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On differentiation

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Homage to the memory of F. Riesz

The ideas developed by F. RIESZ in his proof [1] that a monotonic function is almost everywhere differentiable are used here to prove:

Theorem 1. *If f and φ increase on an open interval (a, b) then $df/d\varphi$ is finite except on a subset of (a, b) of μ_φ-measure zero.*

Theorem 2. *If the increasing function f is absolutely continuous relative to the increasing function φ on (a, b) then*

$$f(b-)-f(a+) = \int_{(a,b)} df/d\varphi \, d\mu_\varphi. \text{ [1]}$$

This closes a gap left by the Radon—Nikodym theorem. The obvious definition

$$(1) \qquad df/d\varphi|_x = \lim_{y\to x} \frac{f(y)-f(x)}{\varphi(y)-\varphi(x)}$$

can not be used for Theorem 1 as the following example shows. Let $f(x)$ be -1 for $x<0$, 0 for $x=0$, 1 for $x>0$, and let $\varphi(x)$ be -1 for $x<0$ and 1 for $x\geqq 0$. Then $df/d\varphi|_0$, by (1), does not exist and $\mu_\varphi(\{0\})=2$. However

$$\lim_{h\downarrow 0,\, k\uparrow 0} \frac{f(h)-f(k)}{\varphi(h)-\varphi(k)} = 1.$$

This suggests that $df/d\varphi$ be defined as the common value, if it exists, of the upper and lower derivates of f relative to φ.

For any real function f on (a, b) and all $I=(u, v)\subset(a, b)$ let $f(I)=f(v)-f(u)$.

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[1]) These theorems seem to by be a part of the oral mathematical tradition but diligent inquiry by the author did not disclose any written record of their proofs.

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Definition. Let f and φ be real functions on (a, b), $x\in(a, b)$ and assume that $\varphi(I)\neq 0$ for sufficiently small I containing x. Set

$$D_\varphi f(x) = \sup_{x\in J}\inf_{x\in I\subset J} f(I)/\varphi(I), \quad D^\varphi f(x) = \inf_{x\in J}\sup_{x\in I\subset J} f(I)/\varphi(I).$$

If $D_\varphi f(x)=d(x)=D^\varphi f(x)$ let $df/d\varphi|_x=d(x)$.

In the manner of Riesz, we consider the Dini derivates of f relative to φ.

Definition. If f and φ are functions on (a, b) and $x\in(a, b)$ let

$$D_l^\varphi f(x) = \sup_{\alpha<x}\inf_{\alpha<y<x}\frac{f(y)-f(x)}{\varphi(y)-\varphi(x)}, \quad D_L^\varphi f(x) = \inf_{\alpha<x}\sup_{\alpha<y<x}\frac{f(y)-f(x)}{\varphi(y)-\varphi(x)},$$

$$D_r^\varphi f(x) = \sup_{x<\beta}\inf_{x<y<\beta}\frac{f(y)-f(x)}{\varphi(y)-\varphi(x)}, \quad D_R^\varphi f(x) = \inf_{x<\beta}\sup_{x<y<\beta}\frac{f(y)-f(x)}{\varphi(y)-\varphi(x)},$$

provided that the denominators do not vanish. If the four derivates have a common value let if be $d_\varphi f(x)$. The following two statements are immediate consequences of the definitions.

Proposition 1. *$df/d\varphi|_x=d(x)$ if and only if for all sequences of open intervals (x_k, y_k) containing x such that $y_k-x_k\to 0$*

$$\lim_k \frac{f(y_k)-f(x_k)}{\varphi(y_k)-\varphi(x_k)} = d(x).$$

Corollary. (a) *If $f(x+)$, $f(x-)$, $\varphi(x+)$, $\varphi(x-)$ are finite and $\varphi(x+)\neq\varphi(x-)$ then $df/d\varphi|_x$ is finite.* (b) *If f and φ increase on (a, b) and φ is not continuous at $x\in(a, b)$ then $0\leqq df/d\varphi|_x<+\infty$.*

