Acta Sci. Math., 38 (1976), 399—416

On some recurrence equations in a Banach algebra

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1. Introduction. The aim of this paper is to find the solutions of the recurrence equations

(1) $$f_n = \mathbf{L}\{f_{n-1}g_1\}+\mathbf{L}^*\{g_2 f_{n-1}\}$$

and

(2) $$f_n = \mathbf{L}\{f_{n-1}g_1\}+\mathbf{L}^*\{f_{n-1}g_2\},$$

and the solution of the system of recurrence equations

(3) $$u_n = \mathbf{L}\{u_{n-1}h_1+v_{n-1}h_2\}$$

(4) $$v_n = \mathbf{L}^*\{u_{n-1}h_3+v_{n-1}h_4\}$$

where $f_0, g_1, g_2, u_0, v_0, h_1, h_2, h_3, h_4$ are elements of a Banach algebra $\mathbf{R}$, $\mathbf{L}$ is a projection in $\mathbf{R}$, and $\mathbf{L}+\mathbf{L}^*$ is the identity transformation in $\mathbf{R}$. The solutions of these recurrence equations make it possible to determine the stochastic laws of the fluctuations of the partial sums for a sequence of independent and identically distributed real random variables and for a semi-Markov sequence of real random variables.

This paper generalizes and extends some earlier results of the author [11].

2. Preliminaries. Let $\mathbf{R}$ be a Banach algebra of elements $f, f_1, f_2, \dots$. We denote by θ the zero element and by e the identity element of R. Denote by $\|f\|$ the norm of f and let $\|e\|=1$.

Throughout this paper we shall consider transformations $\mathbf{T}$ in $\mathbf{R}$ which satisfy the following conditions:

(i) The transformation $\mathbf{T}$ is a bounded linear transformation of $\mathbf{R}$ into itself.

(ii) The transformation $\mathbf{T}$ is a projection, that is,

$$\mathbf{T}^2\{f\} = \mathbf{T}\{f\} \text{ for all } f.$$

Received July 10, 1975.

(iii) If either $\mathbf{T}\{f_i\}=f_i$ or $\mathbf{T}\{f_i\}=\theta$ for $i=1, 2$, then

$$\mathbf{T}\{f_1f_2\} = \mathbf{T}\{f_1\}\mathbf{T}\{f_2\}.$$

We note that (iii) can be expressed in the following equivalent form:

$$\mathbf{T}\{f_1f_2\} = \mathbf{T}\{f_1\mathbf{T}\{f_2\}\}+\mathbf{T}\{\mathbf{T}\{f_1\}f_2\}-\mathbf{T}\{f_1\}\mathbf{T}\{f_2\} \tag{5}$$

for all f_1 and f_2.

The norm of $\mathbf{T}$ is defined as the smallest nonnegative number $\|\mathbf{T}\|$ for which $\|\mathbf{T}\{f\}\| \leqq \|\mathbf{T}\|\,\|f\|$ for all $f\in\mathbf{R}$. If $\mathbf{T}$ is not the zero transformation, then (ii) implies that $\|\mathbf{T}\|\geqq 1$.

We define

$$\mathbf{T}^*\{f\} = f-\mathbf{T}\{f\} \tag{6}$$

for any $\mathbf{T}$ and f. If $\mathbf{T}$ statisfies the conditions (i), (ii), (iii), then $\mathbf{T}^*$ too satisfies these conditions. We have $\|\mathbf{T}^*\|\leqq 1+\|\mathbf{T}\|$.

It will be convenient to introduce here some useful definitions which we shall need later. Let us suppose that $a_0=b_0=e$ and $a_n=\mathbf{T}\{a_{n-1}g\}$ and $b_n=\mathbf{T}^*\{gb_{n-1}\}$ for $n=1, 2, \ldots$ where $g\in\mathbf{R}$. For a nonzero transformation $\mathbf{T}$ let us define $\mu(\mathbf{T})$ as the largest nonnegative number for which

$$\sum_{n=0}^{\infty} \|a_n\|\,|\varrho|^n < \infty$$

whenever $|\varrho|\,\|g\|<\mu(\mathbf{T})$ and $g\in\mathbf{R}$. Similarly for a nonzero transformation $\mathbf{T}^*$ let us define $\bar\mu(\mathbf{T}^*)$ as the largest nonnegative number for which

$$\sum_{n=0}^{\infty} \|b_n\|\,|\varrho|^n < \infty$$

whenever $|\varrho|\,\|g\|<\bar\mu(\mathbf{T}^*)$ and $g\in\mathbf{R}$. Obviously

$$\|\mathbf{T}\|^{-1} \leqq \mu(\mathbf{T}) \leqq 1 \quad \text{and} \quad \|\mathbf{T}^*\|^{-1} \leqq \bar\mu(\mathbf{T}^*) \leqq 1. \tag{7}$$

If $\|\mathbf{T}\|=0$, then we write $\mu(\mathbf{T})=\infty$ and if $\|\mathbf{T}^*\|=0$, then we write $\bar\mu(\mathbf{T}^*)=\infty$. Let us define

$$c(\mathbf{T}) = \min\left(\mu(\mathbf{T}), \bar\mu(\mathbf{T}^*)\right), \tag{8}$$

and

$$\gamma(\mathbf{T}) = \min\left(c(\mathbf{T}), c(\mathbf{T}^*)\right). \tag{9}$$

We note that if $\mathbf{R}$ is a commutative Banach algebra, and if $\mathbf{T}$ satisfies (i), (ii), (iii), then $c(\mathbf{T})=1$. If $\mathbf{R}$ is a commutative Banach algebra, then we can prove by mathematical induction that

$$na_n = \sum_{k=1}^{n} a_{n-k}\mathbf{T}\{g^k\}$$

for $n=1, 2, \ldots$. Hence

(10) $$n\|a_n\| \leqq \|\mathbf{T}\| \sum_{k=1}^{n} \|a_{n-k}\|(\|g\|)^k$$

for $n=1, 2, \ldots$. By (10) it follows by induction that

$$\|a_n\| \leqq \binom{\|\mathbf{T}\|+n-1}{n}(\|g\|)^n$$

for $n=0, 1, 2, \ldots$. This implies that $\mu(\mathbf{T}) \geqq 1$. Since $\bar{\mu}(\mathbf{T}^*) \geqq 1$ also holds, by (7) and (8) we obtain that $c(\mathbf{T})=1$.

3. The method of factorization. In solving various recurrence equations in the space $\mathbf{R}$ we shall use the method of factorization. It seems the method of factorization in Banach spaces was used for the first time in 1956 by P. MASANI [6]. See also G. BAXTER [1], [2] and I. C. GOHBERG [4].

Let $h(\varrho)$ be an element of $\mathbf{R}$ for $|\varrho|<r$ where r is some positive real number. We say that the element $h(\varrho)$ can be represented by a Taylor series about $\varrho=0$ in the circle $|\varrho|<r$ if

$$h(\varrho) = \sum_{n=0}^{\infty} h_n \varrho^n$$

and

$$\sum_{n=0}^{\infty} \|h_n\| \, |\varrho|^n < \infty$$

for $|\varrho|<r$.

Let us suppose that $\mathbf{T}$ is a transformation in $\mathbf{R}$ which satisfies (i), (ii) and (iii). We shall consider various elements $h(\varrho)$ of $\mathbf{R}$ for $|\varrho|<r$ which satisfy one of the following two properties.

Property (a). The element $h(\varrho)$ has an inverse $[h(\varrho)]^{-1}$, $h(0)=e$, $\mathbf{T}\{h(\varrho)-e\}= =h(\varrho)-e$, $\mathbf{T}\{[h(\varrho)]^{-1}-e\}=[h(\varrho)]^{-1}-e$, $h(\varrho)$ and $[h(\varrho)]^{-1}$ can be represented by a Taylor series about $\varrho=0$.

Property (b). The element $h(\varrho)$ has an inverse $[h(\varrho)]^{-1}$, $h(0)=e$, $\mathbf{T}^*\{h(\varrho)-e\}= =h(\varrho)-e$, $\mathbf{T}^*\{[h(\varrho)]^{-1}-e\}=[h(\varrho)]^{-1}-e$, $h(\varrho)$ and $[h(\varrho)]^{-1}$ can be represented by a Taylor series about $\varrho=0$.

