Acta Sci. Math., 38 (1976), 253—260

Lebesgue-type decomposition of positive operators

T. ANDO

1. Introduction

Our main concerns in this paper are bounded (linear) *positive*, i.e. non-negative definite, operators on a Hilbert space $\mathfrak{H}$. Given a positive operator A, we say a positive operator C to be *A-absolutely continuous* if there exists a sequence $\{C_n\}$ of positive operators such that $C_n \uparrow C$ and $C_n \leqq \alpha_n A$ for some $\alpha_n \geqq 0$ $(n=1, 2, \ldots)$. Here $C_n \uparrow C$ means that $C_1 \leqq C_2 \leqq C_3 \leqq \ldots$ and C_n converges strongly to C. A positive operator C is said to be *A-singular* if $0 \leqq D \leqq A$ and $0 \leqq D \leqq C$ imply $D=0$. These definitions are motivated by the corresponding notions in measure theory (cf. [3]). In accordance with a well-known theorem of measure theory (cf. [3] § 32), by an *A-Lebesgue decomposition* of a positive operator B we shall mean a decomposition $B=B_c+B_s$ into positive operators such that B_c and B_s are A-absolutely continuous and A-singular, respectively.

In a recent paper [1] ANDERSON and TRAPP proved that given a (closed) subspace $\mathfrak{G}$, each positive operator B is written uniquely as a sum of two positive operators $B=C+D$ such that $\mathrm{ran}(C^{1/2}) \subseteq \mathfrak{G}$ and $\mathrm{ran}(D^{1/2}) \cap \mathfrak{G} = \{0\}$. Here $C^{1/2}$ is the positive square-root of C, and "ran" stays for "range". If $\mathrm{ran}(A)=\mathfrak{G}$, that is, if A has closed range, then $\mathrm{ran}(C^{1/2}) \subseteq \mathfrak{G}$ implies $C \leqq \alpha A$ for some $\alpha \geqq 0$ while $\mathrm{ran}(D^{1/2}) \cap \mathfrak{G} = \{0\}$ is equivalent to the A-singularity of D (see § 3). The above cited result shows that A-Lebesgue decomposition is always guaranteed and is unique in case A has closed range.

Received April 2, 1976.

The work was done while the author stayed at Bolyai Institute of Mathematics, Szeged University. The research was supported jointly by the Hungarian Institute of Cultural Relations and the Japan Society for the Promotion of Science. The author wishes to express his gratitude to Professor B. Sz.-Nagy and the Bolyai Institute for their hospitality.

The purpose of this paper is to construct an A-Lebesgue decomposition for each positive operator and to find a condition for the uniqueness of A-Lebesgue decompositions.

2. Lebesgue decomposition

Let us recall a useful binary operation in the class $\mathscr{P}$ of all positive operators, which is defined and called parallel addition by ANDERSON and TRAPP [1]. The *parallel sum* $A:B$ of two positive operators A and B is determined by the formula:

$$((A:B)h, h) = \inf_{g \in \mathfrak{H}} \{(Ag, g)+(B(h-g), h-g)\}.$$

The expression on the right side defines really a positive operator. For, define a new scalar product on the direct sum $\mathfrak{H}\oplus\mathfrak{H}$ by

$$\langle g\oplus k, g'\oplus k'\rangle = (Ag, g')+(Bk, k').$$

Let $\mathfrak{K}$ be the associated Hilbert space and $\mathfrak{G}$ the closure of the manifold $\{g\oplus k : g+k=0\}$. The expression is equal to $\langle(I-P)(0\oplus h), 0\oplus h\rangle$ where P is the projection from $\mathfrak{K}$ onto $\mathfrak{G}$.

Obviously, $A, B \geqq A:B \geqq 0$, and $A_1 \geqq A_2$ implies $A_1:B \geqq A_2:B$. Now since $(nA):B$ increases along with n and is bounded by B from above, we can introduce an operation $[A]$ in the class $\mathscr{P}$ by the formula:

$$[A]B = \lim_{n\to\infty} (nA):B,$$

where lim means strong limit. Since $(nA):B\uparrow [A]B$ and $(nA):B\leqq nA$, by definition $[A]B$ is A-absolutely continuous and $[A]B\leqq B$. Remark that the operation $[A]$ is *monotone* in the sense that $B_1\leqq B_2$ implies $[A]B_1\leqq[A]B_2$. This operation is not additive.

The above definition is motivated by a consideration of ANDERSON and TRAPP ([1]; Theorem 12) as well as a proof of the Lebesgue decomposition theorem in measure theory (cf. [3]; § 32).

