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Derivations and translations on lattices

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1. Introduction. Let S be a meet-semilattice and φ a single-valued mapping of S into itself. φ is called a *meet-translation* on S [3], if $\varphi(x\wedge y)=\varphi(x)\wedge y$ for each pair of elements $x, y\in S$. If $S=L$ is a lattice and φ a single-valued mapping of L into L such that

$$\varphi(x\vee y) = \varphi(x)\vee\varphi(y) \quad \text{and} \quad \varphi(x\wedge y) = (\varphi(x)\wedge y)\vee(\varphi(y)\wedge x)$$

for each pair $x, y\in L$, then φ is called a *derivation* on L [5]. As shown by SZÁSZ in [5], a single-valued mapping on a lattice L is a derivation on L if and only if it is a meet-translation as well as an endomorphism on L.

Each meet-translation φ on S has the following properties [3]: $\varphi(x)\leqq x$, $\varphi(\varphi(x))=\varphi(x)$, and $x\leqq y\Rightarrow\varphi(x)\leqq\varphi(y)$. Moreover, in a lattice L the fixedelements of φ, i.e. the elements $t=\varphi(t)$, constitute an ideal K_φ of L [4]. As shown in [4], K_φ determines φ uniquely.

In this note we shall illuminate the dependence of φ from the properties of the ideal K_φ.

A single-valued mapping φ of a join-semilattice V into itself is called a *join-translation* on V, if $\varphi(x\vee y)=\varphi(x)\vee y$ for each pair $x, y\in V$. The results on translations in the papers [1]—[4] are given in terms of join-translations. As we shall consider here meet-translations, we always use the dual of the corresponding result obtained in the papers [1]—[4].

2. Derivations on lattices. We denote by $\mathscr{I}(L)$ the lattice of all ideals of a lattice L; $(z]=\{x|x\leqq z,\ x, z\in L\}$.

Theorem 1. *An ideal I of a lattice L generates a meet-translation φ on L such that $I=K_\varphi$ if and only if for each $y\in L$ there is an element $k\in L$ such that $I\wedge(y]=(k]$.*

Proof. If $I=K_\varphi$ for a meet-translation φ on L, then $I\wedge(y]=(\varphi(y)]$ for each $y\in L$.

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Conversely, let $I\wedge(y]=(k]$ for each $y\in L$. We put $\varphi(y)=k$ and show that φ is a meet-translation on L. Obviously φ is single-valued and $K_\varphi=I$. $I\wedge(x\wedge y]=(I\wedge(x])\wedge(y]$; thus $\varphi(x\wedge y)=\varphi(x)\wedge y$ and the theorem follows.

Theorem 2. *Let D be an ideal of a lattice L generating a meet-translation φ on L. Then φ is a derivation on L if and only if $D\wedge((y]\vee(x])=(D\wedge(y])\vee(D\wedge(x])$ for each pair of elements $x, y\in L$.*

Proof. As D generates a meet-translation φ on L, $D\wedge(y]=(k]$ for each $y\in L$. Let the condition of the theorem be valid for the elements $x, y\in L$. Then $D\wedge(x\vee y]=(D\wedge(x])\vee(D\wedge(y])$, whence $\varphi(x\vee y)=\varphi(x)\vee\varphi(y)$. Furthermore, $D\wedge(x\wedge y]=(D\wedge(x])\wedge(y]=(D\wedge(y])\wedge(x]=\{(D\wedge(x])\wedge(y]\}\vee\{(D\wedge(y])\wedge(x]\}$ which implies that $\varphi(x\wedge y)=(\varphi(x)\wedge y)\vee(\varphi(y)\wedge x)$.

Conversely, let φ be a derivation on L and K_φ the ideal generating it. According to the properties of φ, $K_\varphi\wedge(x]=(\varphi(x)]$. So $\varphi(x\vee y)=\varphi(x)\vee\varphi(y)$ implies that $K_\varphi\wedge(x\vee y]=(\varphi(x\vee y)]=(\varphi(x)]\vee(\varphi(y)]=(K_\varphi\wedge(x])\vee(K_\varphi(y])$. This completes the proof.