The method of factorization is based on the following theorem.

Theorem 1. *If* $g\in\mathbf{R}$ *and if* $|\varrho|\,\|g\|<c(\mathbf{T})$, *then there exist two elements* $g^+(\varrho)\in\mathbf{R}$ *and* $g^-(\varrho)\in\mathbf{R}$ *such that*

(11) $$e-\varrho g = g^+(\varrho)g^-(\varrho)$$

where $g^+(\varrho)$ *satisfies* (a) *and* $g^-(\varrho)$ *satisfies* (b). *The elements* $g^+(\varrho)$ *and* $g^-(\varrho)$ *are uniquely determined by* (a), (b) *and* (11).

Proof. First, we shall construct two elements $g^+(\varrho)$ and $g^-(\varrho)$ which satisfy (a), (b) and (11). Let us suppose that $a_0=b_0=e$ and $a_n=\mathbf{T}\{a_{n-1}g\}$ and $b_n=\mathbf{T}^*\{gb_{n-1}\}$ for $n=1, 2, \dots$. Then

$$a(\varrho) = \sum_{n=0}^{\infty} a_n \varrho^n \in \mathbf{R} \tag{12}$$

for $|\varrho|\,\|g\|<\mu(\mathbf{T})$ and

$$b(\varrho) = \sum_{n=0}^{\infty} b_n \varrho^n \in \mathbf{R} \tag{13}$$

for $|\varrho|\,\|g\|<\bar{\mu}(\mathbf{T}^*)$. From the definitions of $a(\varrho)$ and $b(\varrho)$ it follows immediately that $a(0)=b(0)=e$, $\mathbf{T}\{a(\varrho)-e\}=a(\varrho)-e$, $\varrho\mathbf{T}\{a(\varrho)g\}=a(\varrho)-e$, $\mathbf{T}^*\{b(\varrho)-e\}=b(\varrho)-e$ and $\varrho\mathbf{T}^*\{gb(\varrho)\}=b(\varrho)-e$.

Now we shall prove that

$$(e-\varrho g)b(\varrho)a(\varrho) = b(\varrho)a(\varrho)(e-\varrho g) = e \tag{14}$$

and

$$a(\varrho)(e-\varrho g)b(\varrho) = e \tag{15}$$

for $|\varrho|\,\|g\|<c(\mathbf{T})$. If we take into consideration that $\mathbf{T}\{a(\varrho)(e-\varrho g)\}=\mathbf{T}\{e\}$ and $\mathbf{T}^*\{(e-\varrho g)\,b(\varrho)\}=\mathbf{T}^*\{e\}$, then by (5) it follows that

$$\mathbf{T}\{b(\varrho)a(\varrho)(e-\varrho g)\} = \mathbf{T}\{e\}$$

and

$$\mathbf{T}^*\{(e-\varrho g)b(\varrho)a(\varrho)\} = \mathbf{T}^*\{e\}.$$

If we add these two equations, then we get

$$b(\varrho)a(\varrho) = e+\varrho\mathbf{T}\{b(\varrho)a(\varrho)g\}+\varrho\mathbf{T}^*\{gb(\varrho)a(\varrho)\}. \tag{16}$$

If $|\varrho|\,\|g\|<c(\mathbf{T})$, then $b(\varrho)a(\varrho)\in\mathbf{R}$ and in the above equation we can write that

$$b(\varrho)\,a(\varrho) = \sum_{n=0}^{\infty} \gamma_n \varrho^n$$

where $\gamma_n\in\mathbf{R}$ for $n=0, 1, 2, \dots$. By forming the coefficient of ϱ^n in (16), we get

$$\gamma_n = \mathbf{T}\{\gamma_{n-1}g\}+\mathbf{T}^*\{g\gamma_{n-1}\} \tag{17}$$

for $n=1, 2, \dots$. Since $\gamma_0=e$, it follows from (17) by induction that $\gamma_n=g^n$ for $n=1, 2, \dots$. This implies (14).

By (5) it follows also that

$$\mathbf{T}\{a(\varrho)(e-\varrho g)b(\varrho)\} = \mathbf{T}\{e\} \quad \text{and} \quad \mathbf{T}^*\{a(\varrho)(e-\varrho g)b(\varrho)\} = \mathbf{T}^*\{e\}.$$

If we add these two equations, then we get (15).

We can conclude from (14) and (15) that $[a(\varrho)]^{-1}$ and $[b(\varrho)]^{-1}$ exist and

$$[a(\varrho)]^{-1} = (e-\varrho g)b(\varrho) \tag{18}$$

and

$$[b(\varrho)]^{-1} = a(\varrho)(e-\varrho g) \tag{19}$$

for $|\varrho|\,\|g\|<c(\mathbf{T})$.

If we define

$$g^+(\varrho) = [a(\varrho)]^{-1} \tag{20}$$

and

$$g^-(\varrho) = [b(\varrho)]^{-1} \tag{21}$$

for $|\varrho|\,\|g\|<c(\mathbf{T})$, then $g^+(\varrho)$ and $g^-(\varrho)$ satisfy (a), (b) and (11).

It remains to show that $g^+(\varrho)$ and $g^-(\varrho)$ are uniquely determined by (a), (b) and (11). This fact will be proved as a consequence of Theorem 3.

In exactly the same way as we proved Theorem 1 we can prove the following theorem too.

Theorem 2. *If* $g\in\mathbf{R}$ *and if* $|\varrho|\,\|g\|<c(\mathbf{T}^*)$ *then there exist two elements* $h^+(\varrho)\in\mathbf{R}$ *and* $h^-(\varrho)\in\mathbf{R}$ *such that*

$$e-\varrho g = h^-(\varrho)h^+(\varrho) \tag{22}$$

where $h^+(\varrho)$ *satisfies* (a) *and* $h^-(\varrho)$ *satisfies* (b). *The elements* $h^+(\varrho)$ *and* $h^-(\varrho)$ *are uniquely determined by* (a), (b) *and* (22).

If we suppose that $c_0=d_0=e$, $c_n=\mathbf{T}\{gc_{n-1}\}$ and $d_n=\mathbf{T}^*\{d_{n-1}g\}$ for $n=1, 2, \dots,$

$$c(\varrho) = \sum_{n=0}^{\infty} c_n\varrho^n \tag{23}$$

and

$$d(\varrho) = \sum_{n=0}^{\infty} d_n\varrho^n, \tag{24}$$

then in Theorem 2 we can write that $h^+(\varrho)=[c(\varrho)]^{-1}$ and $h^-(\varrho)=[d(\varrho)]^{-1}$.

We note that if $\mathbf{R}$ is a commutative Banach algebra, then (12), (13), (23) and (24) can be expressed in the following explicit forms

$$a(\varrho) = c(\varrho) = \exp\big\{-\mathbf{T}\{\log(e-\varrho g)\}\big\} \quad \text{and} \quad b(\varrho) = d(\varrho) = \exp\big\{-\mathbf{T}^*\{\log(e-\varrho g)\}\big\}$$

where

$$\log(e-\varrho g) = -\sum_{n=1}^{\infty}\frac{g^n\varrho^n}{n} \quad \text{for} \quad |\varrho|\,\|g\|<1 \quad \text{and} \quad \exp(f) = e+\sum_{n=1}^{\infty}\frac{f^n}{n!}$$

for any $f\in\mathbf{R}$.

4. Some linear transformations in R. In this section we shall consider transformations **L** which satisfy conditions (i), (ii), (iii) and can be represented in the form

$$\mathbf{L}\{f\} = \mathbf{T}\{f\} - \alpha(f)e \tag{25}$$

where **T** is a given transformation satisfying (i), (ii), (iii) and $\alpha(f)$ is a complex (or real) functional on **R**.

We can prove that **L** satisfies the above conditions if and only if $\alpha(f)$ satisfies one of the following three sets of conditions: (1) $\alpha(f)\equiv 0$, (2) $\alpha(cf)=c\alpha(f)$ for any constant c, $\alpha(f_1+f_2)=\alpha(f_1)+\alpha(f_2)$, $\alpha(\mathbf{T}\{f\})=\alpha(f)$, $\alpha(\mathbf{T}\{f_1\}\mathbf{T}\{f_2\})=\alpha(f_1)\alpha(f_2)$, $\alpha(e)e=\mathbf{T}\{e\}$, $|\alpha(f)|\leqq\|\mathbf{T}\|^2\|f\|$, (3) $\alpha(cf)=c\alpha(f)$ for any constant c, $\alpha(f_1+f_2)=$ $=\alpha(f_1)+\alpha(f_2)$, $\alpha(\mathbf{T}^*\{f\})=\alpha(f)$, $\alpha(\mathbf{T}^*\{f_1\}\mathbf{T}^*\{f_2\})=-\alpha(f_1)\alpha(f_2)$, $\alpha(e)e=\mathbf{T}^*\{e\}$, $|\alpha(f)|\leqq\|\mathbf{T}^*\|^2\|f\|$.

