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The lattice of translations on a lattice

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1. Introduction and preliminaries. The purpose of this paper is to consider the lattice of all translations on a lattice and to illuminate the decomposition of lattices generated by translations on lattices. Also some properties of translations on meet-semilattices are given.

Let S be a meet-semilattice and φ a single-valued mapping of S into itself. φ is called a meet-translation, briefly a translation, on S, if $\varphi(x \wedge y)=\varphi(x) \wedge y$ for each pair x, y of elements in S. A translation φ on a lattice L is defined analogously. Each translation φ on S (and on L) has the following properties [7]: $\varphi(x) \leqq x$, $\varphi(x)=\varphi(\varphi(x))$, and $x \leqq y \Rightarrow \varphi(x) \leqq \varphi(y)$. In a lattice L the fixelements of φ, i.e. the elements $t=\varphi(t)$, constitute an ideal K_φ of L, which determines φ uniquely.

A non-empty subset J of a meet-semilattice S is called a semi-ideal of S, if (i) $a \leqq b$ and $b \in J$ imply $a \in J$, and (ii) $a, b \in J$ imply $a \vee b \in J$ whenever $a \vee b$ exists in S. As one can easily conclude from [7, Thm. 1], the fixelements of a translation φ on a meet-semilattice S form a semi-ideal K_φ of S, and K_φ determines φ uniquely [7, Thm. 3].

We denote by $\mathscr{I}(L)$ the lattice of all ideals of a lattice L, $(a]=\{x \mid x \leqq a,\, x, a \in S\}$ is the principal ideal generated by a. The semi-ideals of a meet-semilattice S constitute a lattice $\mathscr{J}(S)$ with respect to the set-theoretical inclusion; $I \vee J$ means the least semi-ideal containing I and J of $\mathscr{J}(S)$.

A translation $s_a(x)=a \wedge x$ is called a specified translation.

The following lemma was proved in [6]:

Lemma 1. *An ideal I of a lattice L generates a translation φ on L, i.e. $K_\varphi=I$, if and only if for each $y \in L$ there is an element $k_y \in I$ such that $I \wedge (y]=(k_y]$.*

A direct analogy holds for translations φ on a meet-semilattice S and semi-ideals J of S.

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2. Translations on a lattice. We denote by $\Phi(L)$ the set of all translations on L. As shown by SZÁSZ and SZENDREI [8, Thm. 3], $\Phi(L)$ is a meet-semilattice.

Theorem 1. *Let φ and λ be two translations on a lattice L. The mapping β on L, defined by $\beta(x)=\varphi(x)\vee\lambda(x)$, is a translation on L if and only if $(K_\varphi\vee K_\lambda)\wedge\wedge(x]=(K_\varphi\wedge(x])\vee(K_\lambda\wedge(x])$ for each $x\in L$.*

Proof. Let $(K_\varphi\vee K_\lambda)$ have the property of the theorem. Then $(K_\varphi\vee K_\lambda)\wedge\wedge(x]=(K_\varphi\wedge(x])\vee(K_\lambda\wedge(x])=(\varphi(x)]\vee(\lambda(x)]=(\varphi(x)\vee\lambda(x)]$, and so $K_\varphi\vee K_\lambda$ generates a translation on L with values $\varphi(x)\vee\lambda(x)$, i.e. $K_\varphi\vee K_\lambda$ generates a translation β on L. Conversely, let β be a translation on L. The fixelements of β are the elements $\varphi(x)\vee\lambda(x)$ $(x\in L)$, and so $K_\beta=K_\varphi\vee K_\lambda$. According to Lemma 1, $(\beta(x)]==(K_\varphi\vee K_\lambda)\wedge(x]=(\varphi(x)]\vee(\lambda(x)]=(K_\varphi\wedge(x])\vee(K_\lambda\wedge(x])$, and the latter part of the theorem follows.

Corollary 1. *Let φ be a translation on L. The mapping $\varphi\vee\lambda$ is a translation on L for each $\lambda\in\Phi(L)$ if and only if K_φ is a standard element of $\mathscr{I}(L)$.*

Proof. If K_φ is standard, then $(K_\varphi\vee K_\lambda)\wedge(x]=(K_\varphi\wedge(x])\vee(K_\varphi\wedge(x])$ for each $\lambda\in\Phi(L)$. Hence $\beta(x)=\varphi(x)\vee\lambda(x)$ is a translation on L. Conversely, if $\varphi\vee\lambda$ is a translation for each $\lambda\in\Phi(L)$, then, in particular the relation $((a]\vee K_\varphi)\wedge\wedge(x]=((a]\wedge(x])\vee(K_\varphi\wedge(x])$ holds for each specified translation s_a, $a\in L$, and for each $x\in L$. But already this equation implies the standardness of K_φ according to [1, Thm. 2(α'')].