An element x of a lattice L is called *distributive*, if $x\wedge(y\vee z)=(x\wedge y)\vee(x\wedge z)$ for each pair $y, z\in L$. The following lemma shows that the condition of Theorem 2 reduces to the distributivity of D in the lattice $\mathscr{I}(L)$.

Lemma 1. *Let T be an ideal of a lattice L such that $T\wedge((x]\vee(y])=(T\wedge(x])\vee(T\wedge(y])$ for each two elements $x, y\in L$. Then $T\wedge(I\vee K)=(T\wedge I)\vee(T\wedge K)$ for each two elements $I, K\in\mathscr{I}(L)$.*

Proof. As is well known, it is sufficient to show that $T\wedge(I\vee K)\subseteq(T\wedge I)\vee(T\wedge K)$ Let $x\in T\wedge(I\vee K)$, i.e. $x\in T$ and $x\leqq i\vee k$ for some $i\in I$ and $k\in K$. Then $(x]\subseteq(i]\vee(k]$ and $x\in(x]=T\wedge(x]\subseteq(T\wedge(i])\vee(T\wedge(k])\subseteq(T\wedge I)\vee(T\wedge K)$, and the lemma follows.

The lattice $\mathscr{I}(L)$ of a modular lattice L is modular. Already the relation $T\wedge(I\vee K)=(T\wedge I)\vee(T\wedge K)$ implies the neutrality of T in a modular lattice [6, Thm. 103 and its corollary]. So we can write

Corollary 1. *A meet-translation φ on a modular lattice L is a derivation on L if and only if K_φ is a neutral element of the lattice $\mathscr{I}(L)$.*

By *the join of two derivations* φ and λ on a lattice L we mean the mapping $\varphi(x)\vee\lambda(x)$ on L and by *the meet* the mapping $\varphi(x)\wedge\lambda(x)$. In the following we consider some conditions under which the join and meet defined above are also derivations on L.

Theorem 3. *The meet of two derivations φ and λ on a lattice L is always a derivation on L. Moreover, the join of φ and λ is a derivation on L if K_φ and K_λ are neutral ideals of L.*

Proof. $(K_\varphi\wedge K_\lambda)\wedge(x]=(K_\varphi\wedge(x])\wedge(K_\lambda\wedge(x])=(\varphi(x)\wedge\lambda(x)]$ and so $K_\lambda\wedge K_\varphi$ generates a meet-translation which is $\varphi(x)\wedge\lambda(x)$. Further, $(K_\varphi\wedge K_\lambda)\wedge(x\vee y]=$

$=K_\varphi\wedge\{K_\lambda\wedge(x\vee y]\}$, and by applying now K_λ and K_φ sequently, $(K_\varphi\wedge K_\lambda)\wedge(x\vee y]=$ $=\{(K_\varphi\wedge K_\lambda)\wedge(x]\}\vee\{(K_\varphi\wedge K_\lambda)\wedge(y]\}$, whence $\varphi(x\vee y)\wedge\lambda(x\vee y)=(\varphi(x)\wedge\lambda(x))\wedge$ $\vee(\varphi(y)\wedge\lambda(y))$. This means that the meet of λ and φ is a join-endomorphism, too, and the first assertion follows.

Let the ideals K_φ and K_λ be neutral and let us consider the ideal $K_\varphi\vee K_\lambda$. $(K_\varphi\vee K_\lambda)\wedge(x]=(K_\varphi\wedge(x])\vee(K_\lambda\wedge(x])=(\varphi(x)]\vee(\lambda(x)]=(\varphi(x)\vee\lambda(x)]$. Thus the ideal $K_\varphi\vee K_\lambda$ generates a meet-translation $\beta(x)=\lambda(x)\vee\varphi(x)$ on L. The join of two neutral ideals is also a neutral ideal, and so $(K_\varphi\vee K_\lambda)\wedge(x\vee y]=\{(K_\varphi\vee K_\lambda)\wedge(x]\}\vee\{(K_\varphi\vee K_\lambda)\wedge$ $\wedge(y]\}$. Hence $\beta(x)$ is a join-endomorphism on L and also a derivation on L.