Later we shall prove that for any **L** defined by (25) we have

$$c(\mathbf{L}) = c(\mathbf{T}) \tag{26}$$

where $c(\mathbf{T})$ is defined by (8).

We shall state here a few general relations which can be deduced from (5). In agreement with (6) we define $\mathbf{L}^*\{f\}=f-\mathbf{L}\{f\}$ for any f.

For any $f\in\mathbf{R}$ we have

$$\mathbf{T}\{\mathbf{T}\{e\}f\} = \mathbf{T}\{e\}\mathbf{T}\{f\} \quad\text{and}\quad \mathbf{T}\{f\mathbf{T}\{e\}\} = \mathbf{T}\{f\}\mathbf{T}\{e\}. \tag{27}$$

By (25) and (27) it follows that if $f\in\mathbf{R}$, $\gamma\in\mathbf{R}$ and $\mathbf{T}\{\gamma\}=\mathbf{T}\{e\}$, then

$$\mathbf{L}\{f\gamma\} = \mathbf{L}\{\mathbf{L}\{f\}\gamma\} \quad\text{and}\quad \mathbf{L}\{\gamma f\} = \mathbf{L}\{\gamma\mathbf{L}\{f\}\}, \tag{28}$$

and if $f\in\mathbf{R}$, $\gamma\in\mathbf{R}$ and $\mathbf{T}^*\{\gamma\}=\mathbf{T}^*\{e\}$, then

$$\mathbf{L}^*\{f\gamma\} = \mathbf{L}^*\{\mathbf{L}^*\{f\}\gamma\} \quad\text{and}\quad \mathbf{L}^*\{\gamma f\} = \mathbf{L}^*\{\gamma\mathbf{L}^*\{f\}\}. \tag{29}$$

If $f\in\mathbf{R}$, $\gamma_i\in\mathbf{R}$ $(i=1, 2)$ and $\mathbf{T}\{\gamma_i\}=\mathbf{T}\{e\}$ $(i=1, 2)$, then we have

$$\mathbf{L}\{\gamma_1\mathbf{L}\{f\}\gamma_2\} = \mathbf{L}\{\gamma_1 f\gamma_2\} \quad\text{and}\quad \mathbf{L}\{\gamma_1\mathbf{L}^*\{f\}\gamma_2\} = \theta. \tag{30}$$

The first equation follows from the repeated applications of (28). The second follows from the first one.

If $f\in\mathbf{R}$, $\gamma_i\in\mathbf{R}$ $(i=1, 2)$ and $\mathbf{T}^*\{\gamma_i\}=\mathbf{T}^*\{e\}$ $(i=1, 2)$, then we have

$$\mathbf{L}^*\{\gamma_1\mathbf{L}^*\{f\}\gamma_2\} = \mathbf{L}^*\{\gamma_1 f\gamma_2\} \quad\text{and}\quad \mathbf{L}^*\{\gamma_1\mathbf{L}\{f\}\gamma_2\} = \theta. \tag{31}$$

The first equation follows from the repeated applications of (29). The second follows from the first one.

Now we shall consider the solutions of the three recurrence equations stated in the Introduction.

5. The first recurrence equation. Let us consider the recurrence equation (1) for $n=1, 2, \ldots$ where $f_0\in\mathbf{R}$, $g_1\in\mathbf{R}$, $g_2\in\mathbf{R}$ and **L** satisfies the conditions (i), (ii), (iii) and can be represented in the form of (25). Obviously, $f_n \in \mathbf{R}$ for $n=1, 2, \ldots$ and our aim is to determine f_n for $n=1, 2, \ldots$.

Denote by $r(\mathbf{L})$ the largest nonnegative number for which

$$\sum_{n=0}^{\infty} \|f_n\| \, |\varrho|^n < \infty$$

whenever $g_1 \in \mathbf{R}$, $g_2 \in \mathbf{R}$ and

$$|\varrho| \max(\|g_1\|, \|g_2\|) < r(\mathbf{L}). \tag{32}$$

The inequalities

$$(\|\mathbf{L}\| + \|\mathbf{L}^*\|)^{-1} \leqq r(\mathbf{L}) \leqq c(\mathbf{L})$$

obviously hold; however, later we shall prove that

$$r(\mathbf{L}) = c(\mathbf{L}) = c(\mathbf{T}) \tag{33}$$

where $c(\mathbf{T})$ is defined by (8).

If (32) is satisfied, then

$$F(\varrho) = \sum_{n=0}^{\infty} f_n \varrho^n \tag{34}$$

belongs to $\mathbf{R}$, and if we multiply (1) by ϱ^n and add for $n=1, 2, \ldots$, then we obtain that

$$\mathbf{L}\{F(\varrho)(e-\varrho g_1)\} + \mathbf{L}^*\{(e-\varrho g_2)F(\varrho)\} = f_0. \tag{35}$$

Conversely, if

$$F(\varrho) = \sum_{n=0}^{\infty} f_n^* \varrho^n \tag{36}$$

belongs to $\mathbf{R}$ for $|\varrho| < r$ where r is some positive number, and if (36) satisfies (35), then by forming the coefficient of ϱ^n for $n=0, 1, 2, \ldots$, we obtain that $f_0^* = f_0$ and f_n^* $(n=1, 2, \ldots)$ satisfies the same recurrence formula as f_n $(n=1, 2, \ldots)$. Thus necessarily $f_n^* = f_n$ for $n \geqq 0$.

We shall demonstrate that $F(\varrho)$ can always be determined by using the method of factorization. Let us assume that

$$e - \varrho g_i = g_i^+(\varrho) g_i^-(\varrho) \tag{37}$$

for $|\varrho| \, \|g_i\| < c(\mathbf{T})$ and $i=1, 2$ where $g_i^+(\varrho)$ and $g_i^-(\varrho)$ satisfy the properties (a) and (b) respectively. We have already proved that such a factorization always exists. By using the factorization (37) which depends only on $\mathbf{T}$, we can determine $F(\varrho)$ not only for $\mathbf{L}=\mathbf{T}$ but for any $\mathbf{L}$ satisfying (i), (ii), (iii) and (25).

Theorem 3. *If $f_0 \in \mathbf{R}$, $g_1 \in \mathbf{R}$, $g_2 \in \mathbf{R}$ and*

$$f_n = \mathbf{L}\{f_{n-1} g_1\} + \mathbf{L}^*\{g_2 f_{n-1}\}$$

for $n=1, 2, \ldots,$ *and if* (32) *is satisfied, then* (34) *belongs to* $\mathbf{R}$ *and we have*

(38) $$F(\varrho) = [g_2^-(\varrho)]^{-1}[\mathbf{L}\{g_2^-(\varrho)f_0[g_1^-(\varrho)]^{-1}\}+\mathbf{L}^*\{[g_2^+(\varrho)]^{-1}f_0 g_1^+(\varrho)\}][g_1^+(\varrho)]^{-1}$$

where $g_i^+(\varrho)$ *and* $g_i^-(\varrho)$ *satisfy* (a), (b) *and* (37).

Proof. If $F(\varrho)$ is defined by (38), then it can be represented in the form of (36). Since $\mathbf{T}\{g_i^-(\varrho)\}=\mathbf{T}\{[g_i^-(\varrho)]^{-1}\}=\mathbf{T}\{e\}$ and $\mathbf{T}^*\{g_i^+(\varrho)\}=\mathbf{T}^*\{[g_i^+(\varrho)]^{-1}\}=\mathbf{T}^*\{e\}$ for $i=1, 2$, by (30) and (31) we obtain that

(39) $$\mathbf{L}\{F(\varrho)(e-\varrho g_1)\} = \mathbf{L}\{f_0\}$$

and

(40) $$\mathbf{L}^*\{(e-\varrho g_2)F(\varrho)\} = \mathbf{L}^*\{f_0\}.$$

If we add (39) and (40), then we get (35). Thus we can conclude that (34) can be expressed in the form of (38). This completes the proof of the theorem.

