Acta Sci. Math., 38 (1976), 325—341

Probability inequalities of exponential type and laws of the interated logarithm

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Introduction

Let $\xi_1, \xi_2, \ldots, \xi_n$ be random variables (in abbreviation: rv); they need not be independent or identically distributed. Set

$$S_k = \sum_{i=1}^{k} \xi_i \quad \text{and} \quad M_n = \max_{1 \leq k \leq n} |S_k|.$$

Further, for each vector $(\xi_{b+1}, \xi_{b+2}, \ldots, \xi_{b+k})$ of k consecutive ξ_i's, let $F_{b,k}$ denote the joint distribution function and let

$$S_{b,k} = \sum_{i=b+1}^{b+k} \xi_i = S_{b+k} - S_b \quad (S_{b,0} = 0)$$

and

$$M_{b,k} = \max \{|S_{b,1}|, |S_{b,2}|, \ldots, |S_{b,k}|\}.$$

Thus $S_k = S_{0,k}$ and $M_n = M_{0,n}$. Set $F_n = F_{0,n}$. The concern of this paper is to provide bounds on $E\{\exp(\lambda M_n)\}$ in terms of given bounds on $E\{\exp(\lambda |S_{b,k}|)\}$, where $\lambda > 0$.

We emphasize that it is *not* assumed that the ξ_i's are independent. The only restrictions on the dependence will be those imposed on the assumed bounds for $E\{\exp(\lambda |S_{b,k}|)\}$. In point of fact, these assumed bounds are guaranteed under a suitable dependence restriction (e.g., mutual independence, martingale differences, weak multiplicativity, or the like).

Bounds on $E\{\exp(\lambda M_n)\}$ are of use in deriving convergence properties of S_n as $n \to \infty$. For development of such results under various dependence restrictions, the theorems of this paper reduce the problem of placing appropriate bounds on $E\{\exp(\lambda M_n)\}$ to the typically easier problem of placing appropriate bounds on $E\{\exp(\lambda |S_{b,k}|)\}$.

Received October 1, 1975.

The proof of our main result (Theorem 1) is based on the "bisection" technique of BILLINGSLEY [1; p. 102] and the treatment is in a setting close to that of SERFLING [9]. The use of Theorem 1 simplifies and extends the method of SERFLING [10] to obtain results such as laws of the iterated logarithm, convergence rates thereof, etc. under probability inequalities of exponential type. For generalities concerning different convergence properties the reader is sent to our main reference [10].

Another extension of Serfling's method based on the study of the moment inequalities of type $E|S_{b,n}|^\nu$ with a fixed $\nu>0$ is dealt with in [6].

§ 1. The main result

In the following the function $g(F_{b,k})$ denotes a non-negative functional depending on the joint distribution function of $\xi_{b+1}, \xi_{b+2}, \dots, \xi_{b+k}$. Examples are: $g(F_{b,k})=k^\alpha$ where $\alpha>0$, or $g(F_{b,k})=\sum_{i=b+1}^{b+k} a_i^2$ where $\{a_i\}$ is a sequence of numbers. (In most cases a_i^2 is the finite variance of ξ_i, but this plays no role in our results.) In the sequel $C, C_1, C_2, \dots$ denote positive constants; b, k, l, n non-negative integers and λ a positive real number.

Theorem 1. *Suppose that there exists a non-negative function $g(F_{b,k})$ satisfying*

$$(1.1)\qquad g(F_{b,k})+g(F_{b+k,l}) \leqq g(F_{b,k+l}) \quad (\textit{all } b \geqq 0,\ k\geqq 1,\ l\geqq 1)$$

such that

$$(1.2)\qquad E\{e^{\lambda|S_{b,k}|}\} \leqq Ce^{\lambda^2 g(F_{b,k})} \quad (\textit{all } b \geqq 0,\ k \geqq 1,\ \lambda > 0).$$

Then

$$(1.3)\qquad E\{e^{\lambda M_n}\} \leqq 8Ce^{12\lambda^2 g(F_n)} \quad (\textit{all } n \geqq 1,\ \lambda > 0).$$

In Theorem 1 the bounds may involve parameters of the joint distribution function of $\xi_1, \xi_2, \dots, \xi_n$, a flexibility particularly useful with non-identically distributed rv.

Proof. We are to find two constants C_1 and C_2 not less then 1, for which

$$(1.4)\qquad E\{e^{\lambda M_n}\} \leqq C_1 e^{C_2\lambda^2 g(F_n)} \quad (n\geqq 1,\ \lambda > 0).$$

The proof goes by induction on n. The result is trivial for $n=1$. Assume now as induction hypotheses that the result holds for each integer less than n. The function $g(F_n)$ being non-negative and non-decreasing in n, we may assume $g(F_n)>0$. There exists an integer h, $1\leqq h\leqq n$, such that

$$(1.5)\qquad g(F_{h-1}) \leqq \tfrac{1}{2} g(F_n) < g(F_h),$$

where $g(F_{h-1})$ on the left is 0 if $h=1$. Then (1.1) and (1.5) imply

$$g(F_{h,n-h}) \leqq g(F_{0,n})-g(F_{0,h}) < \frac{1}{2}\, g(F_n). \tag{1.6}$$

It is obvious that for $1 \leqq k < h$ we have

$$|S_k| \leqq M_{0,h-1},$$

and for $h \leqq k \leqq n$

$$|S_k| \leqq |S_h| + M_{h,n-h}.$$

Also, for $1 \leqq k \leqq n$ and $\lambda > 0$ we have

$$\lambda |S_k| \leqq \lambda |S_h| + \log (e^{\lambda M_{0,h-1}} + e^{\lambda M_{h,n-h}}).$$

