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On a property of strictly logarithmic concave functions

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1. Introduction. In the work [1] by A. PRÉKOPA the following theorem was proved.

Theorem 1. *Let $f(x, y)$ be a function of $n+m$ variables, where x is an n-component and y is an m-component vector. Suppose that f is logarithmic concave in R^{n+m} and let A be a convex subset of R^m. Then the function*

$$I(x) = \int_A f(x, y)\,dy$$

is logarithmic concave in the entire space R^n.

The main result of this work is a similar statement for strictly logarithmic concave functions.

Let f be a non-negative logarithmic concave function in R^{n+m}. We denote

$$D = \{z \in R^{n+m} : f(z) > 0\}, \quad D(x) = \{y \in R^m : f(x, y) > 0\}, \quad B = \{x \in R^n : I(x) > 0\}.$$

The sets $D(x)$ $(x \in R^n)$, D and B are convex in R^m, R^{n+m} and R^n, respectively. The relative interior of a convex set $C \subset R^k$ is denoted by $\operatorname{ri} C$ (see [2] p. 57) and the closure of C by $\bar{C}$. The basic theorem of this work is

Theorem 2. *Let $f(x, y)$ be a function of $n+m$ variables where $x \in R^n$, $y \in R^m$. Suppose f is logarithmic concave in R^{n+m} and strictly logarithmic concave in ri D, and let A be convex subset of the space R^m. If the sets $D(x) \subset R^m$ are bounded for every $x \in R^n$, then the function I is logarithmic concave in the entire space R^n and strictly logarithmic concave in ri B.*

The first part of this statement is just Theorem 1. We shall begin with proving the strictly logarithmic concavity of the function I in $\operatorname{ri} B$ with subsidiary statements.

In this work the terminology has been taken from [2].

2. Auxiliary statements. We define the function $g: R^{n+m} \to R$ as follows

$$g(z) = -\ln f(z), \quad z = (x, y) \in R^{n+m}.$$

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Under the conditions imposed on f, g is a proper convex function with effective domain

$$\operatorname{dom} g = \{z \in R^{n+m} : g(z) < \infty\} = D.$$

We denote

$$f_*(z) = \limsup_{v \to z} f(v),\ v,\ z \in R^{n+m}.$$

Lemma 1. *For all* $z \in R^{n+m}$

$$(\operatorname{cl} g)(z) = -\ln f_*(z),$$

where cl g *is the closure of the convex function* g.

Proof. From the definition of cl g ([2] p. 67—68) and g we have

$$(\operatorname{cl} g)(z) = \liminf_{v \to z} g(v) = \liminf_{v \to z} [-\ln f(v)] = -\limsup_{v \to z} \ln f(v).$$

The continuity and strict monotonicity of the logarithm implies that

$$\limsup_{v \to z} \ln f(v) = \ln [\limsup_{v \to z} f(v)] = \ln f_*(z).$$

The lemma is proved.

Corollary 1. *The function* f_* *is logarithmic concave in* R^{n+m}.

Corollary 2. *The function* f *agrees with* f_* *in* R^{n+m} *except perhaps at relative boundary points of a convex set* D.

Corollaries 1 and 2 follow from Theorem 7.4 [2] and Lemma 1.

Lemma 2. *If* f *is upper semi-continuous on the closed bounded set* $D \subset R^k$, *then there exists* $z_0 \in D$ *such that*

$$\sup_{z \in D} f(z) = f(z_0).$$

Proof. Let $\sup_{z \in D} f(z) = C$ and $\varepsilon_n > 0$, $\varepsilon_n \to 0$ as $n \to \infty$. Then one can find a sequence $\{z_n\} \subset D$ such that for $n = 1, 2, \ldots$

$$f(z_n) > C - \varepsilon_n.$$

Since D is a bounded closed set without loss of generality we may assume that

$$z_n \to z_0 \quad \text{as} \quad n \to \infty, \quad z_0 \in D, \quad \text{and} \quad |z_n - z_0| \leqq \varepsilon_n \quad \text{for} \quad n = 1, 2, \ldots.$$

Hence the inequality

$$(1) \qquad \sup_{|z_0 - z| < \varepsilon_n} f(z) \geqq f(z_n) > C - \varepsilon_n, \quad n = 1, 2, \ldots$$

is valid. Taking into account the upper semi-continuity of the function f we get from (1) that

$$f(z_0) = \lim_{n \to \infty} \sup_{|z - z_0| < \varepsilon_n} f(z) \geqq C.$$

Thus $f(z_0) = C$. The lemma is proved.