We note that if $\mathbf{L}=\mathbf{T}$ and $f_0=e$, then (38) reduces to

$$F(\varrho) = [g_2^-(\varrho)]^{-1}[g_1^+(\varrho)]^{-1}.$$

Now let us suppose that in Theorem 3 we have $g_1=wg$ and $g_2=zg$ where $g\in\mathbf{R}$ and w and z are complex (or real) numbers. In this case by using the factorization in Theorem 1 we can choose $g_1^+(\varrho)=g^+(\varrho w)$, $g_1^-(\varrho)=g^-(\varrho w)$, $g_2^+(\varrho)=g^+(\varrho z)$, and $g_2^-(\varrho)=g^-(\varrho z)$ in Theorem 3. Then by (38) we get

(41) $$F(\varrho) = [g^-(\varrho z)]^{-1}[\mathbf{L}\{g^-(\varrho z)f_0[g^-(\varrho w)]^{-1}\}+\mathbf{L}^*\{[g^+(\varrho z)]^{-1}f_0 g^+(\varrho w)\}][g^+(\varrho w)]^{-1}$$

for $|\varrho| \max(|w|, |z|)\|g\|<r(\mathbf{L})$. If, in particular, $\mathbf{L}=\mathbf{T}$ and $f_0=e$, then (41) reduces to

(42) $$F(\varrho) = [g^-(\varrho z)]^{-1}[g^+(\varrho w)]^{-1}.$$

Now we are going to prove that in Theorem 1 $g^+(\varrho)$ and $g^-(\varrho)$ are uniquely determined by the properties (a) and (b) and by (11).

If $w=1$ and $z=0$ in (42), then the right-hand side becomes $[g^+(\varrho)]^{-1}$. On the other hand in this case by (12) we have $F(\varrho)=a(\varrho)$. Accordingly, $g^+(\varrho)=[a(\varrho)]^{-1}$ necessarily holds. In a similar way, if $w=0$ and $z=1$ in (42), then the right-hand side becomes $[g^-(\varrho)]^{-1}$. On the other hand in this case by (13) we have $F(\varrho)=b(\varrho)$. Accordingly, $g^-(\varrho)=[b(\varrho)]^{-1}$ necessarily holds. This proves that in Theorem 1 $g^+(\varrho)$ and $g^-(\varrho)$ are uniquely determined by the properties (a) and (b) and by (11), and that (20) and (21) necessarily hold.

Having been established that $g_i^+(\varrho)$ and $g_i^-(\varrho)$ ($i=1, 2$) are uniquely determined in (38) we can express $g_i^+(\varrho)$ and $g_i^-(\varrho)$ by formulas (18) and (19) and $[g_i^+(\varrho)]^{-1}$ and $[g_i^-(\varrho)]^{-1}$ by formulas (12) and (13). Proceeding in this way we can conclude from (38) that

$$r(\mathbf{L}) \geqq c(\mathbf{T}) \tag{43}$$

necessarily holds. Since evidently $r(\mathbf{L}) \leqq c(\mathbf{L})$, by (43) we have $c(\mathbf{T}) \leqq c(\mathbf{L})$. If we interchange the roles of $\mathbf{L}$ and $\mathbf{T}$, then it follows that $c(\mathbf{L}) \leqq c(\mathbf{T})$ also holds. This proves that (26) and (33) are indeed true.

In particular, it follows from (26) and (33) that if $\mathbf{L}$ is defined by (25), and if $\mu(\mathbf{T})=1$ and $\bar{\mu}(\mathbf{T}^*)=1$, then $r(\mathbf{L})=c(\mathbf{L})=1$ regardless of the values of $\|\mathbf{L}\|$ and $\|\mathbf{L}^*\|$.

If, instead of (1), we consider the recurrence formula

$$f_n = \mathbf{L}\{g_1 f_{n-1}\} + \mathbf{L}^*\{f_{n-1} g_2\} \tag{44}$$

for $n=1, 2, \dots$ where $f_0 \in \mathbf{R}$, $g_1 \in \mathbf{R}$, $g_2 \in \mathbf{R}$ and $\mathbf{L}$ satisfies the conditions (i), (ii), (iii) and can be represented in the form of (25), and if $|\varrho| \max(\|g_1\|, \|g_2\|) < r(\mathbf{L}^*)$, then

$$F(\varrho) = \sum_{n=0}^{\infty} f_n \varrho^n$$

belongs to $\mathbf{R}$ and can be determined again by the method of factorization. Let us suppose that

$$e - \varrho g_i = h_i^-(\varrho) h_i^+(\varrho)$$

for $|\varrho| \, \|g\| < c(\mathbf{T}^*)$ and $i=1, 2$ where $h_i^+(\varrho)$ satisfies property (a) and $h_i^-(\varrho)$ satisfies property (b). In this case we have

$$F(\varrho) = [h_1^+(\varrho)]^{-1}[\mathbf{L}\{[h_1^-(\varrho)]^{-1} f_0 h_2^-(\varrho)\} + \mathbf{L}^*\{h_1^+(\varrho) f_0 [h_2^+(\varrho)]^{-1}\}][h_2^-(\varrho)]^{-1} \tag{45}$$

whenever $|\varrho| \max(\|g_1\|, \|g_2\|) < r(\mathbf{L}^*)$.

Note. If $\mathbf{R}$ is a commutative Banach algebra and if $f_0 = e$, then (38) and (45) reduce to

$$F(\varrho) = \exp\left\{-\mathbf{L}\{\log(e - \varrho g_1)\} - \mathbf{L}^*\{\log(e - \varrho g_2)\}\right\}$$

for $|\varrho| \max(\|g_1\|, \|g_2\|) < 1$. In some particular cases this last result was demonstrated in 1952 by F. Pollaczek [9] and in 1958 by J. G. Wendel [12].

6. The second recurrence equation. Let us consider the recurrence equation (2) for $n=1, 2, \dots$ where $f_0 \in \mathbf{R}$, $g_1 \in \mathbf{R}$, $g_2 \in \mathbf{R}$ and $\mathbf{L}$ satisfies the conditions (i), (ii), (iii) and can be represented in the form of (25). Obviously $f_n \in \mathbf{R}$ for $n=1, 2, \dots$ and our aim is to determine f_n for $n=1, 2, \dots$.

Denote by $r^*(\mathbf{L})$ the largest nonnegative number for which

$$\sum_{n=0}^{\infty} \|f_n\| |\varrho|^n < \infty \tag{46}$$

whenever $g_1 \in \mathbf{R}$, $g_2 \in \mathbf{R}$ and

$$|\varrho| \max(\|g_1\|, \|g_2\|) < r^*(\mathbf{L}). \tag{47}$$

We shall prove that for every **L**

(48) $$\gamma(\mathbf{T})/3 \leqq r^*(\mathbf{L}) \leqq 1$$

where $\gamma(\mathbf{T})$ is defined by (9). Actually, we shall prove that if

(49) $$|\varrho|[\min(\|g_1\|, \|g_2\|)+\|g_1-g_2\|] < \gamma(\mathbf{T})$$

then (46) is satisfied and this implies (48).

If (47) is satisfied, then

(50) $$F(\varrho) = \sum_{n=0}^{\infty} f_n \varrho^n$$

belongs to **R**, and if we multiply (2) by ϱ^n and add for $n=1, 2, \ldots$, then we obtain that

(51) $$\mathbf{L}\{F(\varrho)(e-\varrho g_1)\}+\mathbf{L}^*\{F(\varrho)(e-\varrho g_2)\} = f_0.$$

Conversely, if

(52) $$F(\varrho) = \sum_{n=0}^{\infty} f_n^* \varrho^n$$

belongs to **R** for $|\varrho|<r$ where r is some positive number, and if (52) satisfies (51), then $f_n^* = f_n$ for all $n \geqq 0$.