Corollary 2. *The meet-semilattice $\Phi(L)$ is a lattice if and only if L is a distributive lattice.*

Proof. If L is a distributive lattice, each $I\in\mathscr{I}(L)$ is a standard element in $\mathscr{I}(L)$, and the first part of the assertion follows. Conversely, if $\Phi(L)$ is a lattice, then each ideal $(a]$ generating a specified translation s_a on L is a standard element of $\mathscr{I}(L)$, from which the distributivity of L follows.

Lemma 2. *$\Phi(L)$ contains always a greatest element ω, and there is a least elemetn τ in $\Phi(L)$ if and only if $0\in L$.*

Proof. The identical mapping $\omega(x)=x$ is a translation on L and $K_\omega=L$; evidently it is the greatest translation on L. The mapping $\tau(x)=0$ is obviously the least translation on L whenever a least element 0 exists in L, and $k_\tau=(0]$. If there is no least element in L, then there exists for each $a_1\in L$ an infinite chain $a_1>a_2>\dots$ and the corresponding specified translations form an infinitely descending chain, whence $\tau\notin\Phi(L)$.

In the following we shall consider a decomposition of a lattice by means of translations on this lattice. In [2] JANOWITZ considered the decomposition of a lattice into a direct sum; this decomposition is generalized for join-semilattices in [5]. Let L be a lattice with 0. $a \triangledown b$ denotes the fact that $a \wedge b = 0$ and $(a \vee x) \wedge b = x \wedge b$ for all $x \in L$. For a subset H of L we denote by $H^{\triangledown}$ the set of elements $a \in L$ such that $a \triangledown b$ for all $b \in H$. In a lattice L with 0, let $H_1, \dots, H_n$ be subsets of L, each of which contains 0. We say that L is the direct sum of $H_1, \dots, H_n$ and write $L = H_1 \oplus \dots \oplus H_n$ when

(1) every element $a \in L$ can be expressed in the form $a = a_1 \vee \dots \vee a_n$, $a_i \in H_i$, $i = 1, \dots, n$, and

(2) $H_i \subset H_j^{\triangledown}$ for $i \neq j$.

The subsets $H_1, \dots, H_n$ are called direct summands of L. If $L = H_1 \oplus \dots \oplus H_n$, then the expression in (1) unique and the sets $H_1, \dots, H_n$ are ideals of L [4, Lemma 4.8]. Moreover, in a lattice L with 0, an ideal J of L is a central element of $\mathscr{I}(L)$ if and only if it is a direct summand of L [2, Thm. 1]. Now we are able to prove a theorem on direct sums of a lattice.

Theorem 2. *A lattice L with 0 has a decomposition into non-trivial direct summands if and only if there are at least two non-trivial translations φ and λ on L such that $\varphi \vee \lambda = \omega$ and $\varphi \wedge \lambda = \tau$, and φ and λ have join with each translation on L.*

Proof. Let $L = J \oplus K$. According to [2, Thm. 1], J and K are standard elements of $\mathscr{I}(L)$, and $J \wedge K = (0]$ and $J \vee K = L$ in $\mathscr{I}(L)$. Consider the meet $J \wedge (x]$, $x \in L$. As $L = J \oplus K$, $x = a_1 \vee a_2$, $a_1 \in J$ and $a_2 \in K$, and the expression $x = a_1 \vee a_2$ is unique. So $J \wedge (x] = (a_1]$, $a_1 \in J$, and hence J generates a translation φ on L. As J is standard in $\mathscr{I}(L)$, the join $\varphi \vee \mu$ exists for each translation $\mu \in \Phi(L)$. Similar facts hold also for the translation λ on L generated by K. $\varphi \wedge \lambda$ corresponds to the translation generated by the ideal $J \wedge K = (0]$, i.e. τ, and $\varphi \vee \lambda$ that of $J \vee K = L$, i.e. ω. As $J, K \neq L, (0]$, φ and λ are non-trivial translations on L, and the first part of the theorem follows.

Conversely, let φ and λ be two translations with the properties given in the theorem. As $\varphi \vee \mu$ exists for each translation $\mu \in \Phi(L)$, the ideal J generating φ is a standard element of the lattice $\mathscr{I}(L)$ (by Corollary 1 to Theorem 1), and this holds also for the ideal K generating λ. As $\varphi \wedge \lambda = \tau$ and $\varphi \vee \lambda = \omega$, $J \wedge K = (0]$ and $J \vee K = L$, respectively. As J and K are standard and complements, they belong to the center of $\mathscr{I}(L)$ [3, Thm. 7.2] and, accordingly, $L = J \oplus K$ [2, Thm. 1]. As φ and λ are non-trivial, $J, K \neq L, (0]$, and the decomposition is also non-trivial.

