =$$

$$= m_j \sum_{k=0}^{p_{j+1}-1} kp_{j+1}^{-1} \sum_{l=0}^{p_{j+1}-1} (\omega_j)^{-lk} \chi_n(x)(\chi_n(e_{j+1}))^l =$$

$$= m_j \chi_n(x) \sum_{k=0}^{p_{j+1}-1} kp_{j+1}^{-1} \sum_{l=0}^{p_{j+1}-1} (\omega_j)^{l(a_j-k)},$$

according to (2). Using (1) we see that this expression can be simplified further into

$$m_j \chi_n(x) a_j p_{j+1}^{-1} p_{j+1} = a_j m_j \chi_n(x).$$

Next, for each $j > r$ we have

$$m_j \sum_{k=0}^{p_{j+1}-1} k p_{j+1}^{-1} \sum_{l=0}^{p_{j+1}-1} (\omega_j)^{-lk} \chi_n(x+le_{j+1}) = m_j \chi_n(x) \sum_{k=0}^{p_{j+1}-1} k p_{j+1}^{-1} \sum_{l=0}^{p_{j+1}-1} (\omega_j)^{-lk} = 0.$$

Therefore,

$$\chi_n^{[1]}(x) = \sum_{j=0}^{r} a_j m_j \chi_n(x) = n \chi_n(x).$$

It is clear that a similar result holds for the strong derivative of χ_n in each of the spaces $X(G)$.

3. Rademacher series on G

In this section we shall consider Rademacher series on G, that is, functions defined by a series $R(x) = \sum_{i=0}^{\infty} c_i \varphi_i(x)$. We shall assume that c_k is real for each $k \geqq 0$ and that $R(x)$ exists for all x in G. The last assumption is equivalent to the condition that $\sum_{i=0}^{\infty} |c_i| < \infty$, as can be seen as follows. Define the element $x = (x_1, x_2, \dots)$ in G by $x_{i+1} = 0$ if $c_i \geqq 0$ and $x_{i+1} = 1$ if $c_i < 0$ and $p_{i+1} = 2$, whereas $x_{i+1} = (p_{i+1} - 1)/2$ if $c_i < 0$ and $p_{i+1} \neq 2$. Then $\varphi_i(x) = 1$ if $c_i \geqq 0$ and $\operatorname{Re}[\varphi_i(x)] \leqq -1/2$ if $c_i < 0$. Consequently, for all $i \geqq 0$ we have $c_i \operatorname{Re}[\varphi_i(x)] \geqq |c_i|/2$ and this shows that $\sum_{i=0}^{\infty} |c_i| < \infty$. We also observe that for Rademacher series on G the following proposition holds. Its proof is similar to the proof for the case of Rademacher series on the dyadic group [7, page 212] and will not be given here.

Proposition 1. *If $R(x) = \sum_{i=0}^{\infty} c_i \varphi_i(x)$ is a Rademacher series on G then* (i) *if $\sum_{i=0}^{\infty} |c_i|^2 < \infty$, then $R(x)$ converges a.e.,* (ii) *if $\sum_{i=0}^{\infty} |c_i|^2 = \infty$, then $R(x)$ diverges a.e.*

Now we turn to the differentiability of such Rademacher series.

Theorem 2. *R is differentiable at a point x in G if and only if $\sum_{k=0}^{\infty} m_k c_k \varphi_k(x)$ converges.*

Proof. For each $j \geqq 0$ and each l with $0 \leqq l < p_{j+1}$ we have

$$R(x+le_{j+1}) = \sum_{i=0}^{\infty} c_i \varphi_i(x+le_{j+1}) = \sum_{i=0}^{j-1} c_i \varphi_i(x) + c_j \varphi_j(x)(\omega_j)^l + \sum_{i=j+1}^{\infty} c_i \varphi_i(x).$$

Hence, for each $j \geqq 0$ we find, using (1), that

$$m_j \sum_{k=0}^{p_{j+1}-1} k p_{j+1}^{-1} \sum_{l=0}^{p_{j+1}-1} (\omega_j)^{-lk} R(x+le_{j+1}) =$$

$$= m_j \sum_{k=0}^{p_{j+1}-1} k p_{j+1}^{-1} \sum_{l=0}^{p_{j+1}-1} (\omega_j)^{-lk} c_j \varphi_j(x) (\omega_j)^l = m_j c_j \varphi_j(x) p_{j+1}^{-1} p_{j+1} = m_j c_j \varphi_j(x).$$

Consequently, R is differentiable at x if and only if $\sum_{j=0}^{\infty} m_j c_j \varphi_j(x)$ converges.