The generating function (50) can be determined by the method of factorization. Let us apply Theorem 1 to $(e-\varrho g_2)^{-1}(e-\varrho g_1) = e-\varrho(e-\varrho g_2)^{-1}(g_1-g_2)$ and Theorem 2 to $(e-\varrho g_1)^{-1}(e-\varrho g_2) = e-\varrho(e-\varrho g_1)^{-1}(g_2-g_1)$. If (49) is satisfied, then we can write that

(53) $$(e-\varrho g_2)^{-1}(e-\varrho g_1) = g^+(\varrho) g^-(\varrho)$$

where $g^+(\varrho)$ and $g^-(\varrho)$ satisfy the properties (a) and (b) respectively.

Theorem 4. *If* $f_0 \in \mathbf{R}$, $g_1 \in \mathbf{R}$, $g_2 \in \mathbf{R}$ *and*

$$f_n = \mathbf{L}\{f_{n-1} g_1\}+\mathbf{L}^*\{f_{n-1} g_2\}$$

for $n=1, 2, \ldots$, *and if* (49) *is satisfied, then* (50) *belongs to* **R** *and we have*

(54) $$F(\varrho) = [\mathbf{L}\{f_0[g^-(\varrho)]^{-1}\}+\mathbf{L}^*\{f_0 g^+(\varrho)\}][g^+(\varrho)]^{-1}(e-\varrho g_2)^{-1}$$

where $g^+(\varrho)$ *and* $g^-(\varrho)$ *satisfy* (a), (b) *and* (53).

Proof. If $F(\varrho)$ is given by (54), then it can be represented in the form of (52) and by using (30) and (31) we can prove that (54) satisfies (51). This proves the theorem.

In a similar way as we proved (53) we can prove that if (49) is satisfied, then we can write that

(55) $$(e-\varrho g_1)(e-\varrho g_2)^{-1} = h^-(\varrho) h^+(\varrho)$$

where $h^+(\varrho)$ and $h^-(\varrho)$ satisfy the properties (a) and (b) respectively. By using (55) we can prove the following result.

If $f_0 \in \mathbf{R}$, $g_1 \in \mathbf{R}$, $g_2 \in \mathbf{R}$ and

(56) $$f_n = \mathbf{L}\{g_1 f_{n-1}\} + \mathbf{L}^*\{g_2 f_{n-1}\}$$

for $n=1, 2, \ldots$, and if (49) is satisfied, then

$$F(\varrho) = \sum_{n=0}^{\infty} f_n \varrho^n$$

belongs to $\mathbf{R}$ and we have

(57) $$F(\varrho) = (e-\varrho g_2)^{-1}[h^+(\varrho)]^{-1}[\mathbf{L}\{[h^-(\varrho)]^{-1} f_0\} + \mathbf{L}^*\{h^+(\varrho) f_0\}]$$

where $h^+(\varrho)$ and $h^-(\varrho)$ satisfy (a), (b) and (55).

If, in particular, $\mathbf{L}=\mathbf{T}$ and $f_0=e$ in (54), then we get $F(\varrho)=[g^+(\varrho)]^{-1}(e-\varrho g_2)^{-1}$. Thus $g^+(\varrho)$ can also be determined by the recurrence formula (2). If, in particular, $\mathbf{L}=\mathbf{T}$ and $f_0=e$ in (57), then we get $F(\varrho)=(e-\varrho g_2)^{-1}[h^+(\varrho)]^{-1}$ and thus $h^+(\varrho)$ can also be determined by the recurrence formula (56).

7. A system of recurrence equations. In this section we shall demonstrate that the system of recurrence equations (3) and (4) can be solved by using Theorem 4 if we apply it to a new Banach algebra $\mathbf{S}$ associated with $\mathbf{R}$. Let us denote by $\mathbf{S}$ the space of matrices

(58) $$\mathbf{f} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$$

where $f_{ij} \in \mathbf{R}$ for $i, j=1, 2$. In $\mathbf{S}$ let us define the operations of addition, multiplication and multiplication by a complex (or real) constant according to the rules of matrix algebra and according to the rules established in $\mathbf{R}$. Define the norm of $\mathbf{f}$ either by

$$\|\mathbf{f}\|_S = \max(\|f_{11}\| + \|f_{12}\|, \|f_{21}\| + \|f_{22}\|)$$

or alternately by

$$\|\mathbf{f}\|_S = \max(\|f_{11}\| + \|f_{21}\|, \|f_{12}\| + \|f_{22}\|).$$

We can easily see that $\mathbf{S}$ is a noncommutative Banach algebra with zero element and identity element

$$\begin{bmatrix} \theta & \theta \\ \theta & \theta \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} e & \theta \\ \theta & e \end{bmatrix},$$

respectively.

If $\mathbf{T}$ is a transformation in $\mathbf{R}$ which satisfies (i), (ii), and (iii), then let us extend the definition of $\mathbf{T}$ to $\mathbf{S}$ in such a way that we form $\mathbf{T}$ element by element for an $\mathbf{f}$ given by (58), that is

$$\mathbf{T}\{\mathbf{f}\} = [\mathbf{T}\{f_{ij}\}]_{i,j=1,2}.$$

We can easily see that $\mathbf{T}$ satisfies (i), (ii) and (iii) in the space $\mathbf{S}$ too.

Now let us consider the system of recurrence equations (3) and (4) for $n=1, 2, \ldots$ where $u_0 \in \mathbf{R}$, $v_0 \in \mathbf{R}$, $h_i \in \mathbf{R}$ $(i=1, 2, 3, 4)$ and $\mathbf{L}$ satisfies the conditions (i), (ii), (iii) and can be represented in the form of (25). We can express (3) and (4) in the following matrix form

$$\begin{bmatrix} u_n & v_n \\ \theta & \theta \end{bmatrix} = \mathbf{L}\left\{\begin{bmatrix} u_{n-1} & v_{n-1} \\ \theta & \theta \end{bmatrix}\begin{bmatrix} h_1 & \theta \\ h_2 & \theta \end{bmatrix}\right\} + \mathbf{L}^*\left\{\begin{bmatrix} u_{n-1} & v_{n-1} \\ \theta & \theta \end{bmatrix}\begin{bmatrix} \theta & h_3 \\ \theta & h_4 \end{bmatrix}\right\}$$

for $n=1, 2, \ldots$. This equation is of type (2). If we apply Theorem 4 to the Banach algebra $\mathbf{S}$, then

$$\sum_{n=0}^{\infty} \begin{bmatrix} u_n & v_n \\ \theta & \theta \end{bmatrix} \varrho^n$$

can be determined by (54).

If, instead of (3) and (4), we consider the recurrence equations

$$u_n = \mathbf{L}\{h_1 u_{n-1} + h_2 v_{n-1}\}$$

and

$$v_n = \mathbf{L}^*\{h_3 u_{n-1} + h_4 v_{n-1}\}$$

for $n=1, 2, \ldots$, then we can write that

(59) $$\begin{bmatrix} u_n & \theta \\ v_n & \theta \end{bmatrix} = \mathbf{L}\left\{\begin{bmatrix} h_1 & h_2 \\ \theta & \theta \end{bmatrix}\begin{bmatrix} u_{n-1} & \theta \\ v_{n-1} & \theta \end{bmatrix}\right\} + \mathbf{L}^*\left\{\begin{bmatrix} \theta & \theta \\ h_3 & h_4 \end{bmatrix}\begin{bmatrix} u_{n-1} & \theta \\ v_{n-1} & \theta \end{bmatrix}\right\}$$

for $n=1, 2, \ldots$. This equation is of type (56). The solution of (59) can be obtained by (57) if we apply it to the Banach algebra $\mathbf{S}$.

By introducing a Banach algebra of finite or countably infinite matrices with elements belonging to $\mathbf{R}$, we can solve a finite or a countably infinite system of recurrence equations in $\mathbf{R}$.

In the next two sections we shall define two Banach algebras $\mathbf{R}_1$ and $\mathbf{R}_2$, and theree transformations $\mathbf{T}, \mathbf{T}_0, \mathbf{T}_1$ satisfying (i), (ii) and (iii). If we apply Theorem 3 and Theorem 4 to these Banach algebras, then we can determine the distributions of several random variables depending on the partial sums of a sequence of independent and identically distributed random variables and of a semi-Markov sequence of real random variables. In particular, we can find the distributions of the maximal partial sum, the ordered partial sums, the number of positive partial sums, the number of changes of sign in the successive partial sums, and the subscript of the first positive partial sum. These applications will be discussed in a subsequent paper.