Therefore,

$$\lambda M_n \leqq \lambda |S_h| + \log (e^{\lambda M_{0,h-1}} + e^{\lambda M_{h,n-h}}),$$

whence

$$e^{\lambda M_n} \leqq e^{\lambda |S_h|} (e^{\lambda M_{0,h-1}} + e^{\lambda M_{h,n-h}})$$

for all $\lambda > 0$. Let p and q be positive numbers with $1/p + 1/q = 1$, whose values will be determined later on. Using Hölder's and then Minkowski's inequalities, we find that

$$\begin{aligned} E\{e^{\lambda M_n}\} &\leqq E\{e^{p\lambda |S_h|}\}^{1/p} E\{(e^{\lambda M_{0,h-1}} + e^{\lambda M_{h,n-h}})^q\}^{1/q} \leqq \\ &\leqq E\{e^{p\lambda |S_h|}\}^{1/p} (E\{e^{q\lambda M_{0,h-1}}\}^{1/q} + E\{e^{q\lambda M_{h,n-h}}\}^{1/q}). \end{aligned} \tag{1.7}$$

Since $h-1 < n$, we may apply the induction hypothesis to the rv $\xi_1, \xi_2, \dots, \xi_{h-1}$ and conclude by (1.4) that

$$E\{e^{q\lambda M_{0,h-1}}\}^{1/q} \leqq C_1^{1/q} e^{qC_2\lambda^2 g(F_{h-1})} \leqq C_1^{1/q} \exp\left[\frac{1}{2}\, qC_2 \lambda^2 g(F_n)\right], \tag{1.8}$$

the last inequality following by (1.5). We note that if $h=1$, then (1.8) is obvious.

If the indices in (1.2) are restricted to $b \geqq h$, $1 \leqq k \leqq n-b$, then only the rv $\xi_{h+1}, \xi_{h+2}, \dots, \xi_n$ are involved. Since $n-h < n$, the induction hypothesis applies to $\xi_{h+1}, \xi_{h+2}, \dots, \xi_n$. Hence (1.4) yields

$$E\{e^{q\lambda M_{h,n-h}}\}^{1/q} \leqq C_1^{1/q} e^{qC_2\lambda^2 g(F_{h,n-h})} \leqq C_1^{1/q} \exp[\tfrac{1}{2}\, qC_2\lambda^2 g(F_n)], \tag{1.9}$$

where the last inequality follows by (1.6). (If $h=n$, (1.9) is trivial.)

Finally, (1.2) implies

$$E\{e^{p\lambda |S_h|}\}^{1/p} \leqq C^{1/p} e^{p\lambda^2 g(F_h)} \leqq C^{1/p} e^{p\lambda^2 g(F_n)}. \tag{1.10}$$

Combining inequalities (1.7)—(1.10), we arrive at

$$E\{e^{\lambda M_n}\} \leqq 2C^{1/p} C_1^{1/q} \exp[(p + \tfrac{1}{2}\, qC_2)\lambda^2 g(F_n)].$$

Assuming $1 < q < 2$, and consequently $p > 2$, we have

$$2C^{1/p} C_1^{1/q} \leqq C_1 \quad \text{and} \quad p + \tfrac{1}{2} qC_2 \leqq C_2,$$

provided

$$C_1 \geqq 2^p C \quad \text{and} \quad C_2 \geqq \frac{2p}{2-q}. \tag{1.11}$$

Choosing, for example, $q=3/2$ and $p=3$, the smallest C_1 and C_2 satisfying (1.11) are given by $C_1=8C$ and $C_2=12$, as they are given in (1.3). This completes the induction step and the proof of Theorem 1.

Although the specific values of C_1 and C_2 will have no importance for us, the best value (provided by the above proof) of C_2 may be taken as $C_2=6+4\sqrt{2}$. (Namely, the expression $2p/(2-q)$ attains its minimum on $(2, \infty)$ at $p=2+\sqrt{2}$.)

The extension of the validity of Theorem 1, when λ^2 in the exponents on the right of (1.2) and (1.3) is replaced by a polynomial in λ, say $r(\lambda)$, is of interest in itself and may be of use in some applications.

Theorem 2. *Suppose that there exist a non-negative function $g(F_{b,k})$ satisfying (1.1) and a polynomial*

$$r(\lambda) = \sum_{i=l}^{m} \alpha_i \lambda^i$$

of at least first degree, strictly positive for $\lambda>0$, such that

$$E\{e^{\lambda|S_{b,k}|}\} \leqq Ce^{r(\lambda)g(F_{b,k})} \quad (\text{all } b \geqq 0,\ k \geqq 1,\ \lambda > 0). \tag{1.12}$$

Then

$$E\{e^{\lambda M_n}\} \leqq C_1 e^{C_2 r(\lambda) g(F_n)} \quad (\text{all } n \geqq 1,\ \lambda > 0), \tag{1.13}$$

where C_1 and C_2 are constants depending only on $r(\lambda)$.

Proof. The proof of Theorem 2 runs along the same lines as that of Theorem 1. The same sort of argument that yielded (1.8)—(1.10) shows that

$$E\{e^{q\lambda M_{0,h-1}}\}^{1/q} \leqq C_1^{1/q} \exp\left[\frac{1}{2q} C_2 r(q\lambda) g(F_n)\right],$$

$$E\{e^{q\lambda M_{h,n-h}}\}^{1/q} \leqq C_1^{1/q} \exp\left[\frac{1}{2q} C_2 r(q\lambda) g(F_n)\right],$$

and

$$E\{e^{p\lambda|S_h|}\}^{1/p} \leqq C^{1/p} \exp\left[\frac{1}{p} r(p\lambda) g(F_n)\right].$$

Combining inequality (1.7) with the last three ones, we arrive at

$$E\{e^{\lambda M_n}\} \leqq 2C^{1/p} C_1^{1/q} \exp\left(\left[\frac{1}{p} r(p\lambda) + \frac{1}{2q} C_2 r(q\lambda)\right] g(F_n)\right). \tag{1.14}$$

Now we have to choose $q<2$ $(p=q/(q-1))$ and the constants C_1, C_2 in such a way that

$$2C^{1/p}C_1^{1/q} \leqq C_1 \tag{1.15}$$

and

$$\frac{1}{p}r(p\lambda)+\frac{1}{2q}C_2 r(q\lambda) \leqq C_2 r(\lambda) \tag{1.16}$$

hold for all $\lambda>0$. Condition (1.15) does not cause any difficulty. On the other hand, (1.16) requires some arguments. Writing

$$s(\lambda) = C_2\left[r(\lambda)-\frac{1}{2q}r(q\lambda)\right]-\frac{1}{p}r(p\lambda),$$

we will prove the existence of q and C_2 such that $s(\lambda)\geqq 0$ for all $\lambda>0$.

