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On thin operators relative to an ideal in a von Neumann algebra

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§ 1. Introduction

Let A be a von Neumann algebra, let Z be the center of A, and let K be a proper closed ideal of A with the property that if $T \in A$ and $TK = \{0\}$, then $T=0$. The set of thin operators of A relative to K, denoted $\mathfrak{I}_K$, is the set of operators of the form $X+T$ where $X \in Z$ and $T \in K$. In the case where $A = B(\mathfrak{H})$, the algebra of all bounded linear operators on a Hilbert space $\mathfrak{H}$, and $K = K(\mathfrak{H})$, the closed ideal of compact operators in $B(\mathfrak{H})$, this definition is due to R. DOUGLAS and C. PEARCY [6]. Let θ_K be the collection of all projections in K. If $P, Q \in \theta_K$, then $P \vee Q \in \theta_K$. This follows from [11, Lemma 2.1] where the proof is given for the more general case when A is an AW^*-algebra. Thus θ_K is upward directed in the usual ordering of projections ($P \leqq Q$ means $PQ = QP = P$). In [6], DOUGLAS and PEARCY characterized the thin operators in $B(\mathfrak{H})$ relative to $K(\mathfrak{H})$ as the set of all operators T that satisfy

$$\lim_{P \in \theta_K} \|PTP - TP\| = 0$$

[6, Theorem 2]. Also in [6], they related the η function of A. BROWN and C. PEARCY [4], [10], to

$$\lim_{P \in \theta_K} \sup \|PTP - TP\|.$$

They asked if there is a suitable extension of these results to the case where A is a general von Neumann algebra.

In a series of papers [7], [8] C. OLSEN proved the Douglas—Pearcy characterization of the thin operators in the general case. Also, she conjectured [8, p. 572]. that the distance from $T \in A$ to $\mathfrak{I}_K$ is given by

$$\lim_{P \in \theta_K} \sup \|PTP - TP\|.$$

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That this conjecture holds when $A=B(\mathfrak{H})$ and $K=K(\mathfrak{H})$ was proved by C. APOSTOL, C. FOIAŞ, and L. ZSIDÓ in [1].

In [2], for A a von Neumann algebra or a C^*-factor, C. APOSTOL and L. ZSIDÓ made a systematic study of the relationship between the distance of an element $T\in A$ from $\mathfrak{J}_K$, the η function evaluated at T, and the norm of the inner derivation induced on A by T.

In this paper we make three contributions to this circle of ideas. First, in § 2 we give a new proof that when A is a von Neumann algebra, then $T\in A$ is in $\mathfrak{J}_K$ if and only if

$$\lim_{P\in\theta_K} \|TP-PT\| = 0.$$

We note in this connection that C. OLSEN proves [8, Theorem 2] that it is always the case that

$$\lim_{P\in\theta_K} \sup \|PTP-TP\| = \lim_{P\in\theta_K} \sup \|TP = PT\|.$$

Our proof depends only on elementary arguments, and is considerably shorter than the proof by OLSEN in [7], [8]. Second, in § 3 we introduce a nonspatial form of the η function of BROWN and PEARCY [4], [10]. The generalized function η is defined on A using pure states of A, and is completely independent of any particular representation of A as a von Neumann algebra of operators on a Hilbert space. We prove some of the elementary properties of η in § 3. Then in § 4 we prove that $\eta(T)$ measures the distance from T to $\mathfrak{J}_K$. This is a generalization of [1, Lemma 1.1]. Third, in § 4 we prove the conjecture of C. Olsen that the distance from T to $\mathfrak{J}_K$ is given by

$$\lim_{P\in\theta_K} \sup \|TP-PT\| = \lim_{P\in\theta_K} \sup \|PTP-TP\|.$$

This result provides another proof of the Douglas—Pearcy—Olsen characterization of $\mathfrak{J}_K$.