8. A commutative Banach algebra $\mathbf{R}_1$. Let us define $\mathbf{R}_1$ as the space of functions $\Phi(s)$ defined for $\text{Re}(s)=0$ on the complex plane which can be represented in the form

(60) $$\Phi(s) = \mathbf{E}\{\zeta e^{-s\eta}\}$$

where η is a real random variable and ζ is a complex (or real) random variable for which $\mathbf{E}\{|\zeta|\} < \infty$. Let us define in $\mathbf{R}_1$ the operations to be the pointwise addition, multiplication and multiplication by a complex (or real) constant. The zero element of $\mathbf{R}_1$ is 0, and the identity element of $\mathbf{R}_1$ is 1. Let us define the norm of $\Phi(s)\in\mathbf{R}_1$ by

$$\|\Phi\| = \inf_{\zeta} \mathbf{E}\{|\zeta|\}$$

by where the infimum is taken for all admissible ζ in the representation (60).

We can easily prove that $\mathbf{R}_1$ is a commutative Banach algebra.

Now we shall consider some transformations in $\mathbf{R}_1$ which satisfy (i), (ii) and (iii).

If $\Phi(s)\in\mathbf{R}_1$ is given by (60), then let us define

$$\Phi^+(s) = \mathbf{E}\{\zeta e^{-s\eta^+}\} \tag{61}$$

for $\mathrm{Re}(s)\geqq 0$ and

$$\Phi^-(s) = \mathbf{E}\{\zeta(e^{-s\eta}-e^{-s\eta^+})\} \tag{62}$$

for $\mathrm{Re}(s)\leqq 0$ where $\eta^+=\max(0,\eta)$. We have $\Phi^+(s)\in\mathbf{R}_1$, $\Phi^-(s)\in\mathbf{R}_1$ and

$$\Phi(s) = \Phi^+(s)+\Phi^-(s) \tag{63}$$

for $\mathrm{Re}(s)=0$, $|\Phi^+(s)|\leqq\|\Phi\|$ for $\mathrm{Re}(s)\geqq 0$ and $|\Phi^-(s)|\leqq 2\|\Phi\|$ for $\mathrm{Re}(s)\leqq 0$.

The function $\Phi^+(s)$ is regular for $\mathrm{Re}(s)>0$, continuous and bounded for $\mathrm{Re}(s)\geqq 0$ and $\Phi^+(0)=\Phi(0)$.

The function $\Phi^-(s)$ is reqular for $\mathrm{Re}(s)<0$, continuous and bounded for $\mathrm{Re}(s)\leqq 0$ and $\Phi^-(0)=0$.

By Liouville's theorem it follows that the above properties uniquely determine $\Phi^+(s)$ and $\Phi^-(s)$ in the representation (63).

If $\Phi(s)\in\mathbf{R}_1$, then for $\mathrm{Re}(s)>0$ we have

$$\Phi^+(s) = \frac{1}{2}\Phi(0)+\lim_{\varepsilon\to 0}\frac{s}{2\pi i}\int_{L_\varepsilon}\frac{\Phi(z)}{z(s-z)}\,dz$$

where $L_\varepsilon=\{z:z=iy,\ -\infty<y\leqq-\varepsilon<\varepsilon\leqq y<\infty\}$. See reference [11].

For any event A let us define $\delta(A)$ as the indicator variable of A, that is, $\delta(A)=1$ if A occurs and $\delta(A)=0$ if A does not occur.

Now we define three transformations $\mathbf{T}$, $\mathbf{T}_0$, $\mathbf{T}_1$ in $\mathbf{R}_1$ which satisfy the conditions (i), (ii) and (iii). If $\Phi(s)\in\mathbf{R}_1$ is given by (60), then let

$$\mathbf{T}\{\Phi(s)\} = \Phi^+(s) = \mathbf{E}\{\zeta e^{-s\eta^+}\}, \tag{64}$$

$$\mathbf{T}_0\{\Phi(s)\} = \Phi^+(s)-\Phi^+(\infty) = \mathbf{E}\{\zeta e^{-s\eta}\delta(\eta>0)\} \tag{65}$$

and

$$\mathbf{T}_1\{\Phi(s)\} = \Phi^+(s)+\Phi^-(-\infty) = \mathbf{E}\{\zeta e^{-s\eta}\delta(\eta\geqq 0)\}. \tag{66}$$

We define $\mathbf{T}^*$, $\mathbf{T}_0^*$ and $\mathbf{T}_1^*$ by (6). We can easily see that these transformations satisfy (i), (ii), (iii), $\|\mathbf{T}\|=\|\mathbf{T}_0\|=\|\mathbf{T}_1\|=\|\mathbf{T}_0^*\|=\|\mathbf{T}_1^*\|=1$ and $\|\mathbf{T}^*\|=2$.

If $\mathbf{L}$ is any one of the transformations $\mathbf{T}$, $\mathbf{T}_0$, $\mathbf{T}_1$, defined by (64), (65), and (66) respectively, then $\mathbf{L}\{\Phi(s)\}$ can be represented in the form of (25), that is,

$$\mathbf{L}\{\Phi(s)\} = \mathbf{T}\{\Phi(s)\}-\alpha(\Phi)$$

where $\mathbf{T}\{\Phi(s)\}$ is defined by (64), and $\alpha(\Phi)\equiv 0$ for $\mathbf{L}=\mathbf{T}$, $\alpha(\Phi)=\Phi^+(\infty)$ for $\mathbf{L}=\mathbf{T}_0$, and $\alpha(\Phi)=-\Phi^-(-\infty)$ for $\mathbf{L}=\mathbf{T}_1$. If $\mathbf{L}$ is any one of the transformations (64), (65), (66), then by (7), (8) and (26) we have $c(\mathbf{L})=1$ and $c(\mathbf{L}^*)=1$.

If we assume that $\mathbf{T}$ is given by (64), then we can formulate the following version of Theorem 1.

Theorem 5. *If $\psi(s)\in\mathbf{R}_1$ and if $|\varrho|\,\|\psi\|<1$, then there exist two functions $\psi^+(s,\varrho)\in\mathbf{R}_1$ and $\psi^-(s,\varrho)\in\mathbf{R}_1$ such that*

$$1-\varrho\psi(s) = \psi^+(s,\varrho)\psi^-(s,\varrho) \tag{67}$$

for $\operatorname{Re}(s)=0$ *where $\psi^+(s,\varrho)$ satisfies property* (α) *and $\psi^-(s,\varrho)$ satisfies property* (β) *stated below.*

Property (α). *The function $\psi^+(s,\varrho)$ is regular for* $\operatorname{Re}(s)>0$, *continuous, bounded and free from zeros for* $\operatorname{Re}(s)\geqq 0$.

Property (β). *The function $\psi^-(s,\varrho)$ is regular for* $\operatorname{Re}(s)<0$, *continuous, bounded and free from zeros for* $\operatorname{Re}(s)\leqq 0$.

Proof. If $\psi^+(s,\varrho)$ satisfies (α), and $\psi^-(s,\varrho)$ satisfies (β), then we say that (67) is a factorization of $1-\varrho\psi(s)$. Such a factorization always exists. For example, if

$$\psi^+(s,\varrho) = C_1(\varrho)\exp\{\mathbf{T}\{\log[1-\varrho\psi(s)]\}\} \tag{68}$$

for $\operatorname{Re}(s)\geqq 0$ and $|\varrho|\,\|\psi\|<1$, and

$$\psi^-(s,\varrho) = C_2(\varrho)\exp\{\mathbf{T}^*\{\log[1-\varrho\psi(s)]\}\} \tag{69}$$

for $\operatorname{Re}(s)\leqq 0$ and $|\varrho|\,\|\psi\|<1$, where $C_1(\varrho)C_2(\varrho)=1$, then ($\alpha$), ($\beta$) and (67) are satisfied. Conversely, it follows from Liouville's theorem that conditions (α), (β) and (67) determine $\psi^+(s,\varrho)$ and $\psi^-(s,\varrho)$ up to a nonvanishing factor depending only on ϱ. Thus (68) and (69) are the general forms of $\psi^+(s,\varrho)$ are $\psi^-(s,\varrho)$ respectively.

If in (68) and (69) we choose $C_1(\varrho)$ and $C_2(\varrho)$ in an appropriate way, then we can easily see that $\psi^+(s,\varrho)$ and $\psi^-(s,\varrho)$ satisfy properties (a) and (b) too.