Lemma 1. *Let A and B be positive operators. Then B is A-absolutely continuous if and only if $[A]B=B$.*

Proof. As remarked above, $[A]B$ is always A-absolutely continuous. Suppose that B is A-absolutely continuous. Then by definition there exists a sequence $\{B_m\}$

such that $B_m \uparrow B$ and $B_m \leqq \alpha_m A$ for some $\alpha_m > 0$. The definition of parallel addition yields, with the convention $0/0=0$, that

$$\begin{aligned}((nA):B_m)h, h) &= \inf_{g\in\mathfrak{H}} \{(nAg, g)+(B_m(h-g), h-g)\} \\ &= (B_m h, h)+\inf_{g\in\mathfrak{H}} \{((nA+B_m)g, g)-2|(B_m g, h)|\} \\ &= (B_m h, h)+\inf_{g\in\mathfrak{H}} \inf_{\lambda>0} \{\lambda^2((nA+B_m)g, g)-2\lambda|(B_m g, h)|\} \\ &= (B_m h, h)-\sup_{g\in\mathfrak{H}} \frac{|(B_m g, h)|^2}{((nA+B_m)g, g)},\end{aligned}$$

hence

$$\begin{aligned}0 &\leqq (B_m h, h)-(((nA):B_m)h, h) \\ &\leqq \sup_{g\in\mathfrak{H}} \frac{(B_m g, g)(B_m h, h)}{(n\alpha_m^{-1}+1)(B_m g, g)} \leqq \frac{\alpha_m}{n+\alpha_m}(Bh, h).\end{aligned}$$

This implies

$$B_m = \lim_{n\to\infty} (nA):B_m \equiv [A]B_m.$$

Now since by the monotonity of the operation $[A]$

$$B \geqq [A]B \geqq [A]B_m = B_m,$$

taking the limit of B_m we have $B=[A]B$. This completes the proof.

Theorem 2. *Let A be a positive operator. Then for each positive operator B the decomposition*

$$B = [A]B+(B-[A]B)$$

is an A-Lebesgue decomposition with A-absolutely continuous $[A]B$ and A-singular $B-[A]B$. Moreover $[A]B$ is the maximum of all A-absolutely continuous positive operators C with $C\leqq B$.

Proof. The operator $[A]B$ is A-absolutely continuous and $[A]B\leqq B$. If a positive operator C is A-absolutely continuous and $C\leqq B$, the monotonity of $[A]$ and Lemma 1 imply that $C=[A]C\leqq[A]B$. Therefore $[A]B$ has the maximum property in question. It remains to show the A-singularity of $B-[A]B$. Suppose that $0\leqq D\leqq A$ and $0\leqq D\leqq B-[A]B$. Since D is obviously A-absolutely continuous, by definition so is the sum $[A]B+D$. On the other hand, the maximum property of $[A]B$ implies $[A]B+D\leqq[A]B$, hence $D=0$. Thus $B-[A]B$ is A-singular by definition. This completes the proof.

Corollary 3. *Let A and B be positive operators. Then B is A-singular if and only if $[A]B=0$.*

3. Characterization of absolute continuity

Some order relations between two positive operators can be expressed in terms of their range spaces. Here a basic tool is supplied by the following lemma due to DOUGLAS ([2] Theorem 2.1).

Lemma 4. *For bounded linear operators S and T the following conditions are mutually equivalent:*
(a) $\mathrm{ran}\,(S) \subseteq \mathrm{ran}\,(T)$,
(b) *There exists $\alpha \geqq 0$ such that $SS^* \leqq \alpha TT^*$,*
(c) *There exists a bounded linear operator R such that $S = TR$. Here R is uniquely determined under the additional requirement that R^* vanishes on the orthocomplement of* $\mathrm{ran}\,(T^*)$.

When applied to the square roots of positive operators A and B, Lemma 4 yields that $\mathrm{ran}\,(B^{1/2}) \subseteq \mathrm{ran}\,(A^{1/2})$ is equivalent to the existence of $\alpha \geqq 0$ such that $B \leqq \alpha A$, a condition stronger than the A-absolute continuity of B. Lemma 4 shows further that $\mathrm{ran}\,(A^{1/2}) \cap \mathrm{ran}\,(B^{1/2}) = \{0\}$ implies the A-singularity of B. Conversely, in view of the general formula

$$\mathrm{ran}\,(A^{1/2}) \cap \mathrm{ran}\,(B^{1/2}) = \mathrm{ran}\,\big((A:B)^{1/2}\big)$$

([1] Theorem 11) and the inequality $0 \leqq A:B \leqq A, B$, the A-singularity of B implies $\mathrm{ran}\,(A^{1/2}) \cap \mathrm{ran}\,(B^{1/2}) = \{0\}$. Our purpose in this section is to find a characterization of A-absolute continuity in this direction.