In [5, Thm. 3] SZÁSZ has shown that the product $\varphi\lambda$ of two derivations on a lattice L is always a derivation, and moreover, $\varphi\lambda(x)=\varphi(\lambda(x))=\varphi(x)\wedge\lambda(x)$.

As shown by SZÁSZ [5, Thm. 2], the derivations of a lattice L are exactly those meet-translations of L that are also endomorphisms on L. As immediate corollary of the construction of KOLIBIAR in [1, Thm. 1], we can write

Theorem 4. *On a modular lattice L there is a one-to-one correspondence between meet-translations φ and congruence relations θ_φ having the property*

(i) *There is in L a neutral ideal T such that every rest class modulo θ_φ contains exactly one element of T.*

The congruence relation θ_φ relating to the meet-translation φ and the meet-translation φ_θ relating to the congruence relation θ_φ are characterized by (ii) *and* (iii), *respectively:*

(ii) $x\theta_\varphi y\Leftrightarrow\varphi(x)=\varphi(y)$, $x, y\in L$;

(iii) $\varphi_\theta(x)=x''\in T$ *for which* $x\theta_\varrho x''$.

Now we can prove an extension of [2, Thm. 1]

Theorem 5. *Let L be a modular lattice. The set of all congruence relations θ_φ relating to the derivations φ on L constitutes a sublattice of the lattice $\theta(L)$ of all congruence relations on L.*

Proof. According to Theorem 4, $x\theta_\varphi y\Leftrightarrow(x]\wedge K_\varphi=(y]\wedge K_\varphi$ for each derivation φ on L. As L is modular, for each derivation φ on L the ideal K_φ is a neutral element of $\mathcal{J}(L)$ (Corollary 1). Hence, for any two derivations φ and λ on L the mappings $\varphi(x)\vee\lambda(x)$ and $\varphi(x)\wedge\lambda(x)$ are derivations on L, too (Theorem 3). Let $\beta(x)=$ $=\varphi(x)\wedge\lambda(x)$. We prove $\theta_\beta=\theta_\varphi\vee\theta_\lambda$ by showing that 1) $\theta_\varphi\vee\theta_\lambda\leqq\theta_\beta$, and 2) $\theta_\varphi\vee\theta_\lambda\geqq$ $\geqq\theta_\beta$.

1) $x\theta_\varphi y\Leftrightarrow(x]\wedge K_\varphi=(y]\wedge K_\varphi\Rightarrow(x]\wedge(K_\varphi\wedge K_\lambda)=(y]\wedge(K_\varphi\wedge K_\lambda)\Leftrightarrow x\theta_\beta y$, and so $\theta_\varphi\leqq\theta_\beta$. Similarly we see that $\theta_\lambda\leqq\theta_\beta$, whence $\theta_\varphi\vee\theta_\lambda\leqq\theta_\beta$.

2) Let $x\theta_\beta y\Leftrightarrow(x]\wedge K_\varphi\wedge K_\lambda=(y]\wedge K_\varphi\wedge K_\lambda\Leftrightarrow x\wedge\varphi(x)\wedge\lambda(x)=y\wedge\varphi(y)\wedge\lambda(y)$. On the other hand, $x\wedge\varphi(x)\theta_\lambda x\wedge\varphi(x)\wedge\lambda(x)$, and moreover, $x\theta_\varphi x\wedge\varphi(x)$. Hence,

$x(\theta_\varphi \vee \theta_\lambda)x \wedge \varphi(x) \wedge \lambda(x)$. Similarly we see that $y(\theta_\varphi \vee \theta_\lambda)y \wedge \varphi(y) \wedge \lambda(y)$, and by combining these results we obtain $x(\theta_\varphi \vee \theta_\lambda)y$. Thus $\theta_\varphi \vee \theta_\lambda \geqq \theta_\beta$.