Lemma 3. *Let* $z_1, z_2 \in R^{n+m}$ *and* $0<\lambda<1$. *If* f *is strictly logarithmic concave in* $\mathrm{ri}\, D \subset R^{n+m}$ *and* $\lambda z_1+(1-\lambda)z_2 \in \mathrm{ri}\, D$, *then the inequality*

$$f_*(\lambda z_1+(1-\lambda)z_2) > f_*^{\lambda}(z_1) f_*^{1-\lambda}(z_2) \tag{2}$$

is valid.

Proof. Two cases are possible.

(i) One of the points, either z_1 or z_2, does not belong to $\bar{D}$. In this case inequality (2) is obviously correct.

(ii) Let $z_1, z_2 \in \bar{D}$. Let us draw a straight line l across the points z_1 and z_2 and choose some point $z \in l \cap \mathrm{ri}\, D$. Let $\varphi(\mu)=g(\mu z_1+(1-\mu)z_2)$. Then $\mathrm{cl}\, \varphi$ is a proper strictly convex function on [0, 1]. From Theorems 7.4 and 7.5 of [2] it follows that $(\mathrm{cl}\, \varphi)(\mu)=\varphi(\mu)$ for $\mu \in (0, 1)$ and

$$(\mathrm{cl}\, \varphi)(1) = \lim_{v \uparrow 1} (v+(1-v)\mu_0) = \lim_{v \uparrow 1} g(vz_1+(1-v)z) = (\mathrm{cl}\, g)(z_1),$$

$$(\mathrm{cl}\, \varphi)(0) = \lim_{v \uparrow 1} (\mu_0-v\mu_0) = \lim_{v \uparrow 1} g(vz_2+(1-v)z) = (\mathrm{cl}\, g)(z_2),$$

where $z=\mu_0 z_1+(1-\mu_0)z_2$. This means that the function $\mathrm{cl}\, g$ is strictly convex on the set $1 \cap \bar{D}$, that is

$$(\mathrm{cl}\, g)(\lambda z_1+(1-\lambda)z_2) < \lambda(\mathrm{cl}\, g)(z_1)+(1-\lambda)(\mathrm{cl}\, g)(z_2), \quad 0<\lambda<1. \tag{3}$$

From (3) and Lemma 1 it can be seen that inequality (2) is true. The lemma is proved.

Corollary 3. *Let* $z_1, z_2 \in R^{n+m}$ *and* $0<\lambda<1$. *If* f *is strictly logarithmic concave in* $\mathrm{ri}\, D \subset R^{n+m}$ *and* $\lambda z_1+(1-\lambda)z_2 \in \mathrm{ri}\, D$, *then we have the inequality*

$$f(\lambda z_1+(1-\lambda)z_2) > f^{\lambda}(z_1) f^{1-\lambda}(z_2).$$

Lemma 4. *If* $x_0 \in \mathrm{ri}\, B$, $y_0 \in \mathrm{int}\, D(x_0)$, *then* $z_0=(x_0 y_0) \in \mathrm{ri}\, D$.

Proof. Let P be the projection $(x, y) \to x$ from R^{n+m} onto R^n. It can be shown that $B \subset PD$ and if B is not empty then the dimension of the set B agrees with that of PD. Hence $\mathrm{ri}\, B \subset \mathrm{ri}(PD)$ and the point $(x_0, y_0) \in \mathrm{ri}\, D$ by Theorem 6.8 of [2]. The lemma is proved.

3. Proof of Theorem 2. We denote

$$D_*(x) = \{y \in R^m : f_*(x, y) > 0\}.$$

For all $x \in \mathrm{ri}\, B$ the sets $D(x)$ and $D_*(x)$ have the same closure and the same interior (see Corollary 2).

Let $x_1, x_2 \in \text{ri } B$, $0 < \lambda < 1$ and $x_0 = \lambda x_1 + (1-\lambda)x_2$. We define the functions f_1 and f_2 as follows:

$$f_1(y) = f_*(x_1, y) \quad \text{if} \quad y \in \bar{A}, \quad \text{and} \quad f_1(y) = 0 \quad \text{otherwise;}$$

$$f_2(y) = f_*(x_2, y) \quad \text{if} \quad y \in \bar{A}, \quad \text{and} \quad f_2(y) = 0 \quad \text{otherwise.}$$

For given $y \in R^m$ and λ, $0 < \lambda < 1$, we shall denote by $S(y; \lambda)$ the set of points (u, v) such that $u, v \in R^m$, $\lambda u + (1-\lambda)v = y$.