Theorem 5. *Let A and B be positive operators. Then B is A-absolutely continuous if and only if the linear manifold $\{h : B^{1/2}h \in \mathrm{ran}\,(A^{1/2})\}$ is dense in $\mathfrak{H}$.*

Proof. Suppose that the linear manifold $\mathfrak{D} \equiv \{h : B^{1/2}h \in \mathrm{ran}\,(A^{1/2})\}$ is dense in $\mathfrak{H}$. Since the orthocomplement of the kernel of $A^{1/2}$ coincides with $\mathrm{ran}\,(A^{1/2})^-$, the closure of $\mathrm{ran}\,(A^{1/2})$, the correspondence $h \mapsto g$ from $\mathfrak{D}$ to $\mathrm{ran}\,(A^{1/2})^-$, defined by $B^{1/2}h = A^{1/2}g$, determines a linear operator T with domain $\mathfrak{D}$. As easily follows from the boundedness of $A^{1/2}$ and $B^{1/2}$ ([2] Theorem 2.1), T is closed. Now since T is a densely defined closed operator, its adjoint T^* is a densely defined closed operator (cf. [4]; V, § 3.1). Since $A^{1/2}T \subseteq B^{1/2}$ by definition, the boundedness of $A^{1/2}$ and $B^{1/2}$ yields $T^*A^{1/2} = B^{1/2}$. Let $T^* = VS$ be the polar decomposition of T^* (cf. [4]; VI, § 2,7); S is an (unbounded) positive self-adjoint operator whose domain coincides with that of T^* and V is a partial isometry with initial space $\mathrm{ran}(S)^-$ and final space $\mathrm{ran}\,(T^*)^-$. Then $\mathrm{ran}\,(A^{1/2})$ is included in the domain of S, and for all $h \in \mathfrak{H}$

$$\|SA^{1/2}h\|^2 = (Bh, h).$$

Consider the spectral representation

$$S = \int_0^\infty \lambda \, dE(\lambda) \quad \text{and let} \quad S_n = \int_0^n \lambda \, dE(\lambda) \qquad (n = 1, 2, \ldots).$$

Then we can readily verify that $A^{1/2}S_n^2A^{1/2}\uparrow B$ and $A^{1/2}S_n^2A^{1/2}\leqq n^2A$, hence B is A-absolutely continuous.

Suppose conversely that B is A-absolutely continuous. Then by definition there exists a sequence $\{B_n\}$ such that $B_n\uparrow B$ and $B_n\leqq \alpha_n A$ for some $\alpha_n\geqq 0$. By Lemma 4 for each n there exists a bounded linear operator R_n such that $B_n^{1/2}=A^{1/2}R_n$ and R_n^* vanishes on the orthocomplement of $\operatorname{ran}(A^{1/2})$. Then $B_n\leqq B_{n+1}$ implies $R_nR_n^*\leqq R_{n+1}R_{n+1}^*$. Let $\mathfrak{D}$ denote the linear manifold of all g with $\sup_n \|R_n^*g\|<\infty$, and define a functional φ on $\mathfrak{D}$ by the formula

$$\varphi(g)\equiv\sup_n \|R_n^*g\|^2=\lim_{n\to\infty}\|R_n^*g\|^2.$$

The functional φ is closed in the sense that if $g_n\in\mathfrak{D}$, $\lim_{n\to\infty} g_n=h$ and if $\lim_{n,m\to\infty}\varphi(g_n-g_m)=0$, then $h\in\mathfrak{D}$ and $\lim_{n\to\infty}\varphi(h-g_n)=0$. Further, since, by definition of $\{B_n\}$, for all $h\in\mathfrak{D}$

$$\sup_n \|R_n^*A^{1/2}h\|^2=\sup_n \|B_n^{1/2}h\|^2=(Bh,h)<\infty$$

and since every R_n^* vanishes on the orthocomplement of $\operatorname{ran}(A^{1/2})$, the linear manifold $\mathfrak{D}$ includes the dense set $\operatorname{ran}(A^{1/2})+(\mathfrak{H}\ominus\operatorname{ran}(A^{1/2}))$. Thus φ is densely defined, closed and expressed as the limit of the bounded quadratic forms $\|R_n^*g\|^2$. Now in view of a theorem on quadratic forms ([4]; VI, § 2,6) there exists an (unbounded) positive self-adjoint operator S such that its domain coincides with $\mathfrak{D}$ and $\|Sg\|^2=\varphi(g)$. Then we have for all $h\in\mathfrak{H}$