If we want to solve a recurrence equation of type (1) in the space $\mathbf{R}_1$, then instead of (11) we can use the factorization (67). Since in (38) only the product $C_1(\varrho)C_2(\varrho)=1$ appears, therefore it does not matter how we choose $C_1(\varrho)$ and $C_2(\varrho)$ in (68) and (69).

Let us mention one example specifically. Let

$$U_n(s) = w\mathbf{L}\{U_{n-1}(s)\psi(s)\}+z\mathbf{L}^*\{U_{n-1}(s)\psi(s)\}$$

for $n=1,2,\ldots$ where $U_0(s)\in\mathbf{R}_1$, $\psi(s)\in\mathbf{R}_1$, w and z are complex (or real) numbers, and $\mathbf{L}$ is any one of the transformations (64), (65), (66). If $|\varrho|\max(|w|,|z|)\,\|\psi\|<1$, then

$$U(s,\varrho) = \sum_{n=0}^{\infty} U_n(s)\varrho^n$$

belongs to $\mathbf{R}_1$ and by Theorem 3 we have

$$U(s, \varrho) = \big[\mathbf{L}\{U_0(s)\psi^-(s, \varrho z)[\psi^-(s, \varrho w)]^{-1}\} + \\ +\mathbf{L}^*\{U_0(s)\psi^+(s, \varrho w)[\psi^+(s, \varrho z)]^{-1}\}\big][\psi^+(s, \varrho w)]^{-1}[\psi^-(s, \varrho z)]^{-1}$$

where $\psi^+(s, \varrho)$ and $\psi^-(s, \varrho)$ are determined by Theorem 5 or by (68) and (69), respectively.

Finally, we note that in properties (α) and (β) the requirement of boundedness can be replaced by the weaker conditions $\lim_{|s|\to\infty} [\log\psi^+(s, \varrho)]/s=0$ $(\mathrm{Re}(s)\geqq 0)$ and $\lim_{|s|\to\infty} [\log\psi^-(s, \varrho)]/s=0$ $(\mathrm{Re}(s)\leqq 0)$, respectively.

9. A noncommutative Banach algebra $\mathbf{R}_2$. Let I be a fixed finite or countably infinite set. We consider complex (or real) matrices $\mathbf{A}=[a_{ij}]$, $i\in I$, $j\in I$, for which

$$\mathbf{M}\{\mathbf{A}\} = \sup_{i\in I} \sum_{j\in I} |a_{ij}| < \infty.$$

We shall denote by $\mathbf{0}$ the zero matrix all of whose elements are zeros, and by $\mathbf{I}$ the identity matrix. ($\mathbf{I}=[\delta_{ij}]$, $i\in I$, $j\in I$, where $\delta_{ij}=1$ for $i=j$ and $\delta_{ij}=0$ for $i\neq j$.) If $\mathbf{M}\{\mathbf{A}\}<\infty$, $\mathbf{M}\{\mathbf{B}\}<\infty$ and $\mathbf{AB}=\mathbf{BA}=\mathbf{I}$, then we say that $\mathbf{A}$ and $\mathbf{B}$ are inverse matrices and write $\mathbf{B}=\mathbf{A}^{-1}$.

We say that a matrix function $\mathbf{A}(s)=[a_{ij}(s)]$, $i\in I$, $j\in I$, is continuous, or regular, or bounded on a set D according to whether every $a_{ij}(s)$ is continuous on D, or every $a_{ij}(s)$ is regular on D, or $\mathbf{M}\{\mathbf{A}(s)\}<K$ for $s\in D$ where K is a positive constant.

Let $\mathbf{R}_2$ be the space of all matrix functions

(70) $$\mathbf{\Phi}(s) = [\Phi_{ij}(s)]_{i,j\in I}$$

defined for $\mathrm{Re}(s)=0$ on the complex plane such that I is a fixed countable set, $\Phi_{ij}(s)\in\mathbf{R}_1$ and

(71) $$\|\mathbf{\Phi}\| = \sup_{i\in I} \sum_{j\in I} \|\Phi_{ij}\|_{\mathbf{R}_1} < \infty.$$

We define the norm of $\mathbf{\Phi}(s)$ by (71). Let us define the operations of addition, multiplication and multiplication by a complex (or real) constant in $\mathbf{R}_2$ according to the rules of matrix algebra. We can easily see that $\mathbf{R}_2$ is a noncommutative Banach algebra with zero element $\mathbf{0}$ and identity element $\mathbf{I}$.

If $\mathbf{\Phi}(s)\in\mathbf{R}_2$ is given by (70), then let

$$\mathbf{\Phi}^+(s) = [\Phi_{ij}^+(s)]_{i,j\in I}$$

for $\mathrm{Re}(s)\geqq 0$ and

$$\mathbf{\Phi}^-(s) = [\Phi_{ij}^-(s)]_{i,j\in I}$$

for $\mathrm{Re}(s)\leqq 0$ where $\Phi_{ij}^+(s)$ is defined by (61) and $\Phi_{ij}^-(s)$ by (62).

Obviously, $\mathbf{\Phi}^+(s)\in\mathbf{R}_2$, $\mathbf{\Phi}^-(s)\in\mathbf{R}_2$ and

(72) $$\mathbf{\Phi}(s) = \mathbf{\Phi}^+(s)+\mathbf{\Phi}^-(s)$$

for $\mathrm{Re}(s)=0$. We have $\mathbf{M}\{\mathbf{\Phi}^+(s)\} \leqq \|\mathbf{\Phi}\|$ for $\mathrm{Re}(s)\geqq 0$ and $\mathbf{M}\{\mathbf{\Phi}^-(s)\}\leqq 2\|\mathbf{\Phi}\|$ for $\mathrm{Re}(s)\leqq 0$.

The matrix function $\mathbf{\Phi}^+(s)$ is regular for $\mathrm{Re}(s)>0$, continuous and bounded for $\mathrm{Re}(s)\geqq 0$ and $\mathbf{\Phi}^+(0)=\mathbf{\Phi}(0)$.

The matrix function $\mathbf{\Phi}^-(s)$ is regular for $\mathrm{Re}(s)<0$, continuous and bounded for $\mathrm{Re}(s)\leqq 0$ and $\mathbf{\Phi}^-(0)=\mathbf{0}$.

By Liouville's theorem it follows that the above properties uniquely determine $\mathbf{\Phi}^+(s)$ and $\mathbf{\Phi}^-(s)$ in the representation (72).

Now let us extend the definition of the transformations (64), (65), (66) from the space $\mathbf{R}_1$ to the space $\mathbf{R}_2$ in such a way that we form these transformations element by element for $\mathbf{\Phi}(s)\in\mathbf{R}_2$, that is,

$$\mathbf{T}\{\mathbf{\Phi}(s)\} = \mathbf{\Phi}^+(s), \tag{73}$$

$$\mathbf{T}_0\{\mathbf{\Phi}(s)\} = \mathbf{\Phi}^+(s)-\mathbf{\Phi}^+(\infty), \tag{74}$$

and

$$\mathbf{T}_1\{\mathbf{\Phi}(s)\} = \mathbf{\Phi}^+(s)+\mathbf{\Phi}^-(-\infty). \tag{75}$$

We define $\mathbf{T}^*$, $\mathbf{T}_0^*$, $\mathbf{T}_1^*$ by (6). We can easily see that these transformations satisfy (i), (ii), (iii), $\|\mathbf{T}\|=\|\mathbf{T}_0\|=\|\mathbf{T}_1\|=\|\mathbf{T}_0^*\|=\|\mathbf{T}_1^*\|=1$ and $\|\mathbf{T}^*\|=2$.