We now mention a number of corollaries of Theorem 2. For a Rademacher series R let

$$\Delta_R = \{x \in G;\ R \text{ differentiable at } x\}.$$

Corollary 1. *If for some x in G we have $x \in \Delta_R$ and if y is a rational element of G, that is, y has the property that there exists a constant K so that $y_k = 0$ for $k > K$, then $x + y \in \Delta_R$.*

Proof. Since for $j \geqq K$ we have $\varphi_j(y) = 1$, we see that

$$\sum_{j=0}^{\infty} m_j c_j \varphi_j(x+y) = \sum_{j=0}^{K-1} m_j c_j \varphi_j(x+y) + \sum_{j=K}^{\infty} m_j c_j \varphi_j(x).$$

Hence, if $x \in \Delta_R$, then Theorem 1 implies that $x + y \in \Delta_R$.

Because the rational elements of G are dense in G we have

Corollary 2. *If Δ_R is not empty, then Δ_R is dense in G.*

In view of Proposition 1 we have

Corollary 3. *For each Rademacher series R we have $m(\Delta_R) = 0$ or $m(\Delta_R) = 1$.*

An argument similar to the one in the beginning of this section shows the following.

Corollary 4. *R is differentiable for all x in G if and only if $\sum_{j=0}^{\infty} m_j |c_j| < \infty$.*

Finally we give the analogue on G of the well-known example of Weierstrass of a continuous nowhere differentiable function on T, namely $f(x) = \sum_{n=0}^{\infty} 2^{-n} \cos 2^n x$.

Corollary 5. *There exists a continuous nowhere differentiable function on G.*

Proof. Let $R(x) = \sum_{k=0}^{\infty} (m_k)^{-1} \varphi_k(x)$. Clearly, R is continuous on G and, according to Theorem 1, R is differentiable at x if and only if $\sum_{k=0}^{\infty} \varphi_k(x)$ converges. Hence, Δ_R is the empty set.

As mentioned earlier, the functions in GBF on G have many properties in common with the functions in BV on T. However, we shall now show that this is not the case with the differentiability property.

Theorem 3. (a) *If R is differentiable at a point x in G then $R\in r$-GBF for all $r\geqq 1$.* (b) *There exists a function R in GBF for which $m(\Delta_R)=0$.*

Proof. (a) Consider a fixed coset $z_{q,n}+G_n$ of G. Since R is continuous on G there are points x, y in $z_{q,n}+G_n$ such that

$$\operatorname{osc}(R;\ z_{q,n}+G_n) = R(x)-R(y) = \sum_{i=0}^{\infty} c_i(\varphi_i(x)-\varphi_i(y)).$$

Since $x_i=y_i$ for $1\leqq i\leqq n$, we have $\varphi_i(x)=\varphi_i(y)$ for $1\leqq i\leqq n$; also, $\varphi_0(x)=\varphi_0(y)=1$. Therefore,

$$\operatorname{osc}(R;\ z_{q,n}+G_n) = \left|\sum_{i=n+1}^{\infty} c_i(\varphi_i(x)-\varphi_i(y))\right| \leqq 2\sum_{i=n+1}^{\infty}|c_i|.$$

Hence,

$$\left\{\sum_{q=0}^{m_n-1}(\operatorname{osc}(R;\ z_{q,n}+G_n))^r\right\}^{1/r} \leqq \left\{m_n\left(2\sum_{i=n+1}^{\infty}|c_i|\right)^r\right\}^{1/r} \leqq 2(m_n)^{1/r}\sum_{i=n+1}^{\infty}|c_i|.$$

Next, if $R^{[1]}(x)$ exists for at least one x in G, then Theorem 2 implies that there exists a natural number K such that for all $i>K$ we have $|c_i|<(m_i)^{-1}$. Hence, if $n\geqq K$, then

$$2(m_n)^{1/r}\sum_{i=n+1}^{\infty}|c_i| \leqq 2(m_n)^{1/r}\sum_{i=n+1}^{\infty}(m_i)^{-1} \leqq 2(m_n)^{1/r}(m_n)^{-1}\sum_{k=1}^{\infty}2^{-k} = 2(m_n)^{(1-r)/r}.$$

Thus, $R\in r$-GBF if $r\geqq 1$.