Proposition 2. $\lim\limits_{y\to x}\dfrac{f(y)-f(x)}{\varphi(y)-\varphi(x)}=d(x)$ *if and only if $d_\varphi f(x)=d(x)$.*

Proposition 3. *If φ increases on (a, b) and $d_\varphi f(x)$ is finite then $df/d\varphi|_x= =d_\varphi f(x)$.*

Proof. For any $\varepsilon>0$ there is some $\delta>0$ such that if $x-\delta<y'<x<y''<x+\delta$ then

$$(1) \qquad d_\varphi f(x)-\varepsilon<\frac{f(y')-f(x)}{\varphi(y')-\varphi(x)}, \quad \frac{f(y'')-f(x)}{\varphi(y'')-\varphi(x)}<d_\varphi f(x)+\varepsilon.$$

Consider the points $P'(\varphi(y'), f(y'))$, $P(\varphi(x), f(x))$, $P''(\varphi(y''), f(y''))$ in the (φ, f)-plane and the slopes S', S, S'' of $P'P$, $P'P''$, PP'' respectively. Since φ increases on (a, b) it follows from (1) that the strict inequalities $\varphi(y')<\varphi(x)<\varphi(y'')$ hold. Hence

$$\min\{S', S''\}\leqq S\leqq\max\{S', S''\}.$$

Consequently

$$d_\varphi f(x)-\varepsilon \leqq D_\varphi f(x) \leqq D^\varphi f(x) \leqq d_\varphi f(x)+\varepsilon \quad \text{for all} \quad \varepsilon > 0.$$

The conclusion follows from the definition of $df/d\varphi|_x$.

It is convenient to fix some notation. We use f and φ for increasing functions on a closed interval $[a, b]$. For $x\in(a, b)$

$$\varphi^\lambda(x) = \sup_{y<x} \varphi(y), \quad \varphi^\varrho(x) = \inf_{y>x} \varphi(y), \quad E(\varphi) = \{x|\varphi^\lambda(x) < \varphi^\varrho(x)\}.$$

Then on (a, b), φ^λ and φ^ϱ increase, $\varphi^\lambda \leqq \varphi \leqq \varphi^\varrho$, $\varphi^{\lambda\lambda}=\varphi^\lambda$, $\varphi^{\varrho\varrho}=\varphi^\varrho$ and, if $(x, y)\neq\emptyset$, $(x, y)-E(\varphi)$ is uncountable since $E(\varphi)$ is the countable set of discontinuities of φ.

The exceptional set $E(f, \varphi)$

The sets

$$E_{l,R}^{\varphi\lambda}(f^\varrho) = \{x\in(a, b)|D_l^{\varphi\lambda} f^\varrho(x) < D_R^{\varphi\lambda} f^\varrho(x)\},$$

$$E_{r,L}^{\varphi\varrho}(f^\lambda) = \{x\in(a, b)|D_r^{\varphi\varrho} f^\lambda(x) < D_L^{\varphi\varrho} f^\lambda(x)\},$$

$$E_{R,\infty}^{\varphi\lambda}(f^\varrho) = \{x\in(a, b)|D_R^{\varphi\lambda} f^\varrho(x) = +\infty\},$$

modeled on the similar sets in [1], are called the *Riesz sets.*

The set $C(\varphi)$, next to be defined, is determined by the intervals on which φ is constant. Let

$$C_x = \{y|\varphi(y) = \varphi(x)\} \quad \text{and} \quad \lambda_x = \inf C_x, \quad \varrho_x = \sup C_x \quad \text{for} \quad x\in(a, b).$$

The sets C_x are disjoint and contain x. The set of non-empty (λ_x, ϱ_x) is countable. Let these open intervals be (λ_n, ϱ_n) and let $[\lambda_n, \varrho_n]$ be their closures, and set

$$C(\varphi) = \bigcup_n [\lambda_n, \varrho_n] \cap (a, b).$$

Proposition 4. *If* $x\in(a, b)-C(\varphi)$ *and* $a<x'<x<x''<b$, $\varphi(x')<\varphi(x)<\varphi(x'')$.

