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A note on quasisimilarity of operators

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1. Introduction. Let $\mathfrak{H}$ be a separable, infinite dimensional complex Hilbert space, and let $\mathcal{L}(\mathfrak{H})$ denote the algebra of all bounded, linear operators on $\mathfrak{H}$. An operator X in $\mathcal{L}(\mathfrak{H})$ is *quasi-invertible* [1]) if X is injective and has dense range (i.e., $\ker(X)=\ker(X^*)=\{0\}$). Operators A and B in $\mathcal{L}(\mathfrak{H})$ are *quasisimilar* if there exist quasi-invertible operators X and Y in $\mathcal{L}(\mathfrak{H})$ such that $AX=XB$ and $YA=BY$. Two operators that are similar are clearly quasisimilar, and similar operators have equal spectra; one purpose of this note is to study the relationships between the spectra of quasisimilar operators.

There are several cases in which the quasisimilarity of two operators A and B implies the equality of their spectra: this is true if A and B are decomposable [7] or if A and B are hyponormal [6]. In section 4 we give necessary and sufficient conditions for two injective weighted shifts to be quasisimilar. We prove that if shifts W_α and W_β are quasisimilar, then they have equal spectra; if, in addition, W_α or W_β is invertible, then W_α is similar to W_β.

Contrasting with these results is an example, due to HOOVER [15], of two quasisimilar non-injective weighted shifts A and B such that $\sigma(A)=\{0\}$ and $\sigma(B)==D=\{z\in\mathbf{C}:|z|\leqq 1\}$, In [18] SZ.-NAGY and FOIAŞ gave necessary and sufficient conditions for a contraction to be quasisimilar to a unitary operator, and they gave an example of such an operator whose spectrum equals the disk D. The general result governing all of these cases is the following well-known corollary of Rosenblum's Theorem: The intersection of the spectra of quasisimilar operators is nonempty [15]. In Theorem 2.5 we prove the following refinement of this result: If $AX=XB$, where X is injective, and S is a part of B, then each non-empty closed-and-open subset of $\sigma(S)$ has non-empty intersection with $\sigma(A)$. In Theorem 2.6, Lemma 2.8, and Lemma 2.11 we give partial analogues of this result for the essential spectra of A and B.

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[1]) „Quasiaffinity" in [18].

In [13] Foiaş and Pearcy established a model for quasinilpotent operators up to similarity, and in [19] Pearcy inquired whether an analogous model could be given for quasinilpotent operators up to quasisimilarity. Since quasisimilarity is a transitive relation, such a model would apply to each operator in $\mathcal{Q}_{qs}=\{T\in\mathcal{L}(\mathfrak{H}): T$ is quasisimilar to some quasinilpotent operator in $\mathcal{L}(\mathfrak{H})\}$; in particular, the hyperinvariant subspace problem for operators in $\mathcal{Q}_{qs}$ is equivalent to the hyperinvariant subspace problem for operators in $\mathcal{Q}$ (see [15]). In section 3 we study properties of operators in $\mathcal{Q}_{qs}$. While quasisimilarity does not, in general, preserve quasitriangularity [24], we prove that each operator in $\mathcal{Q}_{qs}$ is quasitriangular; in addition, $\mathcal{Q}_{qs}$ is a proper subset of the norm closure of the set of all nilpotent operators (i.e., $\mathcal{Q}_{qs}\subsetneqq\mathcal{N}^-$). We prove that $\mathcal{Q}_{qs}$ contains no non-quasinilpotent decomposable or hyponormal operators. On the other hand, $\mathcal{Q}_{qs}$ is closed under countable direct sums (Proposition 3.10), and this result is used to prove that a subset $X\subset\mathbf{C}$ is the spectrum of an operator in $\mathcal{Q}_{qs}$ if and only if X is compact, connected, and contains 0 (Theorem 3.11).

We conclude this section with some terminology and notation. Let $\mathcal{K}$ denote the ideal of all compact operators in $\mathcal{L}(\mathfrak{H})$; if T is in $\mathcal{L}(\mathfrak{H})$, let $\tilde{T}$ denote the image of T in the Calkin algebra $\mathcal{L}(\mathfrak{H})/\mathcal{K}$. The essential spectrum of T, $\sigma_e(T)$, is the spectrum of $\tilde{T}$ with respect to the Calkin algebra [11]. We will use results from [9] about semi-Fredholm operators and quasitriangular operators. We denote by $\mathcal{N}$ and $\mathcal{Q}$ the sets of all nilpotent and, respectively, quasinilpotent operators in $\mathcal{L}(\mathfrak{H})$. If T is in $\mathcal{L}(\mathfrak{H})$, then a *part* of T is an operator S of the form $S=T|\mathfrak{M}$, where $\mathfrak{M}$ is a closed subspace of $\mathfrak{H}$ such that $T\mathfrak{M}\subset\mathfrak{M}$ and $\mathfrak{M}\neq\{0\}$ ($\mathfrak{M}=\mathfrak{H}$ is permitted). We denote the spectrum of T by $\sigma(T)$ and the spectral radius of T by $r(T)=\sup\{|\lambda|:\lambda\in\sigma(T)\}=\lim\|T^n\|^{1/n}$; thus $\mathcal{Q}=\{T \text{ in } \mathcal{L}(\mathfrak{H}): r(T)=0\}$.

2. On the spectra of quasisimilar operators. Let $\mathcal{A}$ denote a complex Banach algebra with identity and let $\mathcal{M}(\mathcal{A})$ denote the Banach algebra consisting of all 2×2 matrices with entries from $\mathcal{A}$ (where the norm of a matrix is its norm as an operator on the Banach space $\mathcal{A}\oplus\mathcal{A}$). Let a, b, and x denote elements of $\mathcal{A}$. Let $\sigma(y)$ denote the spectrum of an element y of $\mathcal{A}$.

Lemma 2.1. *If f is a function that is analytic in a neighborhood of $\sigma(a)\cup\sigma(b)$, and $ax=xb$, then $f(a)x=xf(b)$.*

Proof. Let M and N denote, respectively, the elements of $\mathcal{M}(\mathcal{A})$ whose matrices are

$$\begin{bmatrix} a & 0\\ 0 & b\end{bmatrix} \text{ and } \begin{bmatrix} 0 & x\\ 0 & 0\end{bmatrix}.$$

Since f is analytic in a neighborhood of $\sigma(M)=\sigma(a)\cup\sigma(b)$, then $f(a)$, $f(b)$, and

$f(M)$ are defined by the Riesz functional calculus, and it is easy to prove that

$$f(M)=\begin{bmatrix} f(a) & 0 \\ 0 & f(b) \end{bmatrix} \text{ (see, e.g., the proof of [10, Lemma 2.1]).}$$

Since $ax=xb$, N commutes with M, and Theorem 7.4 of [5, page 33] implies that N commutes with $f(M)$. A matrix calculation now shows that $f(a)x=xf(b)$ and the proof is complete.

The following well-known result is usually proved as a corollary of Rosenblum's Theorem [20, Theorem 0.12, page 8]; we give an elementary proof based on Lemma 2.1.

Lemma 2.2. *If* $ax=xb$ *and* $\sigma(a)\cap\sigma(b)=\emptyset$, *then* $x=0$.

Proof. Without loss of generality we may replace a and b, respectively, by $a-\lambda$ and $b-\lambda$, where λ is any complex number, and we may thus assume that a is invertible. Let $f(z)$ be an analytic function such that $f(z)=z$ in a neighborhood of $\sigma(a)$ and $f(z)=0$ in a neighborhood of $\sigma(b)$. Since $f(a)=a$ and $f(b)=0$, Lemma 2.1 implies that $ax=0$, and the invertibility of a implies that $x=0$.

Using Lemma 2.2 and basic properties of the spectral measure of a normal operator, we can prove the following refinement of Lemma 2.2. The proof, which is not needed in the sequel, will be omitted.

Proposition 2.3. *Suppose that* T, X, *and* N *are in* $\mathcal{L}(\mathfrak{H})$, *where* N *is normal and* $TX=XN$ *or* $XT=NX$. *Let* $E(\cdot)$ *denote the spectral measure of* N. *If* $E(\sigma(T))=0$, *then* $X=0$.

We note that the preceding result is also valid if N is a spectral operator. An element e in $\mathcal{A}$ is said to be *idempotent* if $e^2=e$.

Lemma 2.4. *If* $ax=xb$ *and if there exists no non-zero idempotent* e *such that* $xe=0$, *then each non-empty closed-and-open subset of* $\sigma(b)$ *has non-empty intersection with* $\sigma(a)$.