At this point we introduce some notation. Throughout this paper A, Z, K, θ_K, and $\mathfrak{J}_K$ will be as stated at the beginning of this §. The identity operator in A is denoted by I. If B is a subalgebra of A and P is a projection in A, then $B_P=PBP$. Also, if $T\in A$, then $T_P=PTP$. The distance of $T\in A$ from a subspace $B\subset A$ is denoted $d(T, B)$, i.e.,

$$d(T, B) = \inf\{\|T+S\| : S\in B\}.$$

The set of pure states of A is denoted P_A. If $\alpha\in P_A$, then let Φ_α be the irreducible representation determined by α, and let $\mathfrak{H}_\alpha$ be the corresponding representation space. The inner product of vectors $\xi, \tau\in\mathfrak{H}_\alpha$ is denoted by $\langle\xi, \tau\rangle$.

If $P\in\theta_K$, then let

$$\Delta(P) = \{\alpha\in P_A : \alpha(K) \neq \{0\} \quad\text{and}\quad \alpha(P) = 0\}.$$

The collection $\Delta(P)$ plays an important role in later sections. Now we verify that $\Delta(P)$ is nonempty. For assume that $P\in\theta_K$. If $AP=K$, then $K(I-P)=\{0\}$. This implies that $P=I$, a contradiction. Thus, $AP\subset K$ and $AP\neq K$. By [5, Théorème 2.9.5] there exists a maximal left ideal M of A such that $AP\subset M$ and $K\not\subset M$. By this same result it follows that there exists $\alpha\in P_A$ such that $\alpha(AP)=\{0\}$ and $\alpha(K)\neq\{0\}$. Therefore $\alpha\in\Delta(P)$.

§ 2. The characterization of the thin operators

In this § we give a new proof of the Douglas—Pearcy—Olsen characterization of $\mathfrak{J}_K$ [6], [7], [8]. The main tool in the proof is a result of the present author [3, Lemma 6.1]. Before proving the characterization, we state this result.

2.1. Assume $\alpha\in P_A$ and $T_k\in A$, $1\leqq k\leqq m$. Then there exists a sequence of nonzero projections $\{E_n\}\subset A$ such that for $1\leqq k\leqq m$,

$$\lim_{n\to\infty}\|E_nT_kE_n-\alpha(T_k)E_n\|=0.$$

This result is established in [3] using completely elementary arguments.

Theorem 2.2. *$T\in\mathfrak{J}_K$ if and only if* $\lim_{P\in\theta_k}\|TP-PT\|=0$.

Proof. If $T\in\mathfrak{J}_K$, then it is straigthforward to prove

(1) $$\lim_{P\in\theta_K}\|TP-PT\|=0;$$

see the proof of [7, Proposition 2.1]. We prove the converse. Assume that (1) holds. Let $\varepsilon>0$ be arbitrary. Choose $Q\in\theta_A$ such that $P\in\theta_A$, $P\geqq Q$ implies that $\|TP-PT\|<\varepsilon$. Assume $R\in\theta_K$ and $R\leqq(I-Q)$. Then $R+Q\in\theta_A$ and $R+Q\geqq Q$. Thus, by the choice of Q, we have $\|T(R+Q)-(R+Q)T\|<\varepsilon$ and $\|TQ-QT\|<\varepsilon$. Therefore, $\|TR-RT\|<2\varepsilon$. This proves

(2) $$\text{if}\quad R\in\theta_K\quad\text{and}\quad R\leqq I-Q,\quad\text{then}\quad\|TR-RT\|<2\varepsilon.$$

Let α be any pure state of A such that $\alpha(K)=\{0\}$. Then α restricts to a pure state of A_{I-Q}. Let S be any operator in A. Consider the elements of A_{I-Q}, $T_1=T_{I-Q}$, $T_2=S_{I-Q}$, and $T_3=(TS)_{I-Q}$. Applying (2.1) to the operators $T_k\in A_{I-Q}$, $1\leqq k\leqq 3$, we have that there exists a sequence of nonzero projections $\{E_n\}$ in A_{I-Q} such that for $k=1, 2, 3$

$$\|E_nT_kE_n-\alpha(T_k)E_n\|\to 0\quad\text{as}\quad n\to\infty.$$

Note that since $\alpha(Q)=0$, we have $\alpha(R_{I-Q})=\alpha(R)$ for all $R\in A$. Therefore,