Proof. Otherwise $x'\in C_x$ or $x''\in C_x$. In either case $(\lambda_x, \varrho_x)\neq\emptyset$ and $x\in[\lambda_x, \varrho_x]\subset \subset C(\varphi)$, contrary to hypothesis.

The exceptional set for f and φ on $[a, b]$ is

$$E(f, \varphi) = E(f)\cup E(\varphi)\cup C(\varphi)\cup E_{l,R}^{\varphi\lambda}(f^\varrho)\cup E_{r,L}^{\varphi\varrho}(f^\lambda)\cup E_{R,\infty}^{\varphi\lambda}(f^\varrho).$$

Proposition 5. *If* $x\in(a, b)-(E(f, \varphi)-E(\varphi))$, *then* $0\leqq df/d\varphi|_x< +\infty$.

Proof. Consider $x \in (a, b) - E(f, \varphi)$ and $a < x' < x < x'' < b$. Since $x \notin E(f) \cup \cup E(\varphi) \cup C(\varphi)$ we infer from Proposition 4

$$f^\lambda(x') \leqq f(x') \leqq f^\varrho(x') \leqq f^\lambda(x) = f(x) = f^\varrho(x) \leqq f^\lambda(x'') \leqq f(x'') \leqq f^\varrho(x''),$$
$$\varphi^\lambda(x') \leqq \varphi(x') \leqq \varphi^\varrho(x') < \varphi^\lambda(x) = \varphi(x) = \varphi^\varrho(x) < \varphi^\lambda(x'') \leqq \varphi(x'') \leqq \varphi^\varrho(x'');$$

and hence,

$$0 \leqq \frac{f^\varrho(x) - f^\varrho(x')}{\varphi^\lambda(x) - \varphi^\lambda(x')} \leqq \frac{f(x) - f(x')}{\varphi(x) - \varphi(x')} \leqq \frac{f^\lambda(x) - f^\lambda(x')}{\varphi^\varrho(x) - \varphi^\varrho(x')} < +\infty,$$
$$0 \leqq \frac{f^\lambda(x'') - f^\lambda(x)}{\varphi^\varrho(x'') - \varphi^\varrho(x)} \leqq \frac{f(x'') - f(x)}{\varphi(x'') - \varphi(x)} \leqq \frac{f^\varrho(x'') - f^\varrho(x)}{\varphi^\lambda(x'') - \varphi^\lambda(x)} < +\infty.$$

Therefore,

$$0 \leqq D_l^{\varphi^\lambda} f^\varrho(x) \leqq D_l^\varphi f(x) \leqq D_L^\varphi f(x) \leqq D_L^{\varphi^\varrho} f^\lambda(x) \leqq +\infty,$$
(1)
$$0 \leqq D_r^{\varphi^\varrho} f^\lambda(x) \leqq D_r^\varphi f(x) \leqq D_R^\varphi f(x) \leqq D_R^{\varphi^\lambda} f^\varrho(x) \leqq +\infty.$$

Since the Riesz sets exclude x it follows from their defining inequalities and (1) that

$$0 \leqq D_l^\varphi f(x) = D_L^\varphi f(x) = D_r^\varphi f(x) = D_R^\varphi f(x) = D_R^{\varphi^\lambda} f^\varrho(x) < +\infty.$$

By Proposition 3,

(2) $$0 \leqq df/d\varphi|_x < +\infty \quad \text{for} \quad x \in (a, b) - E(f, \varphi).$$

By the Corollary to Proposition 1

(3) $$0 \leqq df/d\varphi|_x = \frac{f^\varrho(x) - f^\lambda(x)}{\varphi^\varrho(x) - \varphi^\lambda(x)} < +\infty \quad \text{for} \quad x \in E(\Phi).$$

The conclusion follows from (2), (3).