Proof. Suppose that τ is a non-empty closed-and-open subset of $\sigma(b)$ that is disjoint from $\sigma(a)$. Since $\mathcal{A}$ has an identity, $x\neq 0$, and Lemma 2.2 implies that $\tau\neq\sigma(b)$. Thus there exists an analytic function f such that $f(z)=0$ in a neighborhood of $\sigma(a)\cup(\sigma(b)-\tau)$ and $f(z)=1$ in a neighborhood of τ. Then $f(a)=0$ and [5, Prop. 7.9, page 36] implies that $f(b)$ is a non-zero idempotent in $\mathcal{A}$. Lemma 2.1 implies that $0=f(a)x=xf(b)$, and the hypothesis on x implies that $f(b)=0$, which is a contradiction.

Theorem 2.5. *Let* A, B, *and* X *be in* $\mathcal{L}(\mathfrak{H})$. *Suppose that* $AX=XB$, X *is injective, and* P *is a non-zero projection such that* $P\mathfrak{H}$ *is invariant for* B $(P=1$ *is* *permitted). Then each non-empty closed-and-open subset of $\sigma(B|P\mathfrak{H})$ has non-empty intersection with $\sigma(A)$.*

Proof. We may assume from Lemma 2.4 that $P\neq 1$. Suppose that τ is a non-empty closed-and-open subset of $\sigma(B|P\mathfrak{H})$ such that $\tau\cap\sigma(A)=\emptyset$. Let λ be chosen so that $(B-\lambda)|P\mathfrak{H}$ is invertible; since $PBP=BP$, we have $(*)$ $(A-\lambda)(XP)=(XP)(B-\lambda)P$. Let f be an analytic function such that $f(z)=1$ in a neighborhood of $\tau-\lambda$ and $f(z)=0$ in a neighborhood of $\sigma(A-\lambda)\cup(\sigma((B-\lambda)|P\mathfrak{H})-(\tau-\lambda))\cup\{0\}$. (This definition of f is valid since $\tau-\lambda$ is a non-empty closed-and-open subset of $\sigma((B-\lambda)|P\mathfrak{H})$ such that $(\tau-\lambda)\cap\sigma(A-\lambda)=\emptyset$ and $0\notin\sigma((B-\lambda)|P\mathfrak{H})$.) Since $\sigma((B-\lambda)P)=\sigma((B-\lambda)|P\mathfrak{H})\cup\{0\}$, f is defined in a neighborhood of $\sigma(A-\lambda)\cup\sigma((B-\lambda)P)$, and Lemma 2.1 and $(*)$ imply that $f(A-\lambda)(XP)=(XP)f((B-\lambda)P)$. Now $f(A-\lambda)=0$ and $E=f((B-\lambda)P)$ is a non-zero idempotent; thus we have $0=XPE$. Further, [20, Theorem 2.10, page 31] implies that $(B-\lambda)P$ commutes with E, that the range of E is invariant for $(B-\lambda)P$, and that $\sigma((B-\lambda)P|E\mathfrak{H})=\tau-\lambda$. With respect to the decomposition $\mathfrak{H}=P\mathfrak{H}\oplus(1-P)\mathfrak{H}$, the operator matrices of B and P are, respectively,

$$\begin{bmatrix} B_1 & * \\ 0 & * \end{bmatrix} \quad\text{and}\quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus the operator matrix of $(B-\lambda)P$ is

$$\begin{bmatrix} B_1-\lambda & 0 \\ 0 & 0 \end{bmatrix},$$

where $B_1-\lambda$ is invertible in $\mathscr{L}(P\mathfrak{H})$. Let

$$\begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix}$$

denote the operator matrix of E. Since E commutes with $(B-\lambda)P$, a calculation shows that $E_2=0$ and $E_3=0$. We claim that $E_4=0$ in $\mathscr{L}((1-P)\mathfrak{H})$. Indeed, if x is a nonzero vector in $(1-P)\mathfrak{H}$ such that $E_4x\neq 0$, then

$$\begin{bmatrix} B_1-\lambda & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} E_1 & 0 \\ 0 & E_4 \end{bmatrix}\begin{bmatrix} 0 \\ x \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and so $0\in\sigma((B-\lambda)P|E\mathfrak{H})=\tau-\lambda$, which is a contradiction since $\lambda\notin\sigma(B|P\mathfrak{H})$. Now $E_4=0$, so we have $PE=E$ and $0=XPE=XE$. Since X is injective, $E=0$, and we have a contradiction which completes the proof.

Remark. If X is non-injective, then the conclusion of Theorem 2.5 is no longer valid; if X is a projection in $\mathscr{L}(\mathfrak{H})$, $X\neq 0, 1$, then $1X=X^2$.

In contrast to Theorem 2.5, it can be shown that quasisimilarity does not preserve the connectedness of spectra. Indeed, HOOVER [15] gives an example of quasisimilar operators A and B such that $\sigma(A)=\{0\}$ while $\sigma(B)$ equals the closed unit disk. Then $A\oplus(A-1/2)$ is quasisimilar to $B\oplus(B-1/2)$; the spectrum of the first operator is disconnected and the spectrum of the second operator is connected.

The analogue of Theorem 2.5 for essential spectra is false. Let U denote a unilateral (unweighted) shift of multiplicity one in $\mathcal{L}(\mathfrak{H})$ and let W_α denote the unilateral weighted shift defined by $\alpha_n=1/n$ for $n\geqq 1$ (see section 4 for notation). Let X denote the injective diagonalizable operator defined by $Xe_n=\beta_n e_n$, where $\beta_1=\beta_2=1$ and $\beta_n=1/(n-1)!$ for $n\geqq 3$. Now $W_\alpha X=XU$; however, $\sigma_e(W_\alpha)$ and $\sigma_e(U)$ are disjoint, since $\sigma_e(W_\alpha)=\{0\}$ and $\sigma_e(U)$ is the unit circle.

Despite the preceding example we have the following perhaps surprising result.

Theorem 2.6. *If A and B are quasisimilar operators in $\mathcal{L}(\mathfrak{H})$, then $\sigma_e(A)$ and $\sigma_e(B)$ have non-empty intersection.*

Before proceding with the proof of Theorem 2.6, the following observation seems pertinent. If X is in $\mathcal{L}(\mathfrak{H})$, and if $\tilde{X}$ is "injective" in the Calkin algebra (i.e., if there exists no non-zero idempotent $\tilde{E}$ in the Calkin algebra such that $\tilde{X}\tilde{E}=0$), then $\tilde{X}$ is left invertible in the Calkin algebra (see [11, Theorem 1.1]); thus if X is also quasi invertible, then X is invertible. This fact implies that if two operators are quasi-similar but not similar, then the intertwining quasi invertible operators are both non-injective in the Calkin algebra. Thus it appears to be difficult to directly adapt the proof of Lemma 2.4 to the setting of the Calkin algebra in order to prove Theorem 2.6.

Our proof of Theorem 2.6 is instead inspired by the techniques and terminology of [19]. We next summarize some of the results and terminology from [19]. Let T be in $\mathcal{L}(\mathfrak{H})$. A subset $H\subset\mathbf{C}$ is said to be a *hole* in $\sigma_e(T)$ if H is a bounded connected component of $\mathbf{C}-\sigma_e(T)$; thus $\mathrm{bdry}(H)\subset\sigma_e(T)$.

Lemma 2.7. *If H is a hole in $\sigma_e(T)$ and $H\cap\sigma(T)$ is uncountable, then $H\subset\sigma(T)$. In this case, if S is quasisimilar to T, then $H\subset\sigma(S)$ and* $\mathrm{bdry}(H)\subset\sigma_e(T)\cap\sigma(S)$. *If H is a component of $\mathbf{C}-\sigma_e(T)$ and $H\cap\sigma(T)$ is finite or countably infinite, then each point of $H\cap\sigma(T)$ is an isolated point of $\sigma(T)$ and an eigenvalue of finite multiplicity; moreover, if K is the unbounded component of $\mathbf{C}-\sigma_e(T)$, then $K\cap\sigma(T)$ is either empty, finite, or countably infinite.*

Proof. The proof follows immediately from the results of [19].

Lemma 2.8. *If A and B are quasisimilar, then each non-empty closed-and open subset of $\sigma_e(B)$ has non-empty intersection with $\sigma(A)$.*

Proof. Let τ be a non-empty closed-and-open subset of $\sigma_e(B)$. If τ is open in $\sigma(B)$, then Theorem 2.5 implies that $\tau\cap\sigma(A)\neq\emptyset$. Otherwise, there exists t in τ, and a sequence $\{t_n\}\subset\sigma(B)-\tau$, such that $t_n\to t$. Since τ is open in $\sigma_e(B)$, we may assume that each t_n is in $\sigma(B)-\sigma_e(B)$. Thus t_n is an eigenvalue of B (and thus of A) for infinitely many n, or $\bar{t}_n$ is an eigenvalue of B^* (and thus of A^*) for infinitely many n. In either case, t is in $\sigma_e(B)\cap\sigma(A)$, and the proof is complete.