3. Translations on partial lattices. We call a meet-semilattice S a partial lattice if $a \vee b$ exists for any two $a, b \in S$ having a common upper bound in S. At first we consider the structure of meet-semilattices S for which $\Phi(L)$ is a lattice.

Let $\varphi(x)$ and $\lambda(x)$ be translations on a partial lattice S. As in the case of lattices, one can show that $\beta(x)=\varphi(x)\vee\lambda(x)$ is a translation on S if and only if $(K_\varphi \vee K_\lambda)\wedge(x]=(K_\varphi\wedge(x])\vee(K_\lambda\wedge(x])$ for each $x\in S$, $K_\varphi, K_\lambda, (x]\in\mathscr{I}(S)$.

Theorem 3. *Let S be a partial lattice. Then the following three assumptions are equivalent:*

(i) *The meet-semilattice of all translations on S is a lattice.*

(ii) *Each translation on S is a join-endomorphism on S.*

(iii) $(x]$ *is a distributive sublattice of S for each $x\in S$.*

Proof. We shall show that (i)$\Leftrightarrow$(iii) and (ii)$\Leftrightarrow$(iii).

(iii)$\Rightarrow$(i). We shall show that $\mathscr{I}(S)$ is a distributive lattice, from which the validity of the assertion follows.

Let $I, J\in\mathscr{I}(S)$. $I\wedge J=I\cap J$, and $I\vee J=\{x|x\leqq i\vee j, i\in I, j\in J$ and $i\vee j\in S\}$. We must only show that $F\wedge(I\vee J)\subseteq(F\wedge I)\vee(F\wedge J)$ when $F, I, J\in\mathscr{I}(S)$. Clearly, $x\in F\wedge(I\vee J)\Leftrightarrow x\in F$ and $x\leqq i\vee j$, where $i\in I$ and $j\in J$. By assumption, $(i\vee j]$ is a distributive sublattice of S and $i, j, x\in(i\vee j]$. So $x=x\wedge(i\vee j)=(x\wedge i)\vee(x\wedge j)$, where $(x\vee i)\in F\wedge I$ and $x\vee j\in F\wedge J$. Therefore, $x\in(\wedge I)\vee(F\wedge J)$.

(i)$\Rightarrow$(iii). Let $\Phi(S)$ be a lattice and $w, y, z\in(x]$ in S. Then the mapping $s_y\vee s_z$ is a translation on S, whence $(y\vee z]\wedge(u]=(y\wedge u]\wedge(z\vee u]$ for each $u\in S$ by the analogy of Theorem 1. The distributivity of $(x]$ follows now by putting $u=w$.

(iii)$\Rightarrow$(ii). Let J be a semi-ideal of S generating a translation φ on S, and assume that $x\vee y$ exists in S. As $x\vee y$ exists and $x\leqq\varphi(x), y\leqq\varphi(y)$, then $\varphi(x)\vee\varphi(y)$ exists in S. As shown in the proof (iii)$\Rightarrow$(i), $\mathscr{I}(S)$ is a distributive lattice. Let us consider now $\varphi(x\vee y)$, i.e. the meet $J\wedge(x\vee y]=(J\wedge(x])\vee(J\wedge(y])$, which implies that $\varphi(x\vee y)=\varphi(x)\vee\varphi(y)$. Thus φ is also a join-endomorphism on S.

(ii)$\Rightarrow$(iii). Let $u, w, z\in(x]$. As the mapping s_u is also a join-endomorphism, $s_u(w\vee z)=(u]\wedge(w\vee z]=s_u(w)\vee s_u(z)=((u]\wedge(w])\vee((u]\wedge(z])$, from which the distributivity of $(x]$ follows.

As above, one can easily prove that in a partial lattice S each $(x]$ is a modular lattice of S if and only if $\mathscr{I}(S)$ is a modular lattice. The proof of the following theorem is analogous to that of Theorem 3, and hence we omit it.

Theorem 4. *Let S be a partial lattice. Each translation on S has the property that $\varphi(\varphi(z)\vee y)=\varphi(z)\vee\varphi(y)$ when $\varphi(z)\vee y$ exists in S, if and only if $(x]$ is a modular sublattice of S for each $x\in S$.*

The equivalenc (ii)$\Leftrightarrow$(iii) in Theorem 3 and Theorem 4 are generalizations of Theorems 4 and 5 in SzÁsz's paper [7].

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