Toward $\mu_\varphi(E(f, \varphi) - E(\varphi)) = 0$

We summarize the properties of measure which play a role in what follows. For an increasing function φ defined on an open interval I of $\mathbf{R}$ and any $A \subset I$, let

$$\mu_\varphi(A) = \inf\{\sum_n \varphi(I_n) | A \subset \cup I_n,\ I_n = (a_n, b_n) \subset [a_n, b_n] \subset I\}.$$

Proposition 6. *For A, $[a_n b_n]$, (x, y), $(x, y]$, $[x, y]$, $\{x\}$ and A_n subsets of I we have:*

(a) $\mu_\varphi(A) = \inf\{\sum_n \varphi(I_n) | A \subset \bigcup_n I_n,\ I_n = (a_n, b_n),\ a_n, b_n \notin E(\varphi)\}$.

(b) $\mu_\varphi((x, y)) = \varphi^\lambda(y) - \varphi^\varrho(x)$, $\mu_\varphi((x, y]) = \varphi^\varrho(y) - \varphi^\varrho(x)$,

$\mu_\varphi([x, y]) = \varphi^\varrho(y) - \varphi^\lambda(x)$.

(c) $\mu_\varphi(\{x\}) = \varphi^\varrho(x) - \varphi^\lambda(x)$.

(d) If $\mu_\varphi(A_n) = 0$ for $n \in \mathbf{N}$, $\mu_\varphi(\bigcup_n A_n) = 0$.

Proposition 7. *If* φ, ψ *increase on I then* $\mu_\varphi(A)=\mu_\psi(A)$ *for all* $A\subset I$ *if and only if*

$$(1) \qquad E(\varphi)=E(\psi) \quad \text{and} \quad \varphi(x)-\psi(x) \quad \text{is constant on} \quad I-E(\varphi).$$

Proof. Assume (1). Then, by Proposition 6(a), $\mu_\varphi(A)=\mu_\psi(A)$ for $A\subset I$. Conversely, the latter equality implies $E(\varphi)=E(\psi)$ by Proposition 6(c) and then, choosing $a\in I-E(\varphi)$, $\varphi(x)-\varphi(a)=\mu_\varphi([a,x])=\mu_\psi([a,x])=\psi(x)-\psi(a)$ for $x\in I-E(\varphi)$, $x>a$, and a similar argument applies if $x\in I-E(\varphi)$, $x<a$, by Proposition 6(b).

Corollary. *For all* $A\subset I$, $\mu_{\varphi^\lambda}(A)=\mu_\varphi(A)=\mu_{\varphi^\varrho}(A)$.

Proposition 8. $\mu_\varphi((E(f)\cup C(\varphi))-E(\varphi))=0$.

Proof. By the definition of $C(\varphi)$,

$$(E(f)\cup C(\varphi))-E(\varphi)\subset(E(f)-E(\varphi))\cup\Big(\bigcup_n(\lambda_n,\sigma_n)\cup(\{\lambda_n,\varrho_n|n\in\mathbf{N}\}-E(\varphi))\Big).$$

The first and last sets are countable and φ is continuous at each of their points. Since for each n, φ is constant on (λ_n,ϱ_n), $\varphi^\varrho(\lambda_n)=\varphi^\lambda(\varrho_n)$ for all n. The result now follows from Proposition 6(d).

The 'rising sun' theorem [1] is used as a lemma to show that the three Riesz sets are of μ_φ-measure zero.