Remark. Let X denote a non-empty, bounded, open, connected subset of the complex plane; let $\varphi(X)$ denote the unbounded component of the complement of the closure of X, and let $\beta(X)=\text{bdry}(\varphi(X))$; note that $\beta(X)\subset\text{bdry}(X)$. It is a result of the topology of the plane that $\beta(X)$ is connected [23, Theorem 14.2, page 123]. In particular, if T is in $\mathcal{L}(\mathfrak{H})$ and $\beta(X)\subset\sigma(T)-\sigma_e(T)$, then the connectedness of $\beta(X)$ implies that $\beta(X)$ is contained in some component H of $\mathbf{C}-\sigma_e(T)$; further, since $\beta(X)$ is uncountable, Lemma 2.7 implies that H is a hole in $\sigma_e(T)$.

Lemma 2.9. *If A and B are quasisimilar operators in $\mathcal{L}(\mathfrak{H})$, and if there exists a hole H_0 in $\sigma_e(A)$ such that $H_0\subset\sigma(A)$, then $\sigma_e(A)\cap\sigma_e(B)\neq\emptyset$.*

Proof. Suppose to the contrary that $\sigma_e(A)$ and $\sigma_e(B)$ are disjoint. Since $H_0\subset\sigma(A)$, then $H_0\subset\sigma(B)$, and thus $\beta(H_0)\subset\sigma_e(A)\cap\sigma(B)\subset\sigma(B)-\sigma_e(B)$. The above Remark implies that there exists a hole K_1 in $\sigma_e(B)$ such that $\beta(H_0)\subset K_1$, and it follows by a connectedness argument that $\varphi(K_1)^-\subset\varphi(H_0)$. Now $\beta(K_1)$ is an uncountable connected subset of $\sigma_e(B)$; thus, as above, there exists a hole H_1 in $\sigma_e(A)$ such that $\beta(K_1)\subset H_1$, and we also have $\varphi(H_1)^-\subset\varphi(K_1)$. Moreover, H_1 and H_0 are disjoint; indeed, otherwise H_1 and H_0 (components of $\mathbf{C}-\sigma_e(A)$) are equal, and since $\beta(K_1)\subset H_1$, it follows that there is a point in $\varphi(K_1)\cap H_1=\varphi(K_1)\cap H_0$. Since $\varphi(K_1)\subset\varphi(H_0)\subset\mathbf{C}-H_0$, we have a contradiction, and thus $H_1\cap H_0=\emptyset$.

The above procedure may now be used to inductively define two sequences $\{H_i\}$ $(i\geqq 0)$ and $\{K_i\}$ $(i\geqq 1)$ such that:

i) H_i is a hole in $\sigma_e(A)$; $\beta(H_i)\subset\sigma_e(A)$ $(i\geqq 0)$;

ii) K_i is a hole in $\sigma_e(B)$; $\beta(K_i)\subset\sigma_e(B)$ $(i>0)$;

iii) $\beta(H_i)\subset K_{i+1}$, $\beta(K_{i+1})\subset H_{i+1}$ $(i\geqq 0)$;

iv) $\varphi(H_i)^-\subset\varphi(K_i)$, $\varphi(K_i)^-\subset\varphi(H_{i-1})$ $(i\geqq 1)$;

v) $K_i\cap K_j=\emptyset$, $H_i\cap H_j=\emptyset$ for all $i\neq j$.

Now iii) and iv) imply that $\beta(H_i)\cap\beta(H_j)=\emptyset$ for all $i\neq j$. Let $\{h_i\}$ $(i\geqq 0)$ denote a sequence such that h_i is in $\beta(H_i)$ for $i\geqq 0$. Since these points are distinct, there exists a convergent subsequence $h_{i_k}\to h$, and i) implies that h is in $\sigma_e(A)$. Since $i_k>i_{k-1}$, iv) implies that $\varphi(H_{i_k})\subset\varphi(K_{i_k})\subset\varphi(H_{i_k-1})\subset\ldots\subset\varphi(H_{i_{k-1}})$; now if L denotes the line segment from h_{i_k} to $h_{i_{k-1}}$, then L contains a point g_{i_k} from $\beta(K_{i_k})$. Since $|g_{i_k}-h|\leqq|g_{i_k}-h_{i_k}|+|h_{i_k}-h_{i_{k-1}}|+|h_{i_{k-1}}-h|\leqq 2|h_{i_k}-h_{i_{k-1}}|+|h_{i_{k-1}}-h|$, it follows that $g_{i_k} \to h$. Now ii) implies that h is in $\sigma_e(B)$. Since h is also in $\sigma_e(A)$, we have a contradiction, which completes the proof.

Lemma 2.10. *If A and B are quasisimilar, and if there exists an infinite sequence $\{z_n\}$ of distinct isolated points of $\sigma(A)$ such that* $\dim(\ker(A-z_n))>0$ *or* $\dim(\ker((A-z_n)^*))>0$ *for each n, then $\sigma_e(A)\cap\sigma_e(B)\neq\emptyset$.*

Proof. Since A and B are quasisimilar, $\{z_n\}\subset\sigma(B)$; by passing, if necessary, to a subsequence, we may assume that $z_n\to z$, where z is in $\sigma(B)$. Since z is an accumulation point of bdry$(\sigma(A))$, [19, Corollary 1.26] implies that z is in $\sigma_e(A)$, and we claim that z is also in $\sigma_e(B)$. For otherwise, since z is in $\sigma(B)-\sigma_e(B)$ and z is not an isolated point of $\sigma(B)$, Lemma 2.7 implies that there exists an open disk D centered at z, such that $B-w$ or $(B-w)^*$ is non-injective for each w in D. Since $D\subset\sigma(A)$, and since there exists some z_n in D, it follows that z_n is not an isolated point of $\sigma(A)$, which is a contradiction. Thus z is in $\sigma_e(A)\cap\sigma_e(B)$, and with the proof is complete.

Lemma 2.11. *Let A, B, and X be in $\mathcal{L}(\mathfrak{H})$, with X injective and $AX=XB$. If H is a component of $\mathbf{C}-\sigma_e(A)$ such that $K=H\cap\sigma(B)$ is a non-empty closed-and-open subset of $\sigma(B)$, and $K\cap\sigma_e(B)\neq\emptyset$, then $H\subset\sigma(A)$.*

Proof. The hypothesis implies that K is a closed subset of the open set H; thus there exists an open set U such that $K\subset U\subset U^-\subset H$. If we assume that $H\not\subset\sigma(A)$, then Lemma 2.7 implies that H contains no limit points of $\sigma(A)$; in particular, $L=U\cap\sigma(A)$ is a finite set. Since U contains no limit points of $\sigma(A)$, L is an open subset of $\sigma(A)$. Since K is a non-empty closed-and-open subset of $\sigma(B)$, and $L\supset K\cap\sigma(A)$, Lemma 2.4 implies that L is non-empty: moreover, since $L\cap\sigma_e(A)=\emptyset$, then $L\neq\sigma(A)$.

Thus K and L are, respectively, non-empty closed-and-open subsets of $\sigma(B)$ and $\sigma(A)$. Now there exists an analytic function f such that $f(z)=1$ in a neighborhood of $K\cup L$, and $f(z)=0$ in a neighborhood of $(\sigma(A)-L)\cup(\sigma(B)-K)$. As in the proof of Theorem 2.5, $f(A)$ is an idempotent commuting with A, $\sigma(A|f(A)\mathfrak{H})= =L$, and $\sigma(A|(1-f(A))\mathfrak{H})=\sigma(A)-L$. Since each idempotent operator in $\mathcal{L}(\mathfrak{H})$ is similar to an orthogonal projection, there exists an invertible operator J such that $P=J^{-1}f(A)J$ is an orthogonal projection; then $R=J^{-1}AJ$ commutes with P, and $R|P\mathfrak{H}$ is similar to $A|f(A)\mathfrak{H}$. We assert that $P\mathfrak{H}$ is finite dimensional; otherwise, $\sigma_e(R|P\mathfrak{H})$ is a nonempty subset of $\sigma(R|P\mathfrak{H})=\sigma(A|f(A)\mathfrak{H})=L$. Since $R|P\mathfrak{H}$ is a direct summand of R, it follows that some point of L is in $\sigma_e(R)=\sigma_e(A)$, which is a contradiction.