(3) $$\|E_nTE_n-\alpha(T)E_n\|\to 0,\quad \|E_nSE_n-\alpha(S)E_n\|\to 0,\quad \|E_nTSE_n-\alpha(TS)E_n\|\to 0.$$

Now E_nKE_n is a nonzero closed ideal in the von Neumann algebra E_nAE_n. Therefore for each n we can choose a nonzero projection $F_n \in E_nKE_n \subset K$. Thus $F_n \leqq E_n \leqq I-Q$ for each $n \geqq 1$. It follows immediately from (2) that

(4) $$\|TF_n - F_nT\| < 2\varepsilon \qquad (n \geqq 1).$$

Since $F_n \leqq E_n$ for all n, we have by (3) that

(5) $$\|F_nTF_n - \alpha(T)F_n\| \to 0, \quad \|F_nSF_n - \alpha(S)F_n\| \to 0, \quad \|F_nTSF_n - \alpha(TS)F_n\| \to 0.$$

Now,

$$|\alpha(T)\alpha(S) - \alpha(TS)| = \|\alpha(T)\alpha(S)F_n - \alpha(TS)F_n\| \leqq$$
$$\leqq \|F_nTSF_n - \alpha(TS)F_n\| + \|F_nTSF_n - F_nTF_nSF_n\| + \|F_nTF_nSF_n - \alpha(T)\alpha(S)F_n\|.$$

The first and third terms of the sum on the right hand side of this inequality approach zero by (5). Also,

$$\|F_nTSF_n - F_nTF_nSF_n\| = \|F_nT(I-F_n)SF_n\| \leqq \|F_nT - TF_n\| \, \|S\| \leqq 2\varepsilon\|S\|$$

for all $n \geqq 1$, by (4). Therefore, $|\alpha(T)\alpha(S) - \alpha(TS)| < 2\varepsilon\|S\|$, and since $\varepsilon > 0$ is arbitrary,

$$\alpha(TS) = \alpha(T)\alpha(S).$$

A similar proof shows that for all $S \in A$,

$$\alpha(ST) = \alpha(S)\alpha(T) = \alpha(TS).$$

Thus $\alpha(ST - TS) = 0$ for all $S \in A$ and all $\alpha \in P_A$ with $\alpha(K) = \{0\}$. Therefore T commutes with A modulo K, i.e. the natural quotient map of A onto A/K maps T into the center of A/K. Then by [5, Exercise 7, p. 259], $T \in \mathfrak{J}_K$.

§ 3. The nonspatial from of the η function

In [4], A. Brown and C. Pearcy define a function η on the von Neumann algebra $A = B(\mathfrak{H})$ relative to the ideal K of compact operators by the formula

(1) $$\eta(T) = \inf_{P \in \theta_K} (\sup \{\|T\xi - (T\xi, \xi)\xi\| : \xi \in \mathfrak{H}, \ \|\xi\| = 1, \ P\xi = 0\})$$

If $\xi \in \mathfrak{H}$, $\|\xi\| = 1$, then let ω_ξ be the pure state of $B(H)$ given by $\omega_\xi(T) = (T\xi, \xi)$. Observe that

$$\|T\xi - (T\xi, \xi)\xi\|^2 = \omega_\xi(T^*T) - |\omega_\xi(T)|^2.$$

In this case, $\{\omega_\xi : \xi \in H, \|\xi\| = 1\}$ is exactly the set of pure states α of A with the property that $\alpha(K) \neq \{0\}$. If $\alpha \in P_A$ and $P \in \theta_K$, then we use the notations

(2) $$\gamma(\alpha, T) = (\alpha(T^*T) - |\alpha(T)|^2)^{1/2} \quad (T \in A), \quad \Delta(P) = \{\alpha \in P_A : \alpha(K) \neq \{0\}, \alpha(P) = 0\}.$$

Recall from the Introduction that $\Delta(P)$ is nonempty. With the notation above the formula in (1) takes the form

(3) $$\eta(T) = \inf_{P\in\theta_K} (\sup\{\gamma(\alpha, T) : \alpha\in\Delta(P)\}).$$