First we notice that from the assumption on $r(\lambda)$ it immediately follows that $\alpha_m>0$ and $\alpha_l>0$. Then we show that

$$r(\lambda)-\frac{1}{2q}r(q\lambda) \geqq \frac{1}{4}r(\lambda) \tag{1.17}$$

for all $\lambda>0$, provided q is sufficiently close to 1. Inequality (1.17) is equivalent to

$$t(\lambda) = 3r(\lambda)-\frac{2}{q}r(q\lambda) \geqq 0 \tag{1.18}$$

for all $\lambda>0$. We consider only those q's for which $q^{m-1}\leqq 3/2$ minus a small positive number, say let $q^{m-1}\leqq 5/4$. A simple reasoning gives that if

$$\lambda \geqq \max\left(1, \frac{1}{2\alpha_m}\sum_{i=l}^{m-1}|\alpha_i|\right)$$

or

$$0<\lambda \leqq \min\left(1, \frac{\alpha_l}{2\sum_{i=l+1}^{m}|\alpha_i|}\right)$$

then (1.18) is true. Since

$$\lim_{q\to 1+0} t(\lambda) = r(\lambda)$$

uniformly on each finite segment, hence we can choose q, $1<q$ and $q^{m-1}\leqq 5/4$, such that (1.18) holds for all $\lambda>0$. Thus we can and do fix $q>1$ for which (1.17) is satisfied. Let $p=q/(q-1)$ and return to the study of $s(\lambda)$.

The behaviour of $s(\lambda)$ for λ large enough is determined by the coefficient of λ^m. Hence we have to choose C_2 such that

$$\alpha_m(C_2 - \tfrac{1}{2} C_2 q^{m-1} - p^{m-1}) > 0,$$

i.e.,

$$C_2 > \frac{2p^{m-1}}{2-q^{m-1}}. \tag{1.19}$$

This choice implies $s(\lambda) \geqq 0$ for sufficiently large λ, say $\lambda \geqq \Lambda_0$.

In case when λ is small enough, the coefficient of λ^l is decisive for the sign of $s(\lambda)$. In order to ensure that $s(\lambda) \geqq 0$ for sufficiently small λ, say $0 < \lambda \leqq \lambda_0$, we have to require that

$$C_2 > \frac{2p^{l-1}}{2-q^{l-1}}.$$

But condition (1.19) implies this, it suffices to keep in mind only that $m \geqq l$, $p > 2$, $q > 1$, and $q^{m-1} < 2$.

Thus it remains to deal with the case $\lambda_0 \leqq \lambda \leqq \Lambda_0$. Since the polynomial $r(\lambda)$ has no zero on $0 < \lambda < \infty$, it follows that

$$r_1 = \min_{\lambda_0 \leqq \lambda \leqq \Lambda_0} r(\lambda)$$

is a positive number. Further, set

$$R_p = \max_{\lambda_0 \leqq \lambda \leqq \Lambda_0} \frac{1}{p} r(p\lambda).$$

Taking into account that (1.17) holds for all $\lambda > 0$, we have

$$s(\lambda) \geqq \frac{1}{4} C_2 r(\lambda) - \frac{1}{p} r(p\lambda) \geqq \frac{1}{4} C_2 r_1 - R_p \geqq 0$$

for every λ in $[\lambda_0, \Lambda_0]$ provided $C_2 \geqq 4R_p/r_1$. If, in addition, C_2 fulfills (1.19) then we can conclude that $s(\lambda) \geqq 0$, and consequently, (1.16) is satisfied for all $\lambda > 0$. Finally, if $C_1 = 2^p C$ then (1.15) is also satisfied.

Continuing our reasoning with (1.14), by (1.15) and (1.16) we arrive at the desired (1.13). Thus we finished the proof of Theorem 2.

Before coming to the applications, we make a remark on the validity of Theorems 1 and 2. Viewing the proofs, it is striking that we use no full power of a probability space. In fact, Hölder's and Minkowski's inequalities were applied only, which are available in any measure space (X, A, μ). Hence Theorems 1 and 2 are valid on (X, A, μ) taking integrals over X with respect to μ in place of the expectations on the left-hand sides of the corresponding inequalities.

§ 2. Laws of the iterated logarithm as consequences of a probability inequality of exponential type for $S_{b,n}$

Now we will discuss the stochastic convergence properties of S_n under restrictions of type (1.2). The following result, which expresses a form of the law of the iterated logarithm, certainly has a broad scope of application.

Theorem 3. *Suppose that there exist a positive number K and a sequence $\{a_i\}$ of numbers such that*

$$E\{e^{\lambda|S_{b,k}|}\} \leqq C\exp\left(\tfrac{1}{2}K\lambda^2A_{b,k}^2\right) \quad (\text{all } b \geqq 0,\ k \geqq 1,\ \lambda > 0), \tag{2.1}$$

where

$$A_{b,k} = \left(\sum_{i=b+1}^{b+k} a_i^2\right)^{1/2} \quad \text{and} \quad A_n = A_{0,n} \to \infty \quad (n \to \infty). \tag{2.2a}$$

Then it follows a law of the iterated logarithm with K, i.e.

$$P\left\{\limsup_{n\to\infty}\frac{|S_n|}{(2KA_n^2\log\log A_n)^{1/2}} \leqq 1\right\} = 1. \tag{2.3}$$

We note that the conclusion of Theorem 3 in the special case $a_i \equiv 1$, $A_n^2 = n$ was proved by Serfling [10, Theorem 4.1] for *uniformly bounded rv,* $|\xi_i| \leqq B$, *having the following properties:*

(i) *for any $\nu > 2$ there exists a constant C_ν such that*

$$E|S_{b,n}|^\nu \leqq C_\nu n^{\nu/2} \quad (\text{all } b \geqq 0,\ n \geqq 1), \tag{2.4}$$

(ii) *the inequality*

$$P\{|S_n| > y\} \leqq 2\exp\left\{-\frac{y^2}{2B^2n}\right\} \quad (\text{all } n \geqq 1)$$

holds for any $y > 0$.