Since $AX=XB$, Lemma 2.1 implies that $f(A)X=Xf(B)$. Since P has finite rank, so does $f(A)$, and since X is injective it follows that $f(B)$ also has finite rank. In particular, $f(B)\neq 1$ and so $K\neq\sigma(B)$. Now $f(B)$ is a nontrivial idempotent that commutes with B. Proceeding as above, there exists an invertible operator M such that $Q=M^{-1}f(B)M$ is an orthogonal projection, Q commutes with $S=M^{-1}BM$, $\sigma(S|Q\mathfrak{H})=K$, and $\sigma(S|(1-Q)\mathfrak{H})=\sigma(B)-K$ (since $S|Q\mathfrak{H}$ is similar to $B|f(B)\mathfrak{H}$ and $S|(1-Q)\mathfrak{H}$ is similar to $B|(1-f(B))\mathfrak{H}$). If z is in $K\cap\sigma_e(B)$, then with respect to the orthogonal decomposition $\mathfrak{H}=Q\mathfrak{H}\oplus(1-Q)\mathfrak{H}$, we have $S-z= =((1|Q\mathfrak{H})\oplus((S-z)|(1-Q)\mathfrak{H}))+(((S-z-1)|Q\mathfrak{H})\oplus(0|(1-Q)\mathfrak{H}))$. Since the first term on the right is invertible, while the second term in the sum is a finite rank operator, it follows that $S-z$ is a Fredholm operator, which contradicts the assumption that z is in $\sigma_e(B)=\sigma_e(S)$. Thus $H\subset\sigma(A)$, and the proof is complete.

Proof of Theorem 2.6. By Lemma 2.9 we may assume that if there exists a hole H in $\sigma_e(A)$, then $H\not\subset\sigma(A)$, for otherwise the proof is complete. Moreover, we may assume from Lemmas 2.7 and 2.10 that $H\cap\sigma(A)$ is at most finite, and that if K denotes the unbounded component of $\mathbf{C}-\sigma_e(A)$, then $K\cap\sigma(A)$ is at most finite. Let $X=\sigma_e(B)\cap\sigma(A)$; Lemma 2.8 implies that X is non-empty. If we assume that $X\cap\sigma_e(A)=\emptyset$, then there exists a component H of $\mathbf{C}-\sigma_e(A)$ such that $X\cap H\neq\emptyset$; from the preceding remarks we may assume that $H\cap\sigma(A)$ is a finite set. Since $(\sigma_e(B)\cap H)^-\cap\mathrm{bdry}(H)\subset\sigma_e(B)\cap\sigma_e(A)$, we may assume that there is an open set U such that $\sigma_e(B)\cap H\subset U\subset U^-\subset H$; in particular, $Y=\sigma_e(B)\cap H$ is a closed subset of $\sigma(B)$.

We assert that Y is also an open subset of $\sigma(B)$; indeed, if Y is not open in $\sigma(B)$, then there exists an infinite sequence of distinct points $\{z_n\}\subset\sigma(B)-Y$ such that $z_n\to z$, where z is some point in Y. We may assume (excluding at most a finite number of points) that each z_n is in U; thus each z_n is in $\sigma(B)-\sigma_e(B)\subset \subset\sigma(A)$. Now each z_n is in $H\cap\sigma(A)$, which contradicts the fact that $H\cap\sigma(A)$ is finite. Thus Y is a non-empty closed-and-open subset of $\sigma(B)$, and Lemma 2.11 implies that $H\subset\sigma(A)$, which also contradicts the fact that $H\cap\sigma(A)$ is finite. The proof is now complete.

Remark. In a preliminary version of this paper, the author was unable to prove Theorem 2.6, and instead posed it as a question. L. R. Williams, meanwhile, independently found a somewhat different proof of Theorem 2.6, which will appear in his note [22].

Corollary 2.12. *Let A, B, and X be in $\mathscr{L}(\mathfrak{H})$ with X injective and $AX=XB$. If S is a part of B and S is decomposable, then $\sigma(S)\subset\sigma(A)$.*

Proof. Let $S=B|\mathscr{L}$, where $\mathscr{L}\neq\{0\}$ and $B\mathscr{L}\subset\mathscr{L}$. If $\sigma(S)\not\subset\sigma(A)$, then there exists an open subset $U\subset\mathbf{C}$ such that $U\cap\sigma(S)\neq\emptyset$ and $U\cap\sigma(A)=\emptyset$. Since S is decomposable, [7, Lemma 1.2, page 30] implies that there exists an S-invariant closed subspace $\mathfrak{M}\subset\mathscr{L}$ such that $\mathfrak{M}\neq\{0\}$ and $\sigma(S|\mathfrak{M})\subset U$. Since $\mathfrak{M}\subset\mathfrak{H}$ is also invariant for B and $\sigma(A)\cap\sigma(B|\mathfrak{M})\subset\sigma(A)\cap U=\emptyset$, we have a contradiction to Theorem 2.5, and the proof is complete.

3. On quasisimilarity and quasinilpotent operators. In this section we give some properties of operators in $\mathcal{Q}_{qs}$. An operator T in $\mathcal{L}(\mathfrak{H})$ is called a *quasiaffine transform* of the operator S if there exists a quasi-invertible operator X in $\mathcal{L}(\mathfrak{H})$ such that $XT=SX$. Let $\mathcal{Q}_{af}=\{T\in\mathcal{L}(\mathfrak{H})$: T is a quasiaffine transform of some quasinilpotent operator$\}$ and let $\mathcal{Q}_{af}^*=\{T\in\mathcal{L}(\mathfrak{H}): T^*\in\mathcal{Q}_{af}\}$; thus $\mathcal{Q}_{qs}\subset\mathcal{Q}_{af}\cap\mathcal{Q}_{af}^*$.

Theorem 3.1. *If T is in $\mathcal{Q}_{af}\cap\mathcal{Q}_{af}^*$, then T satisfies the following properties:*

i) *If P is a non-zero projection such that $(1-P)TP=0$, then $\sigma(T|P\mathfrak{H})$ is connected and contains* 0; *if additionally $P\neq 1$, then $\sigma((1-P)T|(1-P)\mathfrak{H})$ is connected and contains* 0.

ii) $\sigma(T)-\{0\}\subset\{\lambda\in\mathbf{C}$: $T-\lambda$ and $(T-\lambda)^*$ are injective$\}$.

iii) *If $\lambda\neq 0$ and $T-\lambda$ is semi-Fredholm, then $T-\lambda$ is invertible.*

iv) $\sigma(T)=\sigma_e(T)$.

v) *T is bi-quasitriangular.*

Proof. Let Q and R be quasinilpotent operators and let X and Y be quasi-invertible operators such that $QX=XT$ and $RY=YT^*$.

i) If $P\neq 0$ and $(1-P)TP=0$, then since X is injective, Theorem 2.5. implies that $\sigma(T|P\mathfrak{H})$ is connected and contains 0. If $P\neq 1$, then since $(1-P)\mathfrak{H}$ is invariant for T^* and Y is injective, $\sigma(T^*|(1-P)\mathfrak{H})$ is connected and contains 0. Since $\sigma((1-P)T|(1-P)\mathfrak{H})=\{\lambda\in\mathbf{C}$: $\bar\lambda\in\sigma(T^*|(1-P)\mathfrak{H})\}$, the proof is complete.

ii) Since $(Q-\lambda)X=X(T-\lambda)$, $(R-\bar\lambda)Y=Y(T-\lambda)^*$, and $\sigma(Q)=\sigma(R)=\{0\}$, it is clear that if $\lambda\neq 0$, then $T-\lambda$ and $(T-\lambda)^*$ are injective.

iii) If $T-\lambda$ is semi-Fredholm but not invertible, then either $T-\lambda$ or $(T-\lambda)^*$ is non-injective, so the result follows from ii).

iv) Since $\sigma_e(T)$ is a non-empty subset of $\sigma(T)$, we may assume that T is not quasinilpotent. It is clear from iii) that each non-zero member of $\sigma(T)$ is in $\sigma_e(T)$; now i) implies that 0 is a limit point of $\sigma_e(T)$ and so 0 is in $\sigma_e(T)$.

v) For each vector h in $\mathfrak{H}$, we have $\|T^nY^*h\|^{1/n}=\|Y^*R^{*n}h\|^{1/n}\leqq\|Y^*\|^{1/n}\cdot\|R^{*n}\|^{1/n}\|h\|^{1/n}\to 0$. Since Y^* has dense range, Theorem 3.1 of [1] implies that T is quasitriangular. A similar argument, using the equation $T^{*n}X^*=X^*Q^{*n}$, implies that T^* is quasitriangular.

Corollary 3.2. *If T is in $\mathcal{Q}_{qs}$, then T satisfies properties* i) — v) *of Theorem* 3.1.

Corollary 3.3. *If T is in $\mathcal{Q}_{af}$ and S is a part of T that is decomposable, then S is quasinilpotent.*

Proof. The result follows from Corollary 2.12 or Theorem 3.1—i).