Now θ_K is an upward directed set. For a fixed $T\in A$, the net

$$P \to \sup\{\gamma(\alpha, T) : \alpha\in\Delta(P)\}$$

is decreasing on θ_K. Thus,

$$\eta(T) = \lim_{P\in\theta_K} (\sup\{\gamma(\alpha, T) : \alpha\in\Delta(P)\}).$$

In general, if A is a von Neumann algebra and K is a closed ideal of A, then the definitions in (2) and (3) make sense. In particular, (3) is a generalized nonspatial expression of the useful η function of Brown and Pearcy. At times, in order to indicate the dependence of the function η on the ideal K, we write η_K in place of η. In this § we derive the elementary properties of the function η, while in the next §, we show that $\eta_K(T)$ measures the distance of an operator $T\in A$ from the thin operators relative to K.

Since for any $\alpha\in P_A$ we have $\gamma(\alpha, T)^2\leqq\alpha(T^*T)\leqq\|T\|^2$, it follows that

(3.1) $$\eta(T) \leqq \|T\| \quad (T\in A).$$

Next we show that

(3.2) $$T \to \eta(T) \quad \text{is a seminorm on} \quad A.$$

That $\eta(\lambda T)=|\lambda|\eta(T)$, $T\in A$, λ a scalar, is obvious. Since α is a positive functional on A, we have

(4) $$\alpha((C+B)^*(C+B))^{1/2} \leqq \alpha(C^*C)^{1/2}+\alpha(B^*B)^{1/2},$$

for all $C, B\in A$. Also, note that

$$\gamma(\alpha, T) = \alpha((T^* - \overline{\alpha(T)}I)(T-\alpha(T)I))^{1/2}.$$

Thus, setting $C=T-\alpha(T)I$ and $B=S-\alpha(S)I$ in (4), we have $\gamma(\alpha, T+S)\leqq\gamma(\alpha, T)+$ $+\gamma(\alpha, S)$. Therefore,

$$\sup_{\alpha\in\Delta(P)} \gamma(\alpha, T+S) \leqq (\sup_{\alpha\in\Delta(P)} \gamma(\alpha, T)+ \sup_{\alpha\in\Delta(P)} \gamma(\alpha, S)).$$

Taking limits over $P\in\theta_K$ we have $\eta(T+S)\leqq\eta(T)+\eta(S)$.

(3.3) If $T\in A$ and $S\in K$, then $\eta(T+S) = \eta(T)$.

To prove (3.3) first observe that $\eta(P)=0$ whenever $P\in\theta_K$. Since η is a seminorm, it follows that if L is any finite linear combination of projections in θ_K, then $\eta(L)=0$.

Now assume that $S \in K$. Let $\varepsilon > 0$ be arbitrary. Choose L, a finite linear combination of projections in θ_K such that $\|S-L\| < \varepsilon$. Then

$$\eta(S) = |\eta(S)-\eta(L)| \leqq \eta(S-L) \leqq \|S-L\| < \varepsilon.$$

Thus, $\eta(S)=0$. Then $\eta(T)-\eta(S) \leqq \eta(T+S) \leqq \eta(T)+\eta(S)=\eta(T)$.

$$\eta(T+X) = \eta(T) \quad (T \in A,\ X \in Z). \tag{3.4}$$

To prove (3.4), assume that $\alpha \in P_A$, $T \in A$, and $X \in Z$. Form the irreducible representation $(\Phi_\alpha, \mathfrak{H}_\alpha)$, and choose $\xi \in \mathfrak{H}_\alpha$ $\|\xi\|=1$, such that

$$\alpha(S) = \langle \Phi_\alpha(S)\xi, \xi \rangle \quad (S \in A).$$

Then $\Phi_\alpha(X)$ is the scalar $\alpha(X)$ times the identity operator on H_α. Therefore

$$\alpha(TX) = \langle \Phi_\alpha(T)\Phi_\alpha(X)\xi, \xi \rangle = \alpha(X)\langle \Phi_\alpha(T)\xi, \xi \rangle = \alpha(X)\alpha(T).$$

Thus,

$$\gamma(\alpha, T+X)^2 = \alpha\big((T^*+X^*)(T+X)\big) - |\alpha(T+X)|^2 =$$

$$= \alpha(T^*T) + \overline{\alpha(T)}\alpha(X) + \alpha(T)\overline{\alpha(X)} + |\alpha(X)|^2 - \big(\overline{\alpha(T)+\alpha(X)}\big)\big(\alpha(T)+\alpha(X)\big) =$$

$$= \alpha(T^*T) - |\alpha(T)|^2 = \gamma(\alpha, T)^2.$$

Therefore $\eta(T+X)=\eta(T)$.