The following theorem provides information on the rate of convergence in (2.3).

Theorem 4. *Suppose that* (2.1) *holds, where*

$$A_n \to \infty \quad \text{and} \quad a_n = o(A_n) \quad (n \to \infty). \tag{2.2b}$$

Then, for each $\theta > 2K$, we have

$$\sum_n \frac{a_n^2}{A_n^2\log A_n} P\left\{\sup_{k\geqq n}\frac{|S_k|}{(\theta A_k^2\log\log A_k)^{1/2}} \geqq 1\right\} < \infty. \tag{2.5}$$

If the factor $(\theta\log\log A_k)^{1/2}$ in the expression (2.5) is replaced by a rougher factor $(\log A_k)^\alpha$ with an $\alpha > 0$, then an essentially better rate of convergence depending on α can be achieved, as the following theorem shows.

Theorem 5. *Suppose that* (2.1) *and* (2.2b) *hold. Then setting*

$$P_n = P\left\{\sup_{k\geq n} \frac{|S_k|}{A_k(\log A_k)^\alpha} \geq 1\right\}$$

we have for each choice of $0<\alpha<1/2$ *and* $\beta>0$

$$\sum_n \frac{a_n^2(\log A_n)^\beta}{A_n^2} P_n < \infty,$$

for $\alpha=1/2$ *and* $\beta>0$

$$\sum_n \frac{a_n^2}{A_n^{\beta+(2K-1)/K}} P_n < \infty,$$

and for $\alpha>1/2$ *and* $\beta>0$

$$\sum_n a_n^2 A_n^\beta P_n < \infty.$$

It is instructive to compare Theorem 5 with a result of Serfling [10, Corollary 5.3.1], which reads as follows: *Suppose that in the special case* $a_i\equiv 1$, $A_n^2=n$, *we have* (2.4) *for all* $\nu>2$. *Then*

$$\sum_n \frac{1}{n(\log n)^{1-\beta}} P\left\{\sup_{k\geq n} \frac{|S_k|}{k^{1/2}(\log k)^\alpha} > 1\right\} < \infty$$

holds for each choice of $\alpha>0$ *and* $0<\beta<1$.

The results stated in Theorems 3—5 are obtained by adaption of more or less standard arguments [2], [4], and [7] making use of Theorem 1. More precisely, bounds on $E\{\exp(\lambda M_{b,k})\}$ are of use in deriving bounds on the tail distribution of $M_{b,k}$. By Chebyshev's inequality, (2.1) implies

$$(2.6)\quad P\{|S_n|\geq y\} = P\{e^{\lambda|S_n|}\geq e^{\lambda y}\} \leq C\exp\left(\frac{1}{2}K\lambda^2A_n^2-\lambda y\right) = C\exp\left(-\frac{y^2}{2KA_n^2}\right),$$

if λ is chosen as $\lambda=y/KA$. Here and in the sequel y denotes a positive number. Further, also by Chebyshev's inequality, (2.1) implies via Theorem 1 that

$$(2.7)\qquad P\{M_{b,k}\geq y\} \leq 8C\exp\left(-\frac{y^2}{24KA_{b,k}^2}\right).$$

The proofs below are based on the bounds (2.7) on the tail distribution of $M_{b,k}$, which is of interest in its own right, too. An extra factor of 8 in the coefficient on the right-hand side of (2.7) will not matter for our purposes, and the bounds we derive will decrease with increasing y slowly enough that passing from y^2 to $y^2/12$ in the exponent will have no important effect.

Proof of Theorem 3. We have to prove that, for any $\theta > 2K$, with probability 1 we have

$$|S_n| \leqq (\theta A_n^2 \log\log A_n^2)^{1/2}$$

for all n large enough. It is clear that this implies (2.3).

Let $\delta > 1$ be a fixed number and define a sequence of integers $1 \leqq n_1 \leqq n_2 \leqq \dots$ in the following way:

$$A_{n_k-1}^2 \leqq \delta^k < A_{n_k}^2 \quad (k = 1, 2, \dots;\ A_0 = 0). \tag{2.8}$$

This is possible by (2.2a), and obviously $n_k \to \infty$ as $k \to \infty$.

Set

$$\gamma = \frac{\theta}{2K} \quad \text{and} \quad \mu(n) = (\theta A_n^2 \log\log A_n^2)^{1/2}.$$

By the above assumption $\gamma > 1$. Then (2.6) provides

$$P\{|S_{n_k}| \geqq \mu(n_k)\} \leqq C \exp(-\gamma \log\log A_{n_k}^2) = \frac{C}{(\log A_{n_k}^2)^\gamma}.$$

By (2.8) we get

$$\sum_k{}' P\{|S_{n_k}| \geqq \mu(n_k)\} \leqq \frac{C}{(\log\delta)^\gamma} \sum_{k=1}^{\infty} \frac{1}{k^\gamma} < \infty,$$

where $\sum_k{}'$ means that the summation is taken only once for equal n_k's. In virtue of the Borel—Cantelli lemma, this yields with probability 1 that

$$|S_{n_k}| \leqq (\theta A_{n_k}^2 \log\log A_{n_k}^2)^{1/2} \tag{2.9}$$

for all k large enough.