It can be shown that for all $y \in R^m$

$$\sup_{S(y;\lambda)} f_*^\lambda(x_1, u) f_*^{1-\lambda}(x_2, v) \geqq \sup_{S(y;\lambda)} f_1^\lambda(u) f_2^{1-\lambda}(v)$$

and for $y \bar{\in} \bar{A} \cap \bar{D}(x_0)$

$$\sup_{S(y;\lambda)} f_1^\lambda(u) f_2^{1-\lambda}(v) = 0.$$

Since f_* is logarithmic concave in R^{n+m} (Corollary 1), the following inequality will be valid for all $y \in R^m$:

$$f_*(x_0, y) \geqq \sup_{S(\lambda; y)} f_*^\lambda(x_1, u) f_*^{1-\lambda}(x_2, v).$$

We shall prove that for all $y \in \text{int } D(x_0)$ we have

(4) $$f_*(x_0, y) > \sup_{S(y;\lambda)} f_*^\lambda(x_1, u) f_*^{1-\lambda}(x_2, v).$$

Suppose on the contrary that there could be found a $y_0 \in \text{int } D(x_0)$ such that

$$f_*(x_0, y_0) = \sup_{S(y_0;\lambda)} f_*^\lambda(x_1, u) f_*^{1-\lambda}(x_2, v).$$

In this case $f_*(x_0, y_0) > 0$ as $(x_0, y_0) \in \text{ri } D$ (Lemma 4). According to Lemma 2 there exists a point $(u_0, v_0) \in S(y_0; \lambda)$ such that

$$u_0 \in \bar{D}(x_1), \quad v_0 \in \bar{D}(x_2) \quad \text{and} \quad f_*(x_0, y_0) = f_*^\lambda(x_1, u_0) f_*^{1-\lambda}(x_2, v_0).$$

We have got a contradiction to Lemma 3. So, for all $y \in \text{int } D(x_0)$ inequality (4) is valid.

From the definition of the function I and from Corollary 2 we get

$$I(x_0) = \int_{\bar{A}} f(x_0, y)\, dy = \int_{\bar{A} \cap \bar{D}(x_0)} f_*(x_0, y)\, dy.$$

Taking into account (4) and Theorem 3 of [1] we obtain:

$$\int_{\bar{A}\cap \bar{D}(x_0)} f_*(x_0, y)\,dy > \int_{\bar{A}\cap \bar{D}(x_0)} \sup_{S(y;\lambda)} f_*^{\lambda}(x_1, u) f_*^{1-\lambda}(x_2, v)\,dy \geqq$$

$$\geqq \int_{D(x_0)\cap \bar{A}} \sup_{S(y;\lambda)} f_1^{\lambda}(u) f_2^{1-\lambda}(v)\,dy = \int_{R^m} \sup_{S(y;\lambda)} f_1^{\lambda}(u) f_2^{1-\lambda}(v)\,dy \geqq$$

$$\geqq \Big[\int_{R^m} f_1(y)\,dy\Big]^{\lambda} \Big[\int_{R^m} f_2(y)\,dy\Big]^{1-\lambda} = \Big[\int_{\bar{A}\cap \bar{D}(x_1)} f_*(x_1, y)\,dy\Big]^{\lambda} \Big[\int_{\bar{A}\cap \bar{D}(x_2)} f_*(x_2, y)\,dy\Big]^{1-\lambda} =$$

$$= [I(x_1)]^{\lambda} [I(x_2)]^{1-\lambda}.$$

The theorem is proved.

Corollary 4. *Let* $x_1, x_2 \in R^n$ *and* $0 < \lambda < 1$. *If* $\lambda x_1 + (1-\lambda) x_2 \in \operatorname{ri} B$, *then the inequality*

(5) $$I(\lambda x_1 + (1-\lambda) x_2) > [I(x_1)]^{\lambda} [I(x_2)]^{1-\lambda}$$

is valid.

Proof. It follows from Theorem 2 and Corollary 3.

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References

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