§ 4. The distance from the thin operators

Throughout this §, A is a von Neumann algebra and K is a closed ideal of A with the property that if $T \in A$ and $TK=\{0\}$, then $T=0$. When A is represented spatially, this property of K is equivalent to the property that K is weak operator dense in A. In this § we prove the following theorem.

Theorem 4.1. *Let A and K be as above. Then*

$$\eta_K(T) = \limsup_{P \in \theta_K} \|TP-PT\| = d(T, \mathfrak{I}_K).$$

The first equality in this statement generalizes a result of R. Douglas and C. Pearcy in [6], and the second equality is a conjecture of C. Olsen [8. p. 572].

We prove Theorem 4.1 in several steps. The first of these, the next proposition, is a direct generalization of [6, Theorem 1].

Proposition 4.2. $\eta(T) = \limsup_{P \in \theta_K} \|PT(I-P)\|$.

Proof. Let μ equal the lim sup on the right hand side of the equality above. Fix $P \in \theta_K$. Then

$$(I-P)T^*PT(I-P) \in K_{I-P}.$$

There exists $\beta \in P_A$ such that $\beta(I-P)=1$, and

(1) $$\beta(T^*PT) = \beta((I-P)T^*PT(I-P)) = \|PT(I-P)\|^2.$$

Note that if $PT(I-P)\neq 0$, then $\beta(K)\neq \{0\}$. Also,

(2) $$\beta(T^*(I-P)T)-|\beta(T)|^2 = \beta(T^*(I-P)T)-|\beta((I-P)T)|^2 \geqq 0.$$

Adding (1) and (2) we have

$$\gamma(\beta, T)^2 = \beta(T^*T)-|\beta(T)|^2 \geqq \|PT(I-P)\|^2.$$

Therefore,

$$\sup\{\gamma(\alpha, T) : \alpha\in \Delta(P)\} \geqq \|PT(I-P)\|.$$

Taking the lim sup over $P\in\theta_K$ on both sides of this inequality, it follows that $\eta(T)\geqq\mu$.

Conversely, let $\delta>0$ be arbitrary. Fix $P\in\theta_K$. We proceed to find $Q\in\theta_K$ such that $Q\geqq P$ and

$$\|QT(I-Q)\| \geqq \eta(T)-\delta.$$

Then this suffices to prove the inequality $\mu\geqq\eta(T)$.

Assume $\alpha\in\Delta(P)$ is such that

$$\gamma(\alpha, T_{I-P}) > \eta(T_{I-P})-\delta.$$

Denote by α_0 the restriction of α to A_{I-P}. Then α_0 is a pure state of A_{I-P}. Form the irreducible representation $(\Phi_{\alpha_0}, \mathfrak{H}_{\alpha_0})$ of A_{I-P}. Choose $z\in\mathfrak{H}_{\alpha_0}$, $\|z\|=1$, such that

$$\alpha_0(S) = \langle\Phi_{\alpha_0}(S)z, z\rangle \quad (S\in A_{I-P}).$$

Let $w=\Phi_{\alpha_0}(T_{I-P})z-\alpha_0(T_{I-P})z$. Then

$$\|w\|^2 = \alpha_0((I-P)T^*(I-P)T(I-P))-|\alpha_0(T_{I-P})|^2 = \gamma(\alpha, T_{I-P})^2.$$