$$\|SA^{1/2}h\|^2=(Bh,h)=\|B^{1/2}h\|^2,$$

hence there exists a partial isometry V with initial space $\operatorname{ran}(B^{1/2})^-$ such that $SA^{1/2}=$ $=VB^{1/2}$. This implies $A^{1/2}S\subseteqq B^{1/2}V^*$, and consequently

$$V^*(\mathfrak{D})\subseteqq\{h: B^{1/2}h\in\operatorname{ran}(A^{1/2})\}.$$

Since $\mathfrak{D}$ is dense in $\mathfrak{H}$ and V is a partial isometry with initial space $\operatorname{ran}(B^{1/2})^-$, we can conclude

$$\operatorname{ran}(B^{1/2})^-\subseteqq\{h: B^{1/2}h\in\operatorname{ran}(A^{1/2})\}^-.$$

Finally since $B^{1/2}$ vanishes on the orthocomplement of $\operatorname{ran}(B^{1/2})$, the subspace $\{h: B^{1/2}h\in\operatorname{ran}(A^{1/2})\}^-$ includes this orthocomplement, too, hence coincides with the whole space $\mathfrak{H}$. This completes the proof.

4. Uniqueness condition

Let A be a positive operator. Then A-absolute continuity is *additive* in the sense that the sum of two positive operators is A-absolutely continuous whenever both summands are so. A-singularity is not always additive while it is *hereditary* in the sense that A-singularity of the sum of two positive operators implies A-singularity of both summands. A-absolute continuity is not always hereditary. These discrepancies can cause non-uniqueness in A-Lebesgue decomposition.

Let us say a positive operator B to be *A-strongly continuous* if $B \leqq \alpha A$ for some $\alpha \geqq 0$, or equivalently, as is remarked in § 3, if $\operatorname{ran}(B^{1/2}) \subseteqq \operatorname{ran}(A^{1/2})$. Then *$A$-strong continuity* is additive as well as hereditary.

Theorem 6. *Let A be a positive operator. Then a positive operator B admits a unique A-Lebesgue decomposition if and only if $[A]B$ is A-strongly continuous, that is, $[A]B \leqq \alpha A$ for some $\alpha \geqq 0$.*

Proof. Suppose that $[A]B$ is A-strongly continuous and take an arbitrary A-Lebesgue decomposition $B \doteq C + D$ with A-absolutely continuous C and A-singular D. Theorem 2 implies $D \geqq [A]B - C \geqq 0$. The positive operator $[A]B - C$ is A-strongly continuous as well as A-singular so that it must be equal to 0. Therefore B admits a unique A-Lebesgue decomposition.

Suppose conversely that $[A]B$ is not A-strongly continuous. Then by Lemma 1, Lemma 4 and Theorem 5 the linear manifold $\mathfrak{D} \equiv \{h;\ ([A]B)^{1/2}h \in \operatorname{ran}(A^{1/2})\}$ is dense in $\mathfrak{H}$ but not closed. As in the proof of Theorem 5 there exists a closed operator with domain $\mathfrak{D}$, so that there exists a (bounded) positive operator S with $\operatorname{ran}(S) = \mathfrak{D}$ (cf. [2]; Theorem 1.1). We may assume $S^2 \leqq \frac{1}{2} I$. Since $\operatorname{ran}(S)$ is not closed and $[A]B \neq 0$ by assumption, there exists a separable (closed) subspace $\mathfrak{G}$ such that $SP = PS$, $([A]B) \cdot P = P \cdot ([A]B) \neq 0$ and $\operatorname{ran}(SP)$ is not closed, where P is the orthoprojection onto $\mathfrak{G}$. Then in view of a theorem of von Neumann ([2] Theorem 3.6) there exists a unitary operator U_0 on the separable Hilbert space $\mathfrak{G}$ such that

$$\operatorname{ran}(SP) \cap \operatorname{ran}(U_0 SP) = \{0\}.$$

Let us define a unitary operator U on $\mathfrak{H}$ by $U = U_0 P + (I - P)$. Then it follows from the properties of $\mathfrak{G}$ and U_0 that

$$\mathfrak{D} \cap U^*(\mathfrak{D}) \subseteqq \mathfrak{H} \ominus \mathfrak{G}.$$

Consider the positive operators defined by

$$D \equiv ([A]B)^{1/2} U^* S^2 U ([A]B)^{1/2} \quad \text{and} \quad C \equiv [A]B - D.$$