For an arbitrary n, either $n = n_k$ or $n_k < n < n_{k+1}$ for some k. If $n_k < n < n_{k+1}$, consider

$$\frac{S_n}{\mu(n)} = \frac{S_{n_k}}{\mu(n_k)} \frac{\mu(n_k)}{\mu(n)} + \frac{|S_n - S_{n_k}|}{\bar\mu(n_k)} \frac{\bar\mu(n_k)}{\mu(n)},$$

where

$$\bar\mu(n_k) = (12\,\theta A_{n_k, \nu_k - 1}^2 \log\log A_{n_k}^2)^{1/2} \quad \text{and} \quad \nu_k = n_{k+1} - n_k.$$

Since $\mu(n)$ is non-decreasing, it follows that

$$\frac{|S_n|}{\mu(n)} \leqq \frac{|S_{n_k}|}{\mu(n_k)} + \frac{|S_n - S_{n_k}|}{\bar\mu(n_k)} \frac{\bar\mu(n_k)}{\mu(n)}. \tag{2.10}$$

We will show that with probability 1

$$\max_{n_k < n < n_{k+1}} \frac{|S_n - S_{n_k}|}{\bar\mu(n_k)} = \frac{M_{n_k, \nu_k - 1}}{\bar\mu(n_k)} \leqq 1 \tag{2.11}$$

for all k large enough. To this effect, utilize (2.7). Then

$$P\{M_{n_k, \nu_k - 1} \geqq \bar{\mu}(n_k)\} \leqq 8C \exp(-\gamma \log\log A_{n_k}^2).$$

As above, this implies

$$\sum_k{}'' P\{M_{n_k, \nu_k - 1} \geqq \bar{\mu}(n_k)\} < \infty,$$

where $\sum_k''$ means that the summation is extended to such k's that $n_k < n_{k+1} - 1$. By the Borel—Cantelli lemma we get the wanted (2.11).

Owing to (2.8) we have $A_{n_k}^2 > \delta^k$ and

$$A_{n_k, \nu_k - 1}^2 = A_{n_{k+1}-1}^2 - A_{n_k}^2 \leqq \delta^k(\delta - 1).$$

Thus

$$\frac{\bar{\mu}(n_k)}{\mu(n_k)} = \frac{\sqrt{12}\, A_{n_k, \nu_k - 1}}{A_{n_k}} \leqq [12(\delta - 1)]^{1/2}.$$

The right-most member here can be made as small as needed if $\delta \to 1$. Hence, combining (2.9)—(2.11) it follows that, for any $\varepsilon > 0$, with probability 1

$$|S_n| \leqq [(\theta + \varepsilon) A_n^2 \log\log A_n^2]^{1/2}$$

holds for all n large enough. Since $\theta + \varepsilon$ may be chosen arbitrarily close to $2K$, the conclusion of Theorem 3 is proved.

Proof of Theorem 4. Let $\delta > 1$ be a fixed number. We will show that (2.2b) implies the existence of a strictly increasing sequence $\{n_k\}$ of positive integers such that

$$\delta^k \leqq A_{n_k}^2 < \delta^{k+1} \tag{2.12}$$

for all k large enough. Otherwise, for infinitely many n's, we have

$$A_n^2 < \delta^{k+1} \quad \text{and} \quad A_{n+1}^2 \geqq \delta^{k+2}$$

with suitable k's. This gives that

$$\frac{a_{n+1}^2}{A_{n+1}^2} = 1 - \frac{A_n^2}{A_{n+1}^2} \geqq 1 - \frac{\delta^{k+1}}{\delta^{k+2}} = \frac{\delta - 1}{\delta}$$

for infinitely many n's, which contradicts (2.2b).

In proving the convergence of the series (2.5), we make use of the convergence part of the following assertion, applied widely in the theory of numerical series: *Let $d_i \geqq 0$ be the terms of a divergent series with partial sums D_n. Then the series*

$$\sum_n \frac{d_n}{D_n (\log D_n)^{1+\varepsilon}}$$

converges or diverges according as $\varepsilon > 0$ *or* $\varepsilon \leqq 0$. Hence it is enough to demonstrate that

$$P_n = P\left\{\sup_{l \geqq n} \frac{|S_l|}{(\theta A_l^2 \log\log A_l^2)^{1/2}} \geqq 1\right\} \leqq \frac{C_3}{(\log A_n^2)^{\varepsilon}} \tag{2.13}$$

with an appropriate $\varepsilon > 0$.

To this effect, let us fix a number θ_1 so that

$$2K < \theta_1 < \theta. \tag{2.14}$$

Let $k_0 = k_0(n)$ be defined by $n_{k_0} < n \leqq n_{k_0+1}$. We may assume that n, and consequently k, are large enough, so that (2.12) is satisfied. It is obvious that

$$P_n \leqq \sum_{k=k_0}^{\infty} Q_k \quad \text{where} \quad Q_k = P\left\{\max_{n_k < l \leqq n_{k+1}} \frac{|S_l|}{(\theta A_l^2 \log\log A_l^2)^{1/2}} \geqq 1\right\}. \tag{2.15}$$

It can be easily checked that

$$Q_k \leqq P\left\{\frac{|S_{n_k}|}{[\theta_1 \sigma(n_k)]^{1/2}} \geqq 1\right\} + P\left\{\max_{n_k < l \leqq n_{k+1}} \frac{|S_l - S_{n_k}|}{[2K\sigma(n_k)]^{1/2}} \geqq \eta\right\} = Q_{1,k} + Q_{2,k}, \tag{2.16}$$

where, for the sake of brevity, we put

$$\sigma(n) = A_n^2 \log\log A_n^2 \quad \text{and} \quad \eta = \left[1 - \left(\frac{\theta_1}{\theta}\right)^{1/2}\right] \left(\frac{\theta}{2K}\right)^{1/2}.$$