Lemma. *If g is a real function on* $[a,b]$, $g(a)\geqq g(a+)$, $g(b)\geqq g(b-)$, *and* $g(x)\geqq \max\{g(x+),g(x-)\}$ *for* $a<x<b$, *then there are sequences* (a_n,b_n), (c_n,d_n) *of disjoint subintervals of* (a,b) *such that*

$$\{x\in(a,b)|g(y)>g(x) \text{ for some } y\in(a,x)\}=\bigcup_n(a_n,b_n),$$

$$\{x\in(a,b)|g(y)>g(x) \text{ for some } y\in(x,b)\}=\bigcup_n(c_n,d_n),$$

$$g(a_n)\geqq g(b_n-), \quad g(c_n+)\leqq g(d_n) \quad \text{for all} \quad n.$$

Proposition 9. *If* f,φ *increase on* $[a,b]$, $f(a)=f(a+)$, $f=f^\varrho$, $\varphi(b)=\varphi(b-)$, $\varphi=\varphi^\lambda$, $t>0$, *and* $g=f-t\varphi$ *then g satisfies the hypotheses of the Lemma.*

Proof. Since $\varphi^\lambda=\varphi\leqq\varphi^\varrho$, $f^\lambda\leqq f=f^\varrho$ on (a,b), we have for $x\in(a,b)$

$$g(x+)=f^\varrho(x)-t\varphi^\varrho(x)\leqq f(x)-t\varphi(x)=g(x),$$

$$g(x-)=f^\lambda(x)-t\varphi^\lambda(x)\leqq f(x)-t\varphi(x)=g(x).$$

A similar argument applies for $x=a$ and $x=b$.

In applying the Lemma to the Riesz sets we use Proposition 9 and the fact that $f^{\varrho\varrho}=f^\varrho$, $\varphi^{\lambda\lambda}=\varphi^\lambda$. The next proposition may be called the *Riesz covering theorem*,

Proposition 10. *If $f = f^{\varrho}$, $\varphi = \varphi^{\lambda}$ on $J = (\alpha, \beta) \subset (\alpha, \beta)$ and*

$$E = \{x \in J \mid D_{L}^{\varphi} f(x) < u < v < D_{R}^{\varphi} f(x)\}$$

then there are $N \subset J$ and a countable set S of disjoint subintervals of J such that

$$\mu_{\varphi}(N) = 0, \quad S \text{ covers } E - N, \quad \sum_{I \in S} \varphi(I) \leqq \frac{u}{v} \varphi(J).$$

Proof. If $x \in E$ there is some $y \in (\alpha, x)$ such that $(f(x) - f(y))/(\varphi(x) - \varphi(y)) < u$. Hence

$$g_u(y) = f(y) - u\varphi(y) > f(x) - u\varphi(x) = g_u(x).$$

Since $g_u = f - u\varphi$ satisfies the hypothesis of Proposition 9, it follows from the Lemma that there are disjoint $I_n = (a_n, b_n) \subset J$, $n \in \mathbf{N}$, such that, since $\varphi = \varphi^{\lambda}$ and $f(b_n -) = f^{\lambda}(b_n)$,

$$(1) \qquad E \subset \bigcup_n I_n, \quad g_u(a_n) = f(a_n) - u\varphi(a_n) \geqq f^{\lambda}(b_n) - u\varphi(b_n) = g_u(b_n -).$$

Hence

$$(2) \qquad f^{\lambda}(b_n) - f(a_n) \leqq u(\varphi(b_n) - \varphi(a_n)) = u\varphi(I_n), \quad n \in \mathbf{N}.$$

For each n there is a sequence $b_{n,p} \in I_n - E(\varphi)$ such that $b_{n,p} \uparrow b_n$. Let $b_{n,0} = a_n$, $I_{n,p} = (b_{n,p-1}, b_{n,p})$, $N' = \{b_{n,p} \mid n, p \in \mathbf{N}\}$. Then

$$(3) \quad \mu_{\varphi}(N') = 0, \ I_{n,p}, \quad n, p \in \mathbf{N}, \quad \text{are disjoint}, \quad E - N' \subset \bigcup_{n,p} I_{n,p} \subset \bigcup_n I_n \subset J.$$

Since f increases and $b_{n,0} = a_n$ for all n

$$\sum_p f(I_{n,p}) = \sum_p (f(b_{n,p}) - f(b_{n,p-1})) = \lim_p f(b_{n,p}) - f(a_n) = f^{\lambda}(b_n) - f(a_n).$$