Corollary 3.4. *If T is a decomposable operator in $\mathcal{Q}_{af}$, then is quasinilpotent.*

Corollary 3.5. *If T is in $\mathcal{Q}_{af}$ and S is a part of T that is normal, then $S=0$.*

Theorem 3.6. *If T is a hyponormal operator in $\mathcal{Q}^*_{af}$, then $T=0$.*

Proof. Theorem 1 of [6] implies that if $XA=TX$ and X has dense range, then $\sigma(T)\subset\sigma(A)$; thus, if A is quasinilpotent, then so is T. Now [20, Proposition 1.8, page 24] implies that $\|T\|=r(T)=0$.

Question 3.7. Which injective weighted shifts are in $\mathcal{Q}_{qs}$? This question, which we are unable to answer, motivated the results of section 4. Theorem 4.8 implies that if an injective weighted shift W is quasisimilar to a quasinilpotent injective weighted shift, then W is quasinilpotent.

Corollary 3.8. *$\mathcal{Q}_{qs}$ is a proper subset of $\mathcal{N}^-$.*

Proof. Theorem 3.1 implies that if T is in $\mathcal{Q}_{qs}$, then T is bi-quasitriangular and that $\sigma(T)$ and $\sigma_e(T)$ are connected and contain 0. Now [4] implies that T is in $\mathcal{N}^-$. Theorem 7 of [14] implies that $\mathcal{N}^-$ contains non-zero normal operators, while Corollary 3.5 implies that there are no non-zero normal operators in $\mathcal{Q}_{qs}$; therefore, $\mathcal{Q}_{qs}$ is a proper subset of $\mathcal{N}^-$.

Question 3.9. Is the converse of Corollary 3.2 true?

We note that if T is a noninvertible operator in $\mathcal{L}(\mathfrak{H})$, and if T fails to satisfy properties i) — v) of Theorem 3.1, then T has a nontrivial invariant subspace; moreover, if T fails to satisfy properties ii) — v), then T has a nontrivial hyperinvariant subspace. (These observations are easy to prove except with regard to property v); the fact that a non-bi-quasitriangular operator has a nontrivial hyperinvariant subspace is a result of [3].) Thus, if the converse of Corollary 3.2 is true, and if each quasinilpotent operator does have a nontrivial hyperinvariant subspace, then each operator has a nontrivial invariant subspace. It is therefore of interest to determine whether the converse of Corollary 3.2 is true; we will show in Theorem 3.11 that as regards the topology of the spectra of operators in $\mathcal{Q}_{qs}$, Corollary 3.2 is indeed "best possible".

Proposition 3.10. *$\mathcal{Q}_{qs}$ is closed under countable direct sums.*

Proof. Let $\mathfrak{H}_i$ denote a separable Hilbert space ($i=1, 2, \ldots$), and let T_i be in $\mathcal{Q}_{qs}$ with respect to $\mathcal{L}(\mathfrak{H}_i)$. We seek to prove that if $\{\|T_i\|\}$ is bounded, then $T=\Sigma\oplus T_i$ is in $\mathcal{Q}_{qs}$ with respect to $\mathcal{L}(\mathfrak{H})$, where $\mathfrak{H}=\Sigma\oplus\mathfrak{H}_i$.

For each $i>0$, T_i is quasisimilar to a quasinilpotent operator Q_i in $\mathcal{L}(\mathfrak{H}_i)$; Rota's Theorem [20, Proposition 3.12, page 58] implies that there exists an operator

P_i in $\mathscr{L}(\mathfrak{H}_i)$ such that P_i is similar to Q_i and $\|P_i\|<1/i$. Now [15, Theorem 2.5] implies that T is quasisimilar to $S=\Sigma\oplus P_i$, so it suffices to prove that S is quasinilpotent. Let λ be a non-zero complex number and let n be a positive integer such that $1/n<|\lambda|$. For $i>n$, $\|P_i\|<1/i<1/n<|\lambda|$, and therefore

$$\|(P_i-\lambda)^{-1}\| \leqq (|\lambda|-\|P_i\|)^{-1} < (|\lambda|-1/i)^{-1} < (|\lambda|-1/n)^{-1}.$$

Now $\sup_{i\in N}\|(P_i-\lambda)^{-1}\| \leqq \max\left(\sup_{1\leqq i\leqq n}\|(P_i-\lambda)^{-1}\|, (|\lambda|-1/n)^{-1}\right)<\infty$, and hence $\lambda\notin\sigma(S)$.

Remark. In [13, Theorem 1.1] it is proved that if T is a quasinilpotent operator on $\mathfrak{H}$, then there exists a compact, quasinilpotent backward weighted shift K in $\mathscr{L}(\mathfrak{H})$ and a closed subspace $\mathfrak{M}\subset\mathscr{L}=\mathfrak{H}\oplus\cdots\oplus\mathfrak{H}\oplus\cdots$, such that

i) $\mathfrak{M}$ is invariant for $L=K\oplus\cdots\oplus K\oplus\cdots$;
ii) T is similar to $L|\mathfrak{M}$;
iii) $\|L|\mathfrak{M}\|\leqq\|T\|$ (see [13, Theorem 1.1, inequality 11]).

Using this result and the method of the proof of Proposition 3.10, it is not difficult to prove the following analogue for direct sums operators in $\mathscr{Q}_{qs}$: let $T=\Sigma\oplus T_i$, where T_i is in $\mathscr{Q}_{qs}$ with respect to $\mathfrak{H}_i$, and let $\mathfrak{H}=\Sigma\oplus\mathfrak{H}_i$. Then there exists a compact, quasinilpotent operator K on $\mathfrak{H}$, of arbitrarily small norm, and a closed subspace $\mathfrak{M}\subset\mathscr{L}=\mathfrak{H}\oplus\cdots\oplus\mathfrak{H}\oplus\cdots$, such that
i) $\mathfrak{M}$ is invariant for $L=K\oplus K\oplus\cdots\oplus K\oplus\cdots$;
ii) T is quasisimilar to $L|\mathfrak{M}$.

Theorem 3.11. *A subset $X\subset\mathbf{C}$ is the spectrum of an operator in $\mathscr{Q}_{qs}$ if and only if X is compact, connected, and contains* 0.

Proof. Let X denote a compact, connected subset of the plane that contains 0. Theorem 3.2 of [10] implies that there exists an operator T in $\mathscr{L}(\mathfrak{H})$ such that T is a direct sum of nilpotent operators and $\sigma(T)=X$; Proposition 3.10 implies that T is in $\mathscr{Q}_{qs}$.

The converse is contained in Theorem 3.1—i).

Remark. The proof of Theorem 3.11 did not require the full force of Proposition 3.10, but only the fact that each countable direct sum of nilpotent operators is in $\mathscr{Q}_{qs}$. Using [2, Theoreml] (or [21, Theoreml]), it is not difficult to prove that each countable direct sum of nilpotent operators is quasisimilar to some compact, quasinilpotent operator. On the other hand, not every quasinilpotent operator is quasisimilar to a compact operator (see [13, Prop. 1.5]).

We conclude this section with an additional necessary condition for membership in $\mathscr{Q}_{qs}$. For T in $\mathscr{L}(\mathfrak{H})$, let $\mathfrak{M}(T)=\{x\in\mathfrak{H}: \|T^n x\|^{1/n}\to 0\}$. It is easy to prove that $\mathfrak{M}(T)$ is a linear subspace of $\mathfrak{H}$ and that $\mathfrak{M}(T)^-$ is a (possibly trivial) hyperinvariant subspace for T. For example, if U denotes a unilateral shift of multiplicity one in $\mathcal{L}(\mathfrak{H})$, then $\mathfrak{M}(U)=\{0\}$, and since $\mathfrak{M}(U^*)$ contains an orthonormal basis for $\mathfrak{H}$, then $\mathfrak{M}(U^*)^-=\mathfrak{H}$.

Lemma 3.12. *If T is in $\mathfrak{Q}^*_{af}$, then there exists an orthonormal basis $\{e_k\}$ $(1\leqq k<\infty)$ for $\mathfrak{H}$ such that for each k, $\lim_{n\to\infty}\|T^n e_k\|^{1/n}=0$.*

Proof. Suppose that $XQ=TX$, where X is quasi-invertible and Q is in $\mathfrak{Q}$. For each t in $\mathfrak{H}$, we have $\|T^n Xt\|^{1/n}=\|XQ^n t\|^{1/n}\leqq\|X\|^{1/n}\|Q^n\|^{1/n}\|t\|^{1/n}\to 0$. Theorem 1.1 of [12] and the remarks of [12 ,page 280] imply that for S in $\mathcal{L}(\mathfrak{H})$, $S\mathfrak{H}$ contains an orthonormal basis for $(S\mathfrak{H})^-$. Since X has dense range, $X\mathfrak{H}$ contains an orthonormal basis for $\mathfrak{H}$, and since $X\mathfrak{H}\subset\mathfrak{M}(T)$, the proof is complete.