Repeating the argument that yielded (2.9) in the proof of Theorem 3, we can establish with ease by (2.6) that

$$Q_{1,k} \leqq C \exp(-\gamma_1 \log\log A_{n_k}^2) = \frac{C}{(\log A_{n_k}^2)^{\gamma_1}},$$

where $\gamma_1 = \theta_1/2K$. By (2.14) we have $\gamma_1 > 1$. Thus, using (2.12), we find that

$$\begin{aligned} \sum_{k=k_0}^{\infty} Q_{1,k} &\leqq \frac{C}{(\log \delta)^{\gamma_1}} \sum_{k=k_0}^{\infty} \frac{1}{k^{\gamma_1}} \leqq \frac{C}{(\gamma_1 - 1)(\log \delta)^{\gamma_1}(k_0 - 1)^{\gamma_1 - 1}} \leqq \\ &\leqq \frac{2^{\gamma_1 - 1} C}{(\gamma_1 - 1)(\log \delta)^{\gamma_1}(k_0 + 2)^{\gamma_1 - 1}} \leqq \frac{2^{\gamma_1 - 1} C}{(\gamma_1 - 1) \log \delta \, (\log A_n^2)^{\gamma_1 - 1}}, \end{aligned} \tag{2.17}$$

provided $k_0 + 2 \leqq 2(k_0 - 1)$, i.e., $k_0 \geqq 4$, which we may assume without loss of generality.

Let us now deal with the series $\sum_{k=k_0}^{\infty} Q_{2,k}$. By (2.7) it is bounded from above by the series

$$8C \sum_{k=k_0}^{\infty} \exp\left(-\frac{\eta^2 A_{n_k}^2 \log\log A_{n_k}^2}{12(A_{n_{k+1}}^2 - A_{n_k}^2)}\right),$$

and therefore also by

$$8C\sum_{k=k_0}^{\infty}\exp\left(-\frac{\eta^2\log\log A_{n_k}^2}{12(\delta^2-1)}\right)=8C\sum_{k=k_0}^{\infty}\frac{1}{(\log A_{n_k}^2)^{\gamma_2}}$$

with $\gamma_2=\eta^2/12(\delta^2-1)$, since by (2.12)

$$\frac{A_{n_k}^2}{A_{n_{k+1}}^2-A_{n_k}^2}\geqq\frac{\delta^k}{\delta^{k+2}-\delta^k}=\frac{1}{\delta^2-1}.$$

Since δ may be chosen arbitrary close to 1, fix $\delta>1$ in such a way that $\gamma_2>1$. Then the same sort of argument that yielded (2.17) shows that

$$\sum_{k=k_0}^{\infty}Q_{2,k}\leqq\frac{2^{\gamma_2+2}C}{(\gamma_2-1)\log\delta\,(\log A_n^2)^{\gamma_2-1}}. \tag{2.18}$$

Putting together (2.15)—(2.18), we arrive at (2.13) with $\varepsilon=\min(\gamma_1,\gamma_2)-1$. This completes the proof of Theorem 4.

The proof of Theorem 5 runs along the same lines as that of Theorem 4. We only notice that after the application of (2.6) and (2.7) we have to use the following elementary inequalities:

$$\exp\{-\gamma(\log x)^{2\alpha}\}\leqq\begin{cases}C(\log x)^{-\beta} & \text{if } 0<\alpha<\frac{1}{2} \text{ and } \beta>0,\\ x^{-\gamma} & \text{if } \alpha=\frac{1}{2},\\ Cx^{-\beta} & \text{if } \alpha>\frac{1}{2} \text{ and } \beta>0,\end{cases}$$

where $x\geqq2$ and C depend only on α, β and $\gamma>0$.

In the sequel as a particular case, consider a sequence $\{\varphi_i\}$ of *weakly multiplicative* rv, i.e., we assume that

$$W_r=\Big(\sum_{1\leqq i_1<i_2<\ldots<i_r}E^2\{\varphi_{i_1}\varphi_{i_2}\ldots\varphi_{i_r}\}\Big)^{1/2}<\infty\quad(r=4,6,\ldots), \tag{2.19}$$

where the summation is extended over all integers satisfying only the condition $1\leqq i_1<i_2<\ldots<i_r$, and further

$$W_r^{1/r}=O(1)\quad(r\to\infty).$$

This is a generalization of the concept of *multiplicativity* defined by

$$E\{\varphi_{i_1}\varphi_{i_2}\ldots\varphi_{i_r}\}=0\quad(1\leqq i_1<i_2<\ldots<i_r;\ r=4,6,\ldots). \tag{2.20}$$

The condition (2.20) is stronger than (2.19). Even the former includes the case of a sequence of martingale differences and the case of mutually independent rv and special varieties thereof (see RÉVÉSZ [7]).

We proved in [5, Lemma 3] that (2.1) is valid with a definite K for uniformly bounded sequences of weakly multiplicative rv. More precisely, the following result holds: *Let $\{\varphi_i\}$ be a sequence of rv such that*

$$|\varphi_i| \leqq B(<\infty) \quad (i=1,2,\ldots) \tag{2.21}$$

and

$$\limsup_{r\to\infty} W_r^{1/r} = W(<\infty). \tag{2.22}$$

Then for every $\gamma>0$ there exists a constant C_γ such that for every sequence $\{a_i\}$ of numbers we have

$$E\{e^{\lambda|S_{b,k}|}\} \leqq C_\gamma \exp\left[\tfrac{1}{2}(B^2+W^2+\gamma)\lambda^2 A_{b,k}^2\right] \quad (all\ b\geqq 0,\ k\geqq 1,\ \lambda>0),$$

where

$$S_{b,k}=\sum_{i=b+1}^{b+k} a_i\varphi_i \quad and \quad A_{b,k}^2=\sum_{i=b+1}^{b+k} a_i^2.$$

Hence, via Theorems 3—5, we obtain

Corollary 1. *Let $\{\varphi_i\}$ be a sequence of rv satisfying* (2.21) *and* (2.22). *Let $\{a_i\}$ be a sequence of numbers with* (2.2a). *Then there follows a law of the iterated logarithm for $\{\xi_i=a_i\varphi_i\}$ with $K=B^2+W^2$, i.e.,*

$$P\left\{\limsup_{n\to\infty}\frac{\left|\sum_{i=1}^{n} a_i\varphi_i\right|}{[2(B^2+W^2)A_n^2\log\log A_n]^{1/2}}\leqq 1\right\}=1.$$