By (2), (3), since φ increases,

$$(4) \qquad \sum_{n,p} f(I_{n,p}) = \sum_n (f^{\lambda}(b_n) - f(a_n)) \leqq u \sum_n \varphi(I_n) \leqq u\varphi(J).$$

For each n, p if $x \in E \cap I_{n,p}$ there is some $y \in (x, b_{n,p})$ such that $(f(y) - f(x))/(\varphi(y) - \varphi(x)) > v$. Now

$$g_v(y) = f(y) - v\varphi(y) > f(x) - v\varphi(x) = g_v(x).$$

Since $g_v = f - v\varphi$ satisfies the hypothesis of Proposition 9 it follows from the Lemma that there is a sequence of disjoint $I_{n,p,m} = (c_{n,p,m}, d_{n,p,m}) \subset I_{n,p}$ such that, since $f = f^{\varrho}$ and $\varphi(c_{n,p,m} +) = \varphi^{\varrho}(c_{n,p,m})$,

$$E \cap I_{n,p} \subset \bigcup_m I_{n,p,m}, \quad f(c_{n,p,m}) - v\varphi^{\varrho}(c_{n,p,m}) \leqq f(d_{n,p,m}) - v\varphi(d_{n,p,m}).$$

Hence

$$(5) \qquad v(\varphi(d_{n,p,m}) - \varphi^{\varrho}(c_{n,p,m})) \leqq f(I_{n,p,m}), \quad n, p, m \in \mathbf{N}.$$

For all n, p, m there is a sequence $c_{n,p,m,q} \in I_{n,p,m} - E(\varphi)$ such that $c_{n,p,m,q} \downarrow c_{n,p,m}$. Let $c_{n,p,m,0} = d_{n,p,m}$, $I_{n,p,m,q} = (c_{n,p,m,q}, c_{n,p,m,q-1})$ and

$$N'' = \{c_{n,p,m,q} | n, p, m, q \in \mathbf{N}\}.$$

Then

(6)
$$\mu_\varphi(N'') = 0, \quad I_{n,p,m,q}, \quad n, p, m, q \in \mathbf{N}, \quad \text{are disjoint},$$
$$E - (N' \cup N'') \subset \bigcup_{n,p,m,q} I_{n,p,m,q} \subset \bigcup_{n,p,m} I_{n,p,m} \subset \bigcup_{n,p} I_{n,p}.$$

Since $c_{n,p,m,q} \downarrow c_{n,p,m}$ and $c_{n,p,m,0} = d_{n,p,m}$

(7)
$$\sum_q \varphi(I_{n,p,m,q}) = \sum_q (\varphi(c_{n,p,m,q-1}) - \varphi(c_{n,p,m,q}))$$
$$= \varphi(d_{n,p,m}) - \lim_q \varphi(c_{n,p,m,q}) = \varphi(d_{n,p,m}) - \varphi^\varrho(c_{n,p,m}).$$

Since f increases it follows from (4), (5), (6), (7) that

(8)
$$v \sum_{n,p,m,q} \varphi(I_{n,p,m,q}) \leqq \sum_{n,p,m} f(I_{n,p,m}) \leqq \sum_{n,p} f(I_{n,p}) \leqq u\varphi(J).$$

Let $N = N' \cup N''$ and $S = \{I_{n,p,m,q} | n, p, m, q \in \mathbf{N}\}$. By (3), (6), (8), N and S satisfy the required conditions.

Proposition 11. $\mu_{\varphi^\lambda}(E^{\varphi^\lambda}_{l,R}(f^\varrho)) = 0$.