Observe that $w\perp z$ in $\mathfrak{H}_{\alpha_0}$. Then by Kadison's Transitivity Theorem [5, Théorème 2.8.3] there exists a selfadjoint operator $S\in K_{I-P}$ such that $\Phi_{\alpha_0}(S)z=0$ and $\Phi_{\alpha_0}(S)w=w$. Then $\Phi_{\alpha_0}(S^2)z=0$ and $\Phi_{\alpha_0}(S^2)w=w$. Using the spectral resolution of the identity for S^2, it is not difficult to show that there exists a sequence of projections $\{R_n\}\subset K_{I-P}$ such that

$$\Phi_{\alpha_0}(R_n)z = 0 \quad\text{and}\quad \Phi_{\alpha_0}(R_n)w \to w.$$

Then

$$\alpha_0((I-P)T^*R_nT(I-P)) = \langle\Phi_{\alpha_0}((I-P)T^*R_nT(I-P))z, z\rangle$$
$$= \|\Phi_{\alpha_0}(R_n)\Phi_{\alpha_0}(T_{I-P})z\|^2 = \left\|\Phi_{\alpha_0}(R_n)(\Phi_{\alpha_0}(T_{I-P})z-\alpha_0(T_{I-P})z)\right\|^2$$
$$= \|\Phi_{\alpha_0}(R_n)w\|^2 \to \|w\|^2.$$

Therefore

$$\alpha_0((I-P)T^*R_nT(I-P)) \to \|w\|^2, \quad \|w\|^2 = \gamma(\alpha, T_{I-P})^2 > (\eta(T_{I-P})-\delta)^2.$$

Set $R=R_m$ for some m so large that

$$\alpha_0((I-P)T^*R_mT(I-P)) > (\eta(T_{I-P})-\delta)^2.$$

Now we have

$$\alpha(T^*RT) = \alpha((T^*RT)_{I-P}) = \alpha_0((I-P)T^*RT(I-P)).$$

Also, by (3.3), $\eta(T_{I-P})=\eta(T)$. Thus $PR=RP=0$, and $\alpha(T^*RT)>(\eta(T)-\delta)^2$. Let $Q=P+R$. Then $Q\geqq P$ and $\alpha(Q)=0$. Finally

$$\|QT(I-Q)\|^2 \geqq \alpha((I-Q)T^*QT(I-Q)) = \alpha(T^*QT) \geqq \alpha(T^*RT) > (\eta(T)-\delta)^2.$$

This completes the proof of the proposition.

If $T\in A$, $X\in Z$, and $J\in K$, then by (3.3) and (3.4) we have $\eta(T)=\eta(T+X+J)$. It follows using (3.1) that $\eta(T)\leqq\|T+X+J\|$. Therefore $\eta(T)\leqq d(T, \mathfrak{I}_K)$.

We state this result as a lemma.

Lemma 4.3. $\eta_K(T)\leqq d(T, \mathfrak{I}_K)$.

Our aim now is to prove the reverse of the inequality appearing in Lemma 4.3. First we need a technical result. Let Γ be the set of all primitive ideals B of A such that $K\not\subset B$. For $B\in\Gamma$, let π_B be the natural quotient map of A onto A/B. We show that

$$\|S\| = \sup_{B\in\Gamma} \|\pi_B(S)\| \quad (S\in A). \tag{4.4}$$

Let Φ be the map from A into the C^*-direct product of the C^*-algebras A/B, $B\in\Gamma$, given by

$$\Phi(S) = (\pi_B(S))_{B\in\Gamma}.$$

Since $\bigcap_{B\in\Gamma}(B\cap K)=\{0\}$, Φ is an isomorphism on K. If $S\in A$ and $S\neq 0$, then there exists $J\in K$ such that $SJ\neq 0$. Then $\Phi(SJ)\neq 0$, so $\Phi(S)\neq 0$. Thus Φ, is a *-isomorphism of A, and therefore, an isometry. This proves (4.4).

Lemma 4.5. $\eta_K(T)\geqq d(T, \mathfrak{I}_K)$.