Corollary 2. *Let $\{\varphi_i\}$ be a sequence of rv satisfying* (2.21) *and* (2.22). *Let $\{a_i\}$ be a sequence of numbers with* (2.2b). *Then, for each $\theta>2(B^2+W^2)$, we have*

$$\sum_n \frac{a_n^2}{A_n^2\log A_n} P\left\{\sup_{k\geqq n}\frac{\left|\sum_{i=1}^{k} a_i\varphi_i\right|}{(\theta A_k^2\log\log A_k)^{1/2}}\geqq 1\right\}<\infty.$$

Corollary 3. *Under the conditions of Corollary 2 we have*

$$\sum_n \frac{a_n^2(\log A_n)^\beta}{A_n^2} P\left\{\sup_{k\geqq n}\frac{\left|\sum_{i=1}^{k} a_i\varphi_i\right|}{A_k(\log A_k)^\alpha}\geqq 1\right\}<\infty$$

for each choice of $\alpha>0$ and $\beta>0$.

Corollaries 1 and 2 were proved by the present author [5] in another way, and the latter one under somewhat more restricted conditions stipulated on $\{a_i\}$. Laws of the iterated logarithm, convergence rates in them was proved for multiplicative rv in the special case $a_i\equiv 1$, $A_n^2=n$, by SERFLING [8].

§ 3. Strong convergence and complete convergence

A trivial consequence of the laws of the iterated logarithm is the strong law of large numbers, i.e., under conditions (2.1) and (2.2a) it follows that

$$P\left\{\frac{S_n}{A_n^2} \to 0\right\} = 1. \tag{3.1}$$

It is of interest to obtain information on the rate of convergence in (3.1). Besides, we will give a condition on the sequence $\{c_n\}$ of numbers that

$$\sum_{n=1}^{\infty} P\left\{\frac{|S_n|}{c_n} \geqq \varepsilon\right\}$$

converge for every $\varepsilon>0$, which is referred to as $\{S_n/c_n\}$ *converges completely to zero* in the sense of Hsu and Robbins [3].

Theorem 6. *Suppose that there exist a positive number K and a sequence $\{a_i\}$ of numbers such that* (2.1) *holds. Furthermore, suppose that with some $\beta>0$ we have*

$$A_n \geqq C_4 n^\beta \quad (n \geqq n_0) \quad \textit{and} \quad a_n = o(A_n) \quad (n \to \infty). \tag{3.2}$$

Then, for each $\varepsilon>0$, we have

$$\sum_n \varrho^{A_n^2} P\left\{\sup_{k\geqq n} \frac{|S_k|}{A_k^2} \geqq \varepsilon\right\} < \infty \tag{3.3}$$

for any positive $\varrho<\exp(\varepsilon^2/2K)$; in particular,

$$\sum_n A_n^\alpha P\left\{\sup_{k\geqq n} \frac{|S_k|}{A_k^2} \geqq \varepsilon\right\} < \infty$$

for any $\alpha>0$.

Proof. We use the following elementary inequalities:

(i) If $0<u<1$, $\delta>1$, and k is a positive integer, then

$$u^{\delta^k} + u^{\delta^{k+1}} + u^{\delta^{k+2}} + \ldots \leqq u^{\delta^k}(1-u^{\delta^k(\delta-1)})^{-1}. \tag{3.4}$$

Indeed, if we substitute u^{δ^k} by v then (3.4) becomes

$$v + v^\delta + v^{\delta^2} + \ldots \leqq v(1-v^{\delta-1})^{-1},$$

where $0<v<1$. Now, if $\delta=1+\eta$ with an $\eta>0$, then

$$v + v^\delta + v^{\delta^2} + \ldots \leqq v + v^{1+\eta} + v^{1+2\eta} + \ldots = v(1-v^\eta)^{-1},$$

which makes (3.4) evident.

(ii) If $0<w<1$ and $\beta>0$ then the series

$$w+w^{2^\beta}+w^{3^\beta}+\dots$$

is convergent. This is clear by Bernoulli's inequality, according to which $n^\beta \geqq \beta(n-1)$.

After these preliminaries, let us fix $\varepsilon_1<\varepsilon$ so that $\varrho<\exp(\varepsilon_1^2/2K)$ and fix $\delta>1$ in such a way that

$$\varrho < \exp\left(\frac{\varepsilon_1^2}{2K\delta^2}\right) \quad \text{and} \quad \varepsilon_1 \leqq \frac{\varepsilon-\varepsilon_1}{[12(\delta^2-1)]^{1/2}}. \tag{3.5}$$

Then define a strictly increasing sequence $\{n_k\}$ of integers by (2.12) as we did in the proof of Theorem 4.

By (ii) and (3.5) it is enough to prove that

$$I_n = P\left\{\sup_{k\geqq n} \frac{|S_k|}{A_k^2} \geqq \varepsilon\right\} \leqq C_5 \exp\left(-\frac{\varepsilon_1^2}{2K\delta^2} A_n^2\right) \tag{3.6}$$

for all n large enough. Towards this end, let $n_{k_0}<n\leqq n_{k_0+1}$. We obviously have

$$I_n \leqq \sum_{k=k_0}^{\infty} P\left\{\max_{n_k<l\leqq n_{k+1}} \frac{|S_l|}{A_l^2} \geqq \varepsilon\right\} \leqq \sum_{k=k_0}^{\infty} P\left\{\frac{|S_{n_k}|}{A_{n_k}^2} \geqq \varepsilon_1\right\} +$$

$$+\sum_{k=k_0}^{\infty} P\left\{\max_{n_k<l\leqq n_{k+1}} \frac{|S_l-S_{n_k}|}{A_{n_k}^2} \geqq \varepsilon-\varepsilon_1\right\} = J_1+J_2.$$