Proof. $E^{\varphi^\lambda}_{l,R}(f^\varrho)$ is the union of the countable set of

$$E^J_{u,v} = \{x \in J = (a, b) | D^{\varphi^\lambda}_l f^\varrho(x) < u < v < D^{\varphi^\lambda}_R f^\varrho(x)\}, \quad u, v \text{ rational}.$$

We note that $f^\varrho = f^{\varrho\varrho}$, $\varphi^\lambda = \varphi^{\lambda\lambda}$ and show that for $k \in \mathbf{N}$ there are $N_k \subset J$ and a countable set S_k of disjoint open subintervals of J such that

{k}
$$\mu_{\varphi^\lambda}(N_k) = 0, \quad S_k \text{ covers } E^J_{u,v} - N_k, \quad \sum_{I \in S_k} \varphi(I) \leqq \left(\frac{u}{v}\right)^k \varphi(J).$$

By Proposition 10 with $(\alpha, \beta) = (a, b)$ there are N_1, S_1 satisfying $\{1\}$. Assume that N_k and S_k satisfy $\{k\}$. Let I_p, $p \in \mathbf{N}$, be the intervals of S_k. By Proposition 10 with $(\alpha, \beta) = I_p$ there are $M_p \subset I_p$ and a countable set T_p of disjoint open subintervals of I_p such that

$$\mu_{\varphi^\lambda}(M_p) = 0, \quad T_p \text{ covers } E^J_{u,v} \cap I_p - M_p, \quad \sum_{I \in T_p} \varphi(I) \leqq \frac{u}{v} \varphi(I_p), \quad p \in \mathbf{N}.$$

Let $N_{k+1} = N_k \cup (\bigcup_p M_p)$ and $S_{k+1} = \bigcup_p T_p$. Then $\mu_{\varphi^\lambda}(N_{k+1}) = 0$, S_{k+1} covers $E^J_{u,v} - N_{k+1}$ and

$$\sum_{I \in S_{k+1}} \varphi(I) = \sum_p \sum_{I \in T_p} \varphi(I) \leqq \sum_p \frac{u}{v} \varphi(I_p) \leqq \left(\frac{u}{v}\right)^{k+1} \varphi(J).$$

Thus N_{k+1}, S_{k+1} satisfy $\{k+1\}$, and therefore, $\{k\}$ is satisfied for all $k \in N$.

Let $N=\bigcup_k N_k$. Then $\mu_{\varphi^\lambda}(N)=0$, S_k covers $E^J_{u,v}-N$ for all k and, since $\lim_k (u/v)^k \varphi(J)=0$, $\mu_{\varphi^\lambda}(E^J_{u,v})=0$ for all rational u, v. Hence $\mu_{\varphi^\lambda}(E^{\varphi\lambda}_{l,R}(f^\varrho))=0$.

Proposition 12. $\mu_{\varphi^\varrho}(E^{\varphi\varrho}_{r,L}(f^\lambda))=0$.

Proof. Let $T(x)=-x$ for $x\in\mathbf{R}$. Let $h(T(x))=-f(x)$, $\psi(T(x))=-\varphi(x)$. Then h, ψ increase on $(T(b), T(a))$ $h^\varrho=-f^\lambda$, $\psi^\lambda=-\varphi^\varrho$, and for all $A\subset(T(b), T(a))$, $\mu_{\varphi^\varrho}(T^{-1}(A))=\mu_{\psi^\lambda}(A)$. Since $T(y)<T(x)$ if and only if $x<y$,

$$\frac{h(T(y))-h(T(x))}{\psi(T(y))-\psi(T(x))}=\frac{f(x)-f(y)}{\varphi(x)-\varphi(y)}$$

if either difference quotient is finite. Hence

$$E^{\varphi\varrho}_{r,L}(f^\lambda)=T^{-1}(E^{\psi\lambda}_{l,R}(h^\varrho)).$$

By Proposition 11, $\mu_{\psi^\lambda}(E^{\psi\lambda}_{l,R}(h^\varrho))=0$. Hence $\mu_{\varphi^\varrho}(E^{\varphi\varrho}_{r,L}(f^\lambda))=0$.

Proposition 13. $\mu_{\varphi^\lambda}(E^{\varphi\lambda}_{R,\infty}(f^\varrho))=0$.