Proposition 3.13. *If T is in $\mathfrak{Q}_{qs}$, then $\mathfrak{M}(T)$ and $\mathfrak{M}(T^*)$ contain orthonormal bases for $\mathfrak{H}$; in particular, $\mathfrak{M}(T)^-=\mathfrak{M}(T^*)^-=\mathfrak{H}$.*

Question 3.14. Is the converse of Proposition 3.13 true? It is known that if I is in $\mathcal{L}(\mathfrak{H})$ and $\mathfrak{M}(T)=\mathfrak{H}$, then T is quasinilpotent (see [7, Lemma, page 28]).

Proposition 3.13 is related to Theorem 3.1 by the following result.

Proposition 3.15. *If T is in $\mathcal{L}(\mathfrak{H})$ and $\mathfrak{M}(T)^-=\mathfrak{M}(T^*)^-=\mathfrak{H}$, then T satisfies properties* i)—v) *of Theorem* 3.1.

Proof. Since $\mathfrak{M}(T)^-=\mathfrak{M}(T^*)^-=\mathfrak{H}$, Theorem 3.1 of [1] implies that T is bi-quasitriangular.

Let P be a non-zero projection such that $(1-P)TP=0$ and denote the operator matrix of T with respect to the decomposition $\mathfrak{H}=P\mathfrak{H}\oplus(1-P)\mathfrak{H}$ by

$$\begin{pmatrix} S & A \\ 0 & B \end{pmatrix}.$$

We will first show that $\sigma(S)$ contains 0. If S is invertible, then so is S^*, and there exists $\varepsilon>0$ such that $\|S^*x\|\geqq\varepsilon\|x\|$ for each x in $P\mathfrak{H}$. If z is in $\mathfrak{H}$, then $z=x+y$, where x is in $P\mathfrak{H}$ and y is in $(1-P)\mathfrak{H}$. Now we have $\|T^{*n}z\|^{1/n}\geqq\|S^{*n}x\|^{1/n}\geqq$ $\geqq\varepsilon\|x\|^{1/n}$, which implies that $\mathfrak{M}(T^*)\subset(1-P)\mathfrak{H}$. Since $\mathfrak{M}(T^*)$ is dense, this contradiction implies that $0\in\sigma(S)$; a similar argument, using the relation $\mathfrak{M}(T)^-=$ $=\mathfrak{H}$, implies that if S is a part of T^*, then S is noninvertible. In particular, T and T^* have no non-zero eigenvalues, and thus T satisfies ii)—iv).

To complete the proof we must show that if S is a part of T, then $\sigma(S)$ is connected. Since $0\in\sigma(S)$, if $\sigma(S)$ is not connected, then there exists a non-empty, closed-and-open subset $\tau\subset\sigma(S)$ such that $0\notin\tau$. If E denotes the spectral idempotent for S associated with τ, then $\sigma(T|E\mathfrak{H})=\sigma(S|E\mathfrak{H})=\tau$, which contradicts the fact that $T|E\mathfrak{H}$ is noninvertible.

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Question 3.16. Is $\mathcal{Q}_{qs}=\mathcal{Q}_{af}\cap\mathcal{Q}_{af}^*$?

If the answer to Question 3.9 is affirmative, then it is clear from Proposition 3.15 that the answers to Questions 3.14 and 3.16 would also be affirmative.

4. Quasisimilarity of weighted shifts. In this section we give necessary and sufficient conditions for two injective weighted shifts to be quasisimilar, and we prove that quasisimilar injective weighted shifts have equal spectra. Several authors have considered cases in which quasisimilarity of two operators implies their similarity or the equality of their spectra. Let S, T, and X be in $\mathcal{L}(\mathfrak{H})$ with X quasi-invertible and $SX=XT$. In [6, Theorem 4.4, page 55], Colojoară and Foiaş proved that if S and T are decomposable, then $\sigma(S)=\sigma(T)$. Each normal operator is decomposable [6, Example 1.6—ii, p. 33], and in [8] Douglas proved that if S and T are normal, then S is unitarily equivalent to T. Concerning operators that are not necessarily decomposable, Hoover [15, Theorem 3.1.] proved that if S and T are quasisimilar isometries, then S is unitarily equivalent to T; Clary [6, Theorem 2] proved that if S and T are quasisimilar hyponormal operators, then $\sigma(S)=\sigma(T)$.

Let $I=\mathbf{Z}$ or $\mathbf{Z}^+$ and let $\alpha=\{\alpha_n\}$ $(n\in I)$ denote a bounded sequence of non-zero complex numbers. An operator T in $\mathcal{L}(\mathfrak{H})$ is said to be an *(injective) weighted shift with weight sequence* α if there exists an orthonormal basis $\{e_n\}$ $(n\in I)$ for $\mathfrak{H}$ such that $Te_n=\alpha_n e_{n+1}$ $(n\in I)$. If $I=\mathbf{Z}^+$, T is a *unilateral* shift, while if $I=\mathbf{Z}$, T is a *bilateral* shift.

In [17, Appendix] Lambert proved that if S and T are quasisimilar injective unilateral weighted shifts, then S and T are similar. In the sequel we therefore consider only bilateral weighted shifts; thus we set $I=\mathbf{Z}$ and let $\{e_n\}$ $(n\in\mathbf{Z})$ denote a fixed orthonormal basis for $\mathfrak{H}$. Let W_α denote the bilateral shift with weight sequence α corresponding to this basis. It T is a bilateral shift in $\mathcal{L}(\mathfrak{H})$ with weight sequence α, then T is unitarily equivalent to W_α; moreover, W_α is unitarily equivalent to W_β, where $\beta_n=|\alpha_n|$ $(n\in\mathbf{Z})$. Thus, for questions concerning quasisimilarity of injective bilateral weighted shifts, it suffices to consider shifts of the form W_α, where $\alpha_n>0$ $(n\in\mathbf{Z})$, and in the sequel we implicitly assume that the shifts are of this form.

Lemma 4.1. *The following are equivalent for shifts W_α and W_β:*
i) *There exists an integer k such that*

$$\sup_{i \geqq \max(1-k,1)} (\alpha_0 \dots \alpha_{i-1+k})/(\beta_0 \dots \beta_{i-1}) < \infty$$

and

$$\sup_{i \geqq \max(1-k,1)} (\beta_{-1} \dots \beta_{-(i+k)})/(\alpha_{-1} \dots \alpha_{-i}) < \infty.$$

ii) *There exists a quasi-invertible operator X such that $W_\alpha X = X W_\beta$.*

Proof. Suppose that there is an integer k such that i) is satisfied. We consider five cases for the values of k and define X in each case by giving the values of X on the basis vectors.

Case 1. If $k \geqq 2$ we set

a) $Xe_i = (\alpha_0 \dots \alpha_{i-1+k})/(\beta_0 \dots \beta_{i-1}) e_{i+k}$ for $i \geqq 1$;

b) $Xe_0 = \alpha_0 \dots \alpha_{k-1} e_k$;

c) $Xe_i = (\beta_i \dots \beta_{-1} \alpha_0 \dots \alpha_{k+i-1}) e_{k+i}$ for $-k+1 \leqq i \leqq -1$;

d) $Xe_{-k} = \beta_{-k} \dots \beta_{-1} e_0$;

e) $Xe_{-(k+i)} = (\beta_{-(k+i)} \dots \beta_{-1})/(\alpha_{-i} \dots \alpha_{-1}) e_{-i}$ for $i \geqq 1$.

Case 2. If $k=1$ equation c) may be deleted.

Case 3. If $k=0$, equations b)—d) may be replaced by the equation $Xe_0 = e_0$.

Case 4. If $k \leqq -2$ we set

a) $Xe_i = (\alpha_0 \dots \alpha_{i-1+k})/(\beta_0 \dots \beta_{i-1}) e_{i+k}$ for $i \geqq 1-k$;

b) $Xe_{-k} = 1/(\beta_0 \dots \beta_{-k-1}) e_0$;

c) $Xe_{-k-i} = 1/(\alpha_{-i} \dots \alpha_{-1} \beta_0 \dots \beta_{-k-i-1}) e_{-i}$ for $1 \leqq i \leqq -(k+1)$;

d) $Xe_0 = 1/(\alpha_k \dots \alpha_{-1}) e_k$;

e) $Xe_{-i-k} = (\beta_{-(i+k)} \dots \beta_{-1})/(\alpha_{-i} \dots \alpha_{-1}) e_{-i}$ for $i \geqq 1-k$.