Let $\alpha(x) = \varphi(x) \vee \lambda(x)$; we prove that $\theta_\alpha = \theta_\varphi \wedge \theta_\lambda$ by showing that 3) $\theta_\alpha \geqq \theta_\varphi \wedge \theta_\lambda$ and 4) $\theta_\alpha \leqq \theta_\varphi \wedge \theta_\lambda$.

3) Let $x(\theta_\varphi \wedge \theta_\lambda)y \Leftrightarrow x\theta_\varphi y$ and $x\theta_\lambda y \Leftrightarrow (x] \wedge K_\varphi = (y] \wedge K_\varphi$ and $(x] \wedge K_\lambda = (y] \wedge K_\lambda \Rightarrow$ $\Rightarrow (x] \wedge (K_\varphi \vee K_\lambda) = (y] \wedge (K_\varphi \vee K_\lambda) \Leftrightarrow x\theta_\alpha y$. Thus $\theta_\alpha \geqq \theta_\varphi \wedge \theta_\lambda$.

4) Let $x\theta_\alpha y \Leftrightarrow (x] \wedge (K_\varphi \vee K_\lambda) = (y] \wedge (K_\varphi \vee K_\lambda) \Rightarrow (x] \wedge (K_\varphi \vee K_\lambda) \wedge K_\varphi = (x] \wedge K_\varphi =$ $= (y] \wedge (K_\varphi \vee K_\lambda) \wedge K_\varphi = (y] \wedge K_\varphi$, and so $x\theta_\varphi y$. Similarly we set that $x\theta_\lambda y$, too. Consequently, $x(\theta_\varphi \wedge \theta_\lambda)y$, which implies the desired result.

A meet-translation φ on a lattice L is called a *weak derivation* on L, if $\varphi(\varphi(x) \vee y) =$ $= \varphi(x) \vee \varphi(y)$ for each two elements $x, y \in L$.

Theorem 6. *Let M be an ideal of a lattice L generating a meet-translation φ on L. Then φ is a weak derivation on L if and only if $M \wedge ((x] \vee (y]) = (M \wedge (x]) \vee$ $\vee (M \wedge (y])$ for each two elements $x, y \in L$ and $x \in M$.*

The proof follows the lines of that of Theorem 2, and hence we omit it. Further, the proof of the following lemma is analogous to that of Lemma 1, and hence it is omitted.

Lemma 2. *Let T be an ideal of a lattice L such that $T \wedge ((x] \vee (y]) = (T \wedge (x]) \vee$ $\vee (T \wedge (y])$ for each two elements $x, y \in L$, $x \in T$. Then $T \wedge (I \vee K) = (T \wedge I) \vee (T \wedge K)$ for each two elements $I, K \in \mathscr{I}(L)$, $I \subseteqq T$.*

As shown by SzÁsz [4, Thms. 4 and 5], the distributivity and modularity of a lattice L can be characterized by derivations and weak derivations of L, respectively. It is interesting to see that these characterizations reduce the distributivity (the modularity) of L to the distributivity (the modularity) of $\mathscr{I}(L)$, as one can deduce from Theorem 2 and Lemma 1, and from Theorem 6 and Lemma 2, respectively.

3. Meet-translations on meet-semilattices. In this section we shall show a connection between meet-translations on meet-semilattices and lattices. We shall consider meet-semilattices only, and hence we shall use the brief expression semilattice instead of meet-semilattice. Note that in S a nonvoid set I is an *ideal* if (i) $x \in I$ and $r \geqq x$ imply $r \in I$, and (ii) $x, y \in I$ imply $x \wedge y \in I$. S is *up-directed* if for each pair $x, y \in S$ there is an element $k \in S$ such that $k \geqq x, y$. In particular, if S is up-directed, then $I \wedge J$ is an ideal of S for each two ideals I and J of S.