(b) Let R be defined by

$$R(x) = \sum_{k=1}^{\infty}(-1)^k(k^{1/2}m_k)^{-1}\varphi_k(x).$$

According to Theorem 2, $R^{[1]}(x)$ exists if and only if $\sum_{k=1}^{\infty}(-1)^k k^{-1/2}\varphi_k(x)$ converges. Since $\varphi_k(0)=1$ for all $k\geqq 0$ we see that $R^{[1]}(0)$ exists and, hence, Theorem 3(a) implies that $R\in$GBF. However, it follows from Proposition 1(b) that $m(\Delta_R)=0$.

In case G is the dyadic group we have obtained some slightly stronger results than those of Theorem 3. Since this case is especially interesting we mention these results briefly.

Proposition 2. *If R is a Rademacher function on the dyadic group and if $r \geqq 1$ then*

$$V_r(R) = \sup\left\{2^{(n+r)/r} \sum_{i=n+1}^{\infty} |c_i|;\ n = 0, 1, \ldots\right\}.$$

Proof. Like in the proof of Theorem 3(a) we see that for each coset $z_{q,n}+G_n$ in G we have x, y in $z_{q,n}+G_n$ such that

$$\operatorname{osc}(R;\ z_{q,n}+G_n) = \left|\sum_{i=n+1}^{\infty} c_i(\varphi_i(x)-\varphi_i(y))\right|.$$

Now we observe that if G is the dyadic group we can find elements x and y in this coset so that for $i>n$ we have $x_i=0$ if $c_i \geqq 0$ and $x_i=1$ if $c_i<0$, whereas $y_i=1$ if $c_i \geqq 0$ and $y_i=0$ if $c_i<0$. For this choice of x and y we see that

$$\sum_{i=n+1}^{\infty} c_i(\varphi_i(x)-\varphi_i(y)) = 2\sum_{i=n+1}^{\infty} |c_i|.$$

The rest of the proof is obvious.

In [2, p. 323] J. E. Coury raised the question whether or not there exists a function on [0, 1) which can be expressed as a Rademacher series on [0, 1) and which is differentiable in the classical sense on an uncountable set of measure zero. Though we are unable to solve this problem we have obtained an affirmative answer in the present context of functions and their derivatives on the dyadic group.

Proposition 3. *There exists a Rademacher series on the dyadic group which is differentiable on an uncountable set of measure zero.*

Proof. Let $R(x)=\sum_{k=1}^{\infty} k^{-1/2}2^{-k}\varphi_k(x)$. Clearly, R is well-defined and it follows from Theorem 2 that $x\in\Delta_R$ if and only if $\sum_{k=1}^{\infty} k^{-1/2}\varphi_k(x)$ converges. So, Proposition 1(b) implies that $m(\Delta_R)=0$. Next, in order to show that Δ_R is uncountable we observe that for every real number α we can find a sequence $\{\alpha_n\}$ with $\alpha_n\in\{+1,-1\}$ for all $n\geqq 1$ and so that $\sum_{n=1}^{\infty} \alpha_n n^{-1/2}=\alpha$. Moreover, these sequences can be chosen so that if $\alpha\neq\beta$ then $\{\alpha_n\}\neq\{\beta_n\}$. Also, for every such sequence $\{\alpha_n\}$ there exists a uniquely determined x in the dyadic group such that $\varphi_n(x)=\alpha_n$ for all n. Hence, for each real number α we can find a corresponding x in the dyadic group for which $\sum_{k=1}^{\infty} k^{-1/2}\varphi_k(x)$ converges. This shows that Δ_R is an uncountable set.

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CATHOLIC UNIVERSITY
DEPARTMENT OF MATHEMATICS
TOERNOOIVELD
NIJMEGEN, THE NETHERLANDS

(presently:)

UNIVERSITY OF NEW MEXICO
DEPARTMENT OF MATHEMATICS
ALBUQUERQUE, NM 87131
USA