Case 5. If $k=-1$ equation c) of case 4) may be deleted. Condition i) implies that X may be extended to a quasi-invertible operator X in $\mathscr{L}(\mathfrak{H})$, and a calculation shows that $W_\alpha X = X W_\beta$.

For the converse, let X denote a quasi-invertible operator such that $W_\alpha X = X W_\beta$, and denote the matrix of X with respect to the basis (e_n) $(n \in Z)$ by (x_{ij}) $(-\infty < i, j < \infty)$. X has dense range, so there exists an integer m such that $x_{0,m} \neq 0$. An easy matrix calculation shows that for each pair of integers i and j we have (*) $\alpha_{i-1} x_{i-1,j-1} = x_{ij} \beta_{j-1}$. Successive application of (*) gives the identity (**)

$x_{-i,m-i}=x_{0,m}(\beta_{m-1}\cdots\beta_{m-i})/(\alpha_{-1}\cdots\alpha_{-i})$ for $i\geqq 1$. We consider the case $m\leqq 0$; if we set $k=-m$, then for $i\geqq 1$ we have

$$(\beta_{-1}\dots\beta_{-(i+k)})/(\alpha_{-1}\dots\alpha_{-i})=(\beta_{-1}\dots\beta_{-k})(\beta_{m-1}\dots\beta_{m-i})/(\alpha_{-1}\dots\alpha_{-i})=$$
$$=(\beta_{-1}\dots\beta_{-k})x_{-i,m-i}/x_{0,m}\leqq\|W_\beta\|^k\|X\|/x_{0,m}.$$

Now (*) also implies that (***) $x_{i,m+i}=(\alpha_{i-1}\cdots\alpha_0)/(\beta_{m+i-1}\cdots\beta_m)x_{0,m}$ for $i\geqq 1$, and therefore

$$(\alpha_0\dots\alpha_{k+i-1})/(\beta_0\dots\beta_{i-1})=(x_{k+i,i}/x_{0,m})(\beta_m\dots\beta_{-1})\leqq\|X\|\,\|W_\beta\|^k/x_{0,m},$$

which completes the proof when $m\leqq 0$. The proof for the case $m>0$ may be given similarly, by dividing (**) and (***) by $(\beta_0\cdots\beta_{m-1})$.

Theorem 4.2. *The following are equivalent for shifts W_α and W_β:*

i) *W_α is quasisimilar to W_β;*

ii) *There exists an integer k such that*

$$\sup_{i\geqq\max(1,1-k)}(\alpha_0\dots\alpha_{i-1+k})/(\beta_0\dots\beta_{i-1})<\infty$$

and

$$\sup_{i\geqq\max(1,1-k)}(\beta_{-1}\dots\beta_{-(i+k)})/(\alpha_{-1}\dots\alpha_{-i})<\infty,$$

and there exists an integer m such that

$$\sup_{i\geqq\max(1,1-m)}(\beta_0\dots\beta_{i-1+m})/(\alpha_0\dots\alpha_{i-1})<\infty,$$
$$\sup_{i\geqq\max(1,1-m)}(\alpha_{-1}\dots\alpha_{-(i+m)})/(\beta_{-1}\dots\beta_{-i})<\infty.$$

We state for ease of reference the following result concerning similarity of bilateral shifts.

Theorem 4.3. (Kelley [16]) *The shifts W_α and W_β are similar if and only if there exist an integer k and constants M and N such that*

$$0<M\leqq\prod_{j=0}^{n-1}(\alpha_{j+k}/\beta_j)\leqq N<\infty\quad\text{for}\quad n>0$$

and

$$0<M\leqq\prod_{j=1}^{-n}(\beta_{-j}/\alpha_{-j+k})\leqq N<\infty\quad\text{for}\quad n<0.$$

The next example shows that there exist shifts W_α and W_β that are quasisimilar but not similar.

Example 4.4. Let α be defined by $\alpha_n=1/2^{2n}$ for $n\geqq 0$ and $\alpha_n=1$ for $n<0$; let β be defined by $\beta_n=1/2^{2n-1}$ for $n\geqq 0$ and $\beta_n=1$ for $n<0$. With the values $k=0$ and $m=1$, α and β satisfy the inequalities of Theorem 4.2 ii), and thus W_α is quasisimilar to W_β.

If W_α is similar to W_β, let k, M, and N be as in Theorem 4.3. If $k \geqq 0$ and $n > 0$, then

$$0 < M \leqq (\alpha_k \dots \alpha_{n-1+k})/(\beta_0 \dots \beta_{n-1}) \leqq (\alpha_0 \dots \alpha_{n-1})/(\beta_0 \dots \beta_{n-1})$$

$= 1/2^n$; if $k < 0$ and $n > -k$, then

$$\infty > N \geqq (\alpha_k \dots \alpha_{n-1+k})/(\beta_0 \dots \beta_{n-1+k} \dots \beta_{n-1}) =$$
$$= 1/(2^{n+k} \beta_{n+k} \dots \beta_{n-1}) > 1/(2^{n+k} \beta_{n+k}) = 2^{n+k-1}.$$

In either case, since n is arbitrary, we have a contradiction, and Theorem 4.3 implies that W_α is not similar to W_β.

In Theorem 4.8 (below) we prove that quasisimilar shifts have equal spectra. We now show that this equality of spectra is not a consequence of the results of [6] or [7] by proving that both W_β and W_β^* are non-hyponormal and non-decomposable. Since $\beta_{-1} = 1, \beta_0 = 2$, and $\beta_1 = 1/2$, the weight sequence β is neither increasing nor decreasing and thus neither W_β nor W_β^* is hyponormal.

Let U denote a unilateral (unweighted) shift of multiplicity one in $\mathcal{L}(\mathfrak{H})$. Since $\lim_{n \to \infty} \beta_n = 0$, it is clear that W_β^* is unitarily equivalent to a compact perturbation of $T = U \oplus 0_{\mathfrak{H}}$. The results of [9] imply that T is non-quasitriangular, and thus W_β^* is non-quasitriangular. Theorem 3.1 of [1] states that each decomposable operator is quasitriangular, and it follows that W_β^* is non-decomposable.

To prove that W_β is non-decomposable, we recall from [7, Corollary 1.4, p. 31] that each decomposable operator has the single-valued extension property (in the sense of [7]). Let $D = \{\lambda \in \mathbb{C} \mid 0 < |\lambda| < 1\}$ and for $\lambda \in D$ let

$$f(\lambda) = e_1 + \sum_{n=1}^{\infty} (1/\beta_0) \lambda^n e_{-n+1} + \sum_{n=1}^{\infty} (\beta_1 \dots \beta_n) \lambda^{-n} e_{n+1}.$$

A straightforward series calculation shows that $f(\lambda)$ converges in $\mathfrak{H}$ and that $f: D \to \mathfrak{H}$ is analytic. Since $(W_\beta - \lambda) f(\lambda) = 0$ for each λ in D, W_β does not satisfy the single-valued extension property, and is thus non-decomposable. (Note, however, that W_β is quasitriangular.)

Lemma 4.5. *If W_α is quasisimilar to W_β and W_α is invertible, then W_β is invertible.*

Proof. Since W_α is invertible, $\epsilon \equiv \inf_{i \in Z} \alpha_i > 0$, and it clearly suffices to prove that $\inf_{j \in Z} \beta_j > 0$. Theorem 4.2 implies that there are integers k and m, and a constant $M > 0$, such that

i) $(\alpha_0 \dots \alpha_{i+k}) < M(\beta_0 \dots \beta_i), \quad i \geqq \max(0, -k);$

ii) $(\beta_{-1} \dots \beta_{-i-k}) < M(\alpha_{-1} \dots \alpha_{-i}), \quad i > \max(0, -k);$

iii) $(\beta_0 \dots \beta_{j+m}) < M(\alpha_0 \dots \alpha_j), \quad j \geqq \max(0, -m);$

iv) $(\alpha_{-1} \dots \alpha_{-j-m}) < M(\beta_{-1} \dots \beta_{-j}), \quad j > \max(0, -m).$

We consider first the case when $k+m \geqq 0$. For $j > \max(-m-1, 0)$, let $i = j+m+1$; now i) and iii) imply that $(\alpha_{j+1} \dots \alpha_{j+1+m+k})/\beta_{j+m+1} = (\alpha_0 \dots \alpha_{j+1+m+k} \times \beta_0 \dots \beta_{j+m})/(\beta_0 \dots \beta_{j+m+1}\alpha_0 \dots \alpha_j) < M^2$, and thus $\beta_{j+m+1} > (1/M^2)\epsilon^{m+k+1}$. For $j \geqq \max(1-m, k+2, 2)$, let $i = j-k-1$; now ii) and iv) imply that $(\alpha_{-j+k} \dots \alpha_{-j-m})/\beta_{-j} = (\alpha_{-1} \dots \alpha_{-j-m}\beta_{-1} \dots \beta_{-j+1})/(\beta_{-1} \dots \beta_{-j}\alpha_{-1} \dots \alpha_{-j+k+1}) < M^2$, and thus $\beta_{-j} > (1/M^2)\epsilon^{m+k+1}$. It now follows that $\inf_{j \in Z} \beta_j > 0$ in case $k+m \geqq 0$.