Theorem 7. *Let S be an up-directed semilattice and φ a meet-translation on S. Then φ generates a meet-translation φ^g on the lattice $\mathscr{I}(L)$ of all ideals of S defined as follows: $\varphi^g(I) = \{x \mid x \geqq \varphi(y);\, y \in I \in \mathscr{I}(S)\}$.*

Proof. At first we show that $\varphi^g(I)$ is an ideal of S. Let $x \in \varphi^g(I)$ and $r \geqq x$. Then there exists an $y \in I$ such that $r \geqq x \geqq \varphi(y)$, and so $r \in \varphi^g(I)$. Let $a, b \in \varphi^g(I)$. Thus $a \wedge b \geqq \varphi(y_a) \wedge \varphi(y_b) = \varphi(y_a \wedge y_b)$, where $y_a \wedge y_b \in I$; therefore $a \wedge b \in \varphi^g(I)$.

Clearly φ^g is a single-valued mapping on $\mathscr{I}(S)$; thus it remains to show that $\varphi^g(I\wedge J)=\varphi^g(I)\wedge J$. Let $x\in\varphi^g(I\wedge J)$. Then there is an element $y\in I\wedge J$ such that $x\geqq\varphi(y)$. On the other hand, $y\geqq i\wedge j$ with some $i\in I$ and $j\in J$, and $\varphi(y)\geqq\varphi(i\wedge j)=$ $=\varphi(i)\wedge j$. Thus $x\geqq\varphi(i)\wedge j$ with $\varphi(i)\in\varphi^g(I)$ and $j\in J$, whence $x\in\varphi^g(I)\wedge J$. This shows that $\varphi^g(I\wedge J)\subseteqq\varphi^g(I)\wedge J$.

Let now $x\in\varphi^g(I)\wedge J$. Then $x\geqq r\wedge j$ for some $r\in\varphi^g(I)$ and $j\in J$. Furthermore, there exists an $i\in I$ such that $r\geqq\varphi(i)$, and so $x\geqq\varphi(i)\wedge j=\varphi(i\wedge j)$, where $i\wedge j\in I\wedge J$. Therefore, $x\in\varphi^g(I\wedge J)$, and the relation $\varphi^g(I)\wedge J\subseteqq\varphi^g(I\wedge J)$ holds. Consequently, $\varphi^g(I\wedge J)=\varphi^g(I)\wedge J$, and the theorem follows.

Let $[z)=\{x|x\geqq z,\ x, z\in S\}$. The validity of the following assertion is obvious.

Theorem 8. *A meet-translation φ on $\mathscr{I}(S)$ is generated by a meet-translation λ on S, i.e. $\varphi=\lambda^g$, if and only if for each $x\in S$ there is an element $k\in S$ such that $\varphi([x))=[k)$.*

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References

[1] M. Kolibiar, Bemerkungen über Translationen der Verbände, *Acta Fac. R. N. Univ. Comenianane Math.*, **5** (1961), 455—458.

[2] J. Nieminen, A note on relations of distributive lattices, *Acta Fac. R. N. Univ. Comenianae Math.*, to appear.

[3] G. Szász, Die Translationen der Halbverbände, *Acta Sci. Math.*, **17** (1956), 165—169.

[4] G. Szász, Translationen der Verbände, *Acta Fac. R. N. Univ. Comenianae Math.*, **5** (1961), 449—453.

[5] G. Szász, Derivations of lattices, *Acta Sci. Math.*, 36 (1975), 149—154.

[6] G. Szász, *Introduction to lattice theory*, Academic. Press and Akadémiai Kiadó (New York—London and Budapest, 1963).

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