Applying (2.6) with $y=\varepsilon_1 A_{n_k}^2$ gives

$$J_1 \leqq C\sum_{k=k_0}^{\infty} \exp\left(-\frac{\varepsilon_1^2}{2K} A_{n_k}^2\right) \leqq C\sum_{k=k_0}^{\infty} \exp\left(-\frac{\varepsilon_1^2\delta^k}{2K}\right),$$

while the application of (2.7) with $y=(\varepsilon-\varepsilon_1)A_{n_k}^2$ and (3.5) leads us to

$$J_2 \leqq 8C\sum_{k=k_0}^{\infty} \exp\left(-\frac{(\varepsilon-\varepsilon_1)^2 A_{n_k}^4}{24K(A_{n_{k+1}}^2-A_{n_k}^2)}\right) \leqq$$

$$\leqq 8C\sum_{k=k_0}^{\infty} \exp\left(-\frac{(\varepsilon-\varepsilon_1)^2\delta^k}{24K(\delta^2-1)}\right) \leqq 8C\sum_{k=k_0}^{\infty} \exp\left(-\frac{\varepsilon_1^2\delta^k}{2K}\right),$$

where we used that by (2.12)

$$\frac{A_{n_k}^4}{A_{n_{k+1}}^2-A_{n_k}^2} \geqq \frac{\delta^{2k}}{\delta^{k+2}-\delta^k} = \frac{\delta^k}{\delta^2-1}.$$

To sum up,

$$I_n \leqq J_1+J_2 \leqq 9C\sum_{k=k_0}^{\infty} \exp\left(-\frac{\varepsilon_1^2\delta^k}{2K}\right).$$

Now making use of (3.4) with $v=\exp(-\varepsilon_1^2/2K)$ and of (2.12), we get that

$$(3.7)\qquad \begin{aligned} J_n &\leqq 9C\exp\left(-\frac{\varepsilon_1^2}{2K}\delta^{k_0}\right)\left(1-\exp\left[-\frac{\varepsilon_1^2}{2K}\delta^{k_0}(\delta-1)\right]\right)^{-1} \leqq \\ &\leqq 18C\exp\left(-\frac{\varepsilon_1^2}{2K\delta^2}A^2_{n_{k_0+1}}\right) \leqq 18C\exp\left(-\frac{\varepsilon_1^2}{2K\delta^2}A_n^2\right), \end{aligned}$$

provided

$$\exp\left[-\frac{\varepsilon_1^2}{2K}\delta^{k_0}(\delta-1)\right] \leqq \frac{1}{2},$$

which is the case if n (and a fortiori k_0) is large enough.

Observe that (3.6) and (3.7) coincide if C_5 is taken to $18C$. This completes the proof of Theorem 6.

Finally, we consider the question of norming S_n in such a way that S_n/c_n converge completely to zero. The following theorem may be derived.

Theorem 7. *Suppose that there exist a positive number K and a sequence $\{a_i\}$ of numbers such that* (2.1) *holds. Furthermore, suppose that with some $\beta>0$ we have* (3.2). *Then $M_n/(A_n^2\log A_n)^{1/2}g(n)$, and hence also $S_n/(A_n^2\log A_n)^{1/2}g(n)$, converges completely to* 0 *if $g(n)\to\infty$ as $n\to\infty$.*

Proof. Let $\varepsilon>0$ be given. Then we obtain immediately by (2.7) that

$$\Sigma = \sum_n P\left\{\frac{M_n}{(A_n^2\log A_n)^{1/2}g(n)} \geqq \varepsilon\right\} \leqq 8C\sum_n \exp\left(-\frac{\varepsilon^2 g^2(n)\log A_n}{24K}\right) = 8C\sum_n A_n^{-v_n},$$

where $v_n=\varepsilon^2g^2(n)/24K$. Taking into account (3.2), it follows that

$$\Sigma \leqq 8C\sum_n n^{-\beta v_n} < \infty,$$

since βv_n with $g(n)$ tends to ∞ as $n\to\infty$. Here we suppose that $C_4\geqq 1$, but this does not bother generality. The proof of Theorem 7 is ready.

Condition (3.2) stipulated on the growth of A_n, plays a crucial role in the proofs of Theorems 6 and 7. Namely, (3.2) ensures the convergence of the series $\sum q^{A_n}$ for $0<q<1$ (in the proof of Theorem 6) and that of the series $\sum A_n^{-g(n)}$ for $g(n)\to\infty$ (in the proof of Theorem 7), which fail if, for example, $A_n=\log n$, $q=1/2$, and $g(n)=\log\log n$. Of course, it might be some relaxation of (3.2) using another technique, but we are unable to do so.

Confining attention to a uniformly bounded sequence of weakly multiplicative rv, we get the following

Corollary 4. *Let* $\{\varphi_i\}$ *be a sequence of rv satisfying* (2.21) *and* (2.22). *Let* $\{a_i\}$ *be a sequence of numbers with* (3.2). *Then, for each* $\varepsilon>0$, *we have*

$$\sum_n \varrho^{A_n^2} P\left\{\sup_{k\geqq n} \frac{1}{A_k^2}\left|\sum_{i=1}^k a_i\varphi_i\right| \geqq \varepsilon\right\} < \infty$$

for any $\varrho<\exp[\varepsilon^2/2(B^2+W^2)]$.

Corollary 5. *Let* $\{\varphi_i\}$ *be a sequence of rv satisfying* (2.21) *and* (2.22). *Under conditions* (3.2) *we have*

$$\sum_n P\left\{\frac{1}{(A_n^2\log A_n)^{1/2}g(n)} \max_{1\leqq k\leqq n}\left|\sum_{i=1}^k a_i\varphi_i\right| \geqq \varepsilon\right\} < \infty,$$

provided $g(n)\to\infty$ *as* $n\to\infty$.

We note that Theorem 6 in the special case $a_i\equiv 1$, $A_n^2=n$, was proved by SERFLING [10, Theorem 5.2]. Furthermore, Corollaries 4 and 5 were proved also by SERFLING [8] for sequences of uniformly bounded multiplicative rv and for $a_i\equiv 1$. The proofs given above essentially differ from those of Serfling, since in the case of general sequences $\{a_i\}$ (satisfying merely (3.2)) not only (2.6) but also (2.7) are employed.

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