Proof. Let Δ be the set of all $\alpha\in P_A$ such that $\alpha(K)\neq\{0\}$. Assume $T\in A$. We prove

$$\sup_{\alpha\in\Delta} \gamma(\alpha, T) \geqq d(T, Z). \tag{1}$$

Assume $\alpha\in\Delta$, and let $(\Phi_\alpha, \mathfrak{H}_\alpha)$ be the irreducible representation of A determined by α. If $\xi\in\mathfrak{H}_\alpha$, $\|\xi\|=1$, let

$$\omega_\xi(S) = \langle\Phi_\alpha(S)\xi, \xi\rangle \quad (S\in A).$$

By definition [9, p. 216], ω_ξ is representable by $(\Phi_\alpha, \mathfrak{H}_\alpha)$. Then by [9, Lemma (4.5.8)] the *-representation of A associated with ω_ξ is unitarily equivalent to $(\Phi_\alpha, \mathfrak{H}_\alpha)$. Thus, ω_ξ is a pure state of A [9, Theorem (4.6.4)]. Since $\Phi_\alpha(K)$ acts irreducibly on $\mathfrak{H}_\alpha$, we have $\omega_\xi\in\Delta$. Let D_T and $D_{\alpha,T}$ be the inner derivations determined by T on A, and by $\Phi_\alpha(T)$ on $B(\mathfrak{H}_\alpha)$, respectively. Observe that

$$\gamma(\omega_\xi, T) = \|\Phi_\alpha(T)\xi-\langle\Phi_\alpha(T)\xi, \xi\rangle\xi\|.$$

Then by [2, Corollary 1.3]

(2) $$\sup_{\xi \in H_\alpha, \|\xi\|=1} \gamma(\omega_\xi, T) = \frac{1}{2} \|D_{\alpha,T}\|.$$

Let $\varepsilon > 0$ be arbitrary. Choose $S \in A$, $\|S\| = 1$, such that

$$\|TS - ST\| \geqq \|D_T\| - \varepsilon.$$

Then by (2)

(3) $$\sup_{\xi \in \mathfrak{H}_\alpha, \|\xi\|=1} \gamma(\omega_\xi, T) \geqq \frac{1}{2} \|\Phi_\alpha(TS - ST)\|.$$

Let B_α be the primitive ideal that is the kernel of Φ_α, and let π_α be the natural quotient map of A onto A/B_α. If $R \in A$, then $\|\Phi_\alpha(R)\| = \|\pi_\alpha(R)\|$. Therefore by (4.4)

$$\|R\| = \sup_{\alpha \in \Delta} \|\pi_\alpha(R)\| = \sup_{\alpha \in \Delta} \|\Phi_\alpha(R)\|.$$

Applying this equality to (3), we have

$$\sup_{\alpha \in \Delta} \gamma(\alpha, T) \geqq \frac{1}{2} \sup_{\alpha \in \Delta} \|\Phi_\alpha(TS - ST)\| = \frac{1}{2} \|TS - ST\| \geqq \frac{1}{2} (\|D_T\| - \varepsilon).$$

This proves that

$$\sup_{\alpha \in \Delta} \gamma(\alpha, T) \geqq \frac{1}{2} \|D_T\|.$$

Then by [12, Corollary, p. 148]

$$\sup_{\alpha \in \Delta} \gamma(\alpha, T) \geqq d(T, Z).$$

This completes the proof of (1).

Now fix $P \in \theta_K$. The center of A_{I-P} is Z_{I-P}. Applying (1) to the algebra A_{I-P} and the element $(I-P)T(I-P)$, we have

$$\sup_{\alpha \in \Delta(P)} \gamma(\alpha, T) \geqq d((I-P)T(I-P), Z_{I-P}).$$

Also,

$$d((I-P)T(I-P), Z_{I-P}) = \inf_{X \in Z} \|(I-P)T(I-P) + (I-P)X(I-P)\|$$
$$\geqq d(T, \mathfrak{J}_K).$$

Therefore, $\eta_K(T) \geqq d(T, \mathfrak{J}_K)$.

By [8, Theorem 2]

$$\lim_{P \in \theta_K} \sup \|PT(I-P)\| = \lim_{P \in \theta_K} \sup \|TP - PT\|.$$

This equality in conjunction with Proposition 4.2, Lemma 4.3, and Lemma 4.5, proves Theorem 4.1.

Corollary 4.6. *Let A and K be as before. Then the following are equivalent for $T\in A$:*

$$\lim_{P\in\theta_K}\|TP-PT\|=0,\quad \eta_K(T)=0,\quad \text{and}\quad T\in\mathfrak{I}_K.$$

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