First we shall show that C is A-absolutely continuous. Since

$$[A]B \geqq C = ([A]B)^{1/2} U^* (I - S^2) U ([A]B)^{1/2} \geqq \frac{1}{2} [A]B,$$

by Lemma 4 (cf. [2]; Corollary 2.1.1) there exists a bounded invertible operator R such that $C^{1/2} R = ([A]B)^{1/2}$. Then we have

$$\{h : C^{1/2} h \in \operatorname{ran}(A^{1/2})\} = R(\mathfrak{D}).$$

Since $\mathfrak{D}$ is dense in $\mathfrak{H}$ and R is invertible, $R(\mathfrak{D})$ is dense in $\mathfrak{H}$ too, so that the above relation implies the A-absolute continuity of C by Theorem 5.

Let us prove that D is not A-absolutely continuous. Suppose the contrary. Then Theorem 5 implies that $\operatorname{ran}(D^{1/2})\cap\operatorname{ran}(A^{1/2})$ is dense in $\operatorname{ran}(D^{1/2})$. On the other hand, by Lemma 4 and definition of D we have

$$\operatorname{ran}(D^{1/2})\cap\operatorname{ran}(A^{1/2}) = \operatorname{ran}\big(([A]B)^{1/2}U^*S\big)\cap\operatorname{ran}(A^{1/2}).$$

Take an arbitrary h such that $([A]B)^{1/2}U^*Sh\in\operatorname{ran}(A^{1/2})$. This requirement is equivalent to $U^*Sh\in\mathfrak{D}$ by the definition of $\mathfrak{D}$. Since $\operatorname{ran}(S)=\mathfrak{D}$, it follows that

$$([A]B)^{1/2}U^*Sh\in([A]B)^{1/2}(\mathfrak{D}\cap U^*(\mathfrak{D}))\subseteqq([A]B)^{1/2}(\mathfrak{H}\ominus\mathfrak{G}).$$

Since $\mathfrak{H}\ominus\mathfrak{G}$ reduces $[A]B$, we can conclude

$$\operatorname{ran}(D^{1/2})\cap\operatorname{ran}(A^{1/2})\subseteqq\mathfrak{H}\ominus\mathfrak{G}.$$

Finally since P commutes with S, U and $[A]B$, the subspace $\mathfrak{G}$ reduces $D^{1/2}$ and $D^{1/2}(\mathfrak{G})\neq\{0\}$ according to $([A]B)P\neq 0$. Therefore the above inclusion relation leads to a contradiction that $\operatorname{ran}(D^{1/2})\cap\operatorname{ran}(A^{1/2})$ is not dense in $\operatorname{ran}(D^{1/2})$.

Now consider a decomposition $B=C_1+D_1$, where $C_1=C+[A]\{D+(B-[A]B)\}$ and $D_1=B-C_1$. This is an A-Lebesgue decomposition. In fact, obviously C_1 is positive A-absolutely continuous while D_1 is positive A-singular by Theorem 2, because

$$D_1=\{D+(B-[A]B)\}-[A]\{D+(B-[A]B)\}.$$

Finally C_1 does not coincide with $[A]B$. For otherwise the relation

$$[A]\{D+(B-[A]B)\}=[A]B-C=D$$

would imply the A-absolute continuity of D by Theorem 2, which is a contradiction. Thus B admits an A-Lebesgue decomposition different from the one given in Theorem 2. This completes the proof of the theorem.

Corollary 7. *The following conditions for a positive operator A are mutually equivalent:*

(a) *$\operatorname{ran}(A)$ is closed,*

(b) *A-absolute continuity is hereditary,*

(c) *Each positive operator admits a unique A-Lebesgue decomposition.*

Proof. (a)$\Rightarrow$(b) is immediate, because under the closedness of $\operatorname{ran}(A)$ it is easy to prove the equivalence of A-absolute continuity and A-strong continuity. (b)$\Rightarrow$(c) is proved just as in the first part of the proof of Theorem 6. (c)$\Rightarrow$(a): Let P be the orthoprojection onto the closure of $\operatorname{ran}(A)$. Then obviously P is A-absolutely continuous. Now (c) implies by Theorem 6 that $P\leqq\alpha A$ for some $\alpha\geqq 0$, which is equivalent to the closedness of $\operatorname{ran}(A)$. This completes the proof.

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DIVISION OF APPLIED MATHEMATICS
RESEARCH INSTITUTE OF APPLIED ELECTRICITY
HOKKAIDO UNIVERSITY
SAPPORO, JAPAN