If $\mathbf{L}$ is any one of the transformations (73), (74), (75) and if $\mathbf{\Phi}(s)\in\mathbf{R}_2$, then $\mathbf{L}\{\mathbf{C\Phi}(s)\}=\mathbf{CL}\{\mathbf{\Phi}(s)\}$ and $\mathbf{L}\{\mathbf{\Phi}(s)\mathbf{C}\}=\mathbf{L}\{\mathbf{\Phi}(s)\}\mathbf{C}$ for any constant matrix $\mathbf{C}$ for which $\mathbf{M}\{\mathbf{C}\}<\infty$. Furthermore, $\mathbf{L}\{\mathbf{\Phi}(s)\}$ can be represented in the following form

$$\mathbf{L}\{\mathbf{\Phi}(s)\} = \mathbf{T}\{\mathbf{\Phi}(s)\}-\boldsymbol{\alpha}(\mathbf{\Phi}) \tag{76}$$

where $\mathbf{T}\{\mathbf{\Phi}(s)\}$ is defined by (73), $\boldsymbol{\alpha}(\mathbf{\Phi})=\mathbf{0}$ for $\mathbf{L}=\mathbf{T}$, $\boldsymbol{\alpha}(\mathbf{\Phi})=\mathbf{\Phi}^+(\infty)$ for $\mathbf{L}=\mathbf{T}_0$, and $\boldsymbol{\alpha}(\mathbf{\Phi})=-\mathbf{\Phi}^-(-\infty)$ for $\mathbf{L}=\mathbf{T}_1$. If $\mathbf{L}$ is any one of the transformations (73), (74), (75), then by (7), (8) and (26) we have $c(\mathbf{L})=1$ and $c(\mathbf{L}^*)=1$.

If we assume that $\mathbf{T}$ is defined by (73), then we can formulate the following version of Theorem 1.

Theorem 6. *If* $\mathbf{\Psi}(s)\in\mathbf{R}_2$ *and if* $|\varrho|\,\|\mathbf{\Psi}\|<1$, *then there exist two matrices* $\mathbf{\Psi}^+(s,\varrho)\in\mathbf{R}_2$ *and* $\mathbf{\Psi}^-(s,\varrho)\in\mathbf{R}_2$ *such that*

$$\mathbf{I}-\varrho\mathbf{\Psi}(s) = \mathbf{\Psi}^+(s,\varrho)\mathbf{\Psi}^-(s,\varrho) \tag{77}$$

for $\mathrm{Re}(s)=0$ *where* $\mathbf{\Psi}^+(s,\varrho)$ *satisfies property* (α) *and* $\mathbf{\Psi}^-(s,\varrho)$ *satisfies property* (β) *stated below.*

Property (α). *The matrix* $\mathbf{\Psi}^+(s,\varrho)$ *has an inverse* $[\mathbf{\Psi}^+(s,\varrho)]^{-1}$ *for* $\mathrm{Re}(s)\geqq 0$, *and* $\mathbf{\Psi}^+(s,\varrho)$ *and* $[\mathbf{\Psi}^+(s,\varrho)]^{-1}$ *are bounded and continuous for* $\mathrm{Re}(s)\geqq 0$ *and regular for* $\mathrm{Re}(s)>0$.

Property (β). *The matrix* $\mathbf{\Psi}^-(s, \varrho)$ *has an inverse* $[\mathbf{\Psi}^-(s, \varrho)]^{-1}$ *for* $\mathrm{Re}(s) \leqq 0$, *and* $\mathbf{\Psi}^-(s, \varrho)$ *and* $[\mathbf{\Psi}^-(s, \varrho)]^{-1}$ *are bounded and continuous for* $\mathrm{Re}(s) \leqq 0$ *and regular for* $\mathrm{Re}(s) < 0$.

Proof. The factorization (77) satisfying (α) and (β) always exists. By the method described in the proof of Theorem 1 we can construct two matrices $\mathbf{A}(s, \varrho)$ and $\mathbf{B}(s, \varrho)$ such that

$$\mathbf{I} - \varrho \mathbf{\Psi}(s) = [\mathbf{A}(s, \varrho)]^{-1} [\mathbf{B}(s, \varrho)]^{-1}$$

for $\mathrm{Re}(s) = 0$ and $\mathbf{A}(s, \varrho)$ satisfies (a) and $\mathbf{B}(s, \varrho)$ satisfies (b).

If we define

(78) $$\mathbf{\Psi}^+(s, \varrho) = [\mathbf{A}(s, \varrho)]^{-1} \mathbf{C}_1(\varrho)$$

for $\mathrm{Re}(s) \geqq 0$ and

(79) $$\mathbf{\Psi}^-(s, \varrho) = \mathbf{C}_2(\varrho) [\mathbf{B}(s, \varrho)]^{-1}$$

for $\mathrm{Re}(s) \leqq 0$ where $\mathbf{M}\{\mathbf{C}_1(\varrho)\} < \infty$, $\mathbf{M}\{\mathbf{C}_2(\varrho)\} < \infty$ and $\mathbf{C}_1(\varrho)\mathbf{C}_2(\varrho) = \mathbf{I}$, then all the properties stated in Theorem 6 are satisfied. Conversely, it follows from Liouville's theorem that conditions (α), (β) and (77) determine $\mathbf{\Psi}^+(s, \varrho)$ and $\mathbf{\Psi}^-(s, \varrho)$ up to a matrix factor independent of s. This implies that (78) and (79) are the general forms of $\mathbf{\Psi}^+(s, \varrho)$ and $\mathbf{\Psi}^-(s, \varrho)$ respectively.

In a similar way as we proved Theorem 6, we can prove a corresponding version of Theorem 2.

If we want to solve recurrence equations of type (1) and (2) in the space $\mathbf{R}_2$, then instead of (11), we can use the factorization (77). Since in (38) and in (54) only the product $\mathbf{C}_1(\varrho)\mathbf{C}_2(\varrho) = \mathbf{I}$ appears, it is immaterial how we choose $\mathbf{C}_1(\varrho)$ and $\mathbf{C}_2(\varrho)$ in (78) and (79). We can easily see that although in (76) $\boldsymbol{\alpha}(\mathbf{\Phi})$ is a matrix, not a scalar, we can use formulas (38) and (54) unchangeably. Recurrence equations of types (44) and (56) in the space $\mathbf{R}_2$ can be solved in a similar way by using an analogous version of Theorem 6.

Let us mention one example specefically. Let

$$\mathbf{U}_n(s) = w\mathbf{L}\{\mathbf{U}_{n-1}(s)\mathbf{\Psi}(s)\} + z\mathbf{L}^*\{\mathbf{U}_{n-1}(s)\mathbf{\Psi}(s)\}$$

for $n = 1, 2, \ldots$ where $\mathbf{U}_0(s) \in \mathbf{R}_2$, $\mathbf{\Psi}(s) \in \mathbf{R}_2$, w and z are complex (or real) numbers, and $\mathbf{L}$ is any one of the transformations (73), (74), (75). If $|\varrho|\,[\min(|w|, |z|) + |w - z|] \cdot \cdot \|\mathbf{\Psi}\| < 1$, then

$$\mathbf{U}(s, \varrho) = \sum_{n=0}^{\infty} \mathbf{U}_n(s) \varrho^n$$

belongs to $\mathbf{R}_2$ and by Theorem 4 we have

$$\mathbf{U}(s, \varrho) = \big[\mathbf{L}\{\mathbf{U}_0(s)[\mathbf{\Psi}^-(s, \varrho w, \varrho z)]^{-1}\} + \\ + \mathbf{L}^*\{\mathbf{U}_0(s)\mathbf{\Psi}^+(s, \varrho w, \varrho z)\}\big] \cdot [\mathbf{\Psi}^+(s, \varrho w, \varrho z)]^{-1} [\mathbf{I} - \varrho z \mathbf{\Psi}(\varrho)]^{-1}$$

where

$$[\mathbf{I}-\varrho z\mathbf{\Psi}(s)]^{-1}[\mathbf{I}-\varrho w\mathbf{\Psi}(s)] = \mathbf{\Psi}^{+}(s, \varrho w, \varrho z)\mathbf{\Psi}^{-}(s, \varrho w, \varrho z)$$

for $\mathrm{Re}(s)=0$ and $\mathbf{\Psi}^{+}(s, \varrho w, \varrho z)$ satisfies property (α) and $\mathbf{\Psi}^{-}(s, \varrho w, \varrho z)$ satisfies property (β) in Theorem 6.

We note that in the case of finite matrices the method of matrix factorization has already been used in several fields of mathematics, namely, in the theory of systems of integral equations, in the theory of linear prediction of multivariate stationary stochastic processes and in the theory of Markov chains. We refer to the works of G. D. Birkhoff [3], N. Wiener [13], P. Masani [6], N. Wiener and P. Masani [14], I. C. Gohberg and M. G. Krein [5], M. D. Miller [7], [8] and É. L. Presman [10].

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