For $\delta > 0$, the shifts δW_α and δW_β are quasisimilar, and are invertible if and only if W_α and, respectively, W_β are invertible. We may thus assume that $\|W_\alpha\| \leqq 1$ and $\|W_\beta\| \leqq 1$; since $\alpha_n \leqq 1$ and $\beta_n \leqq 1$ $(n \in Z)$, we may also assume in i)—iv) that $k \geqq 0$ and $m \geqq 0$. Since the result is true when $k+m \geqq 0$, the proof is now complete.

Theorem 4.6. *If W_α is quasisimilar to W_β, and W_α is invertible, then W_α is similar to W_β.*

Proof. From Lemma 4.5, we may assume that W_β is also invertible. It is now straightforward to show that the inequalities of Theorem 4.2—ii) imply that the inequalities of Theorem 4.3 are satisfied for suitable values of k, M, and N, and thus W_α is similar to W_β. (The value for k in Theorem 4.3 may be taken to be that of either k or m from Theorem 4.2—ii); we omit the details.)

Lemma 4.7. *Let A and B be in $\mathscr{L}(\mathfrak{H})$. Suppose that there exist positive integers p and N, integers $a_1, \dots, a_p$, and positive numbers $c_1, \dots, c_p$, such that for each $n \geqq N, n+a_i > 0$ $(1 \leqq i \leqq p)$ and $\|A^n\| \leqq \max_{1 \leqq i \leqq p} c_i \|B^{n+a_i}\|$. Then $r(A) \leqq r(B)$.*

Proof. If T is in $\mathscr{L}(\mathfrak{H})$, then $r(T) = \lim \|T^n\|^{1/n}$, and it suffices to verify that for each integer a, $r(T) = \lim \|T^{n+a}\|^{1/n}$ $(n > -a)$. If $r(T) > 0$, then $\lim (\|T^{n+a}\|^{1/(n+a)})^{1/n} = 1$, so $\lim \|T^{n+a}\|^{1/n} = \lim \|T^{n+a}\|^{1/(n+a)} (\|T^{n+a}\|^{1/(n+a)})^{a/n} = r(T)$. If $r(T) = 0$, then $0 \leqq \overline{\lim} \|T^{n+a}\|^{1/n} \leqq \overline{\lim} \|T^{n+a}\|^{1/(n+a)} \|T\|^{a/n} = 0 = r(T)$, and the proof is complete.

Theorem 4.8. *If W_α is quasisimilar to W_β, then $\sigma(W_\alpha) = \sigma(W_\beta)$.*

Proof. From Theorem 4.6, we may assume that both W_α and W_β are non-invertible. In this case the spectra of W_α and W_β consist of closed disks centered at 0 [16], and therefore, by symmetry, it suffices to prove that $r(W_\alpha) \leqq r(W_\beta)$. For each $\epsilon > 0$, the shifts ϵW_α and ϵW_β are quasisimilar; moreover $r(\epsilon W_\alpha) =$

$=\epsilon r(W_\alpha)$ and $r(\epsilon W_\beta)=\epsilon r(W_\beta)$. We may therefore assume that $\|W_\alpha\|\leqq 1$ and $\|W_\beta\|\leqq 1$. Theorem 4.2 implies that there exists $M>0$ and integers k and m such that $\alpha_0\dots\alpha_{i-1+k}\leqq M\beta_0\dots\beta_{i-1}$ and $\beta_{-1}\dots\beta_{-(i+k)}\leqq M\alpha_{-1}\dots\alpha_{-i}$ for $i\geqq\max(1, 1-k)$, and such that $\beta_0\dots\beta_{i-1+m}\leqq M\alpha_0\dots\alpha_{i-1}$ and $\alpha_{-1}\dots\alpha_{-(i+m)}\leqq M\beta_{-1}\dots\beta_{-i}$ for $i\geqq\max(1, 1-m)$. Since $\alpha_j\leqq 1$ and $\beta_j\leqq 1$ for each j, we may assume that $k\geqq 0$ and $m\geqq 0$. To prove that $r(W_\alpha)\leqq r(W_\beta)$ we will show that the hypothesis of Lemma. 4.7 is satisfied with $A=W_\alpha$ and $B=W_\beta$. Since $\|W_\alpha^n\|=\sup_{j\in Z}\alpha_{j+1}\dots\alpha_{j+n}$, we may replace $\|A^n\|$ in Lemma 4.7 by an arbitrary product $\alpha_{j+1}\dots\alpha_{j+n}$, and we now estimate these products.

Let $N=k+m+1$ and $n>N$. We consider several special cases for the product $\alpha_{j+1}\dots\alpha_{j+n}$.

i) Suppose that $j\geqq 0$. Since $j\geqq 0\geqq k-n, j+m\geqq 0$, and $n-k\geqq m+1$, then

$$\alpha_{j+1}\dots\alpha_{j+n}\leqq M\big((\beta_0\dots\beta_{j+m})/(\alpha_0\dots\alpha_j)\big)(\beta_{j+m+1}\dots\beta_{j+n-k})\leqq M^2\|W_\beta^{n-k-m}\|.$$

ii) Suppose that $j\geqq 1$. Since $j=0\geqq -n+1+m$, we have $-j-n+m<-1$, and since $-j-k\leqq -1$ and $n\geqq m+k+1$, then

$$\alpha_{-j-1}\dots\alpha_{-j-n}\leqq M\big((\beta_{-1}\dots\beta_{-j-k})/(\alpha_{-1}\dots\alpha_{-j})\big)(\beta_{-j-k-1}\dots\beta_{-j-n+m})\leqq M^2\|W_\beta^{n-k-m}\|.$$

iii) We also have $\alpha_0\dots\alpha_{n-1}\leqq M\beta_0\dots\beta_{n-1-k}\leqq M\|W_\beta^{n-k}\|$, and $\alpha_{-1}\dots\alpha_{-n}\leqq$ $\leqq M\beta_{-1}\dots\beta_{-n+m}\leqq M\|W_\beta^{n-m}\|$.

The remaining products are of the form $\alpha_{j+1}\dots\alpha_{-1}\alpha_0\dots\alpha_{j+n}$ for $-n\leqq j\leqq -2$. Since $j+n\geqq 0$ and $j+1\leqq -1$, there are $p=-j-1\geqq 1$ factors with a negative subscript and $q=j+n+1\geqq 1$ factors with a nonnegative subscript. We consider the possible values of p and q.

iv) If $p>m$ and $q>k$, then $-1-j>m$ and $j+n-k>-1$, and therefore

$$\alpha_{j+1}\dots\alpha_{-1}\alpha_0\dots\alpha_{j+n}\leqq M^2\beta_{j+1+m}\dots\beta_{-1}\beta_0\dots\beta_{j+n-k}\leqq M^2\|W_\beta^{n-k-m}\|.$$

v) If $p=-j-1\leqq m$, then $q=j+n+1>k$ since $p+q=n>m+k$. Now

$$\alpha_{j+1}\dots\alpha_{j+n}\leqq\alpha_0\dots\alpha_{j+n}\leqq M\beta_0\dots\beta_{j+n-k}\leqq M\|W_\beta^{j+n-k+1}\|,\ \text{where}\ -1-m\leqq j\leqq -2.$$

Thus $\alpha_{j+1}\dots\alpha_{j+n}\leqq(M)\max\{\|W_\beta^{n+a}\|: -m-k\leqq a\leqq -1-k\}$.

vi) If $q=n+j+1\leqq k$, then $p=-j-1>m$, and $\alpha_{j+1}\dots\alpha_{-1}\alpha_0\dots\alpha_{j+n}\leqq$ $=M\beta_{j+1+m}\dots\beta_{-1}\alpha_0\dots\alpha_{j+n}\leqq M\|W_\beta^{-j-m-1}\|$. Since $n+1-k\leqq -j\leqq n$, then $n-k-m\leqq$ $\leqq -j-m-1\leqq n-m-1$, and so $\alpha_{j+1}\dots\alpha_{j+n}\leqq(M)\max\{\|W_\beta^{n+a}\|: -k-m\leqq a\leqq$ $\leqq -m-1\}$. The proof is now complete.

Remark. The example just before Corollary 2.6 shows that the conclusion of Theorem 4.8 is false if we only have a single equation $SX=XT$ (where S and T are injective weighted shifts and X is quasi-invertible).

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