Proof. For each $m\in\mathbf{N}$ let

$$E_m=\{x\in(a,b)|D^{\varphi\lambda}_R f^\varrho(x)>m\}.$$

Then $E_{m+1}\subset E_m\subset(a,b)$ for all m. If $x\in E_m$ there is some $y\in(x,b)$ such that

$$g_m(y)=f^\varrho(y)-m\varphi^\lambda(y)>f^\varrho(x)-m\varphi^\lambda(x)=g_m(x).$$

By Proposition 9 and the Lemma there is a sequence of disjoint $I_p=(c_p,d_p)\subset(a,b)$ such that, since $f^\varrho(c_p+)=f^\varrho(c_p)$,

$$E_m\subset\bigcup_p I_p,\quad f^\varrho(c_p)-m\varphi^\lambda(c_p+)\leqq f^\varrho(d_p)-m\varphi^\lambda(d_p),\quad p\in\mathbf{N}.$$

For each p there is a sequence $c_{p,q}\in I_p-E(\varphi)$ such that $c_{p,q}\downarrow c_p$. Let $c_{p,0}=d_p$, $I_{p,q}=(c_{p,q},c_{p,q-1})$ and $N=\{c_{p,q}|p,q\in\mathbf{N}\}$. Then $\mu_{\varphi^\lambda}(N)=0$, $E_m-N\subset\bigcup_{p,q}I_{p,q}\subset\bigcup_p I_p\subset(a,b)$ for all m,

$$m\sum_{p,q}\varphi^\lambda(I_{p,q})=m\sum_p\sum_q(\varphi^\lambda(c_{p,q-1})-\varphi^\lambda(c_{p,q}))=m\sum_p(\varphi^\lambda(d_p)-\varphi^\lambda(c_p+))\leqq$$
$$\leqq\sum_p(f^\varrho(d_p)-f^\varrho(c_p))\leqq f^\varrho((a,b))<+\infty.$$

Hence, $\mu_\varphi(E_m)\leqq f^\varrho((a,b))/m$ for all m. Since $E^{\varphi\lambda}_{R,\infty}(f^\varrho)\subset\bigcap_m E_m\subset(a,b)$,

$$0\leqq\mu_{\varphi^\lambda}(E^{\varphi\lambda}_{R,\infty}(f^\varrho))\leqq\lim_m\mu_{\varphi^\lambda}(E_m)=0.$$

Theorem 1. *If f and φ increase on (a,b) there is some $A\subset I$ such that*

$$0\leqq df/d\varphi|_x<+\infty\quad\textit{for}\quad x\in A\quad\textit{and}\quad\mu_\varphi((a,b)-A)=0.$$

Proof. By representing (a, b) as a union of countably many closed subintervals, we may consider one of them and assume that f and φ increase on $[a, b]$. By the definition of the exceptional set $E(f, \varphi)$

$$E(f, \varphi)-E(\varphi)\subset((E(f)\cup C(\varphi))-E(\varphi))\cup E^{\varphi^\lambda}_{l,R}(f^\varrho)\cup E^{\varphi^\varrho}_{r,L}(f^\lambda)\cup E^{\varphi^\lambda}_{R,\infty}(f^\varrho).$$

Since $E(\varphi)$ is the set of discontinuities of φ, φ^λ, φ^ϱ and $\varphi=\varphi^\lambda=\varphi^\varrho$ on $(a, b)-E(\varphi)$ it follows from Proposition 7 that μ_φ, μ_{φ^λ}, μ_{φ^ϱ} are identical measures.

Let $A=(a, b)-(E(f, \varphi)-E(\varphi))$. The conclusion follows from Propositions 6, 8, 11, 12, 13.

Toward Theorem 2

FUBINI's theorem [2] on the derivative of a function represented by a convergent series of increasing functions is extended in the following proposition.

Proposition 14. *If f_n, $n\in\mathbf{N}$, and φ increase on (a, b) and*

$$\sum_n f_n(x) = f(x) \quad \textit{is finite on} \quad (a, b)$$

then there is some $A\subset(a, b)$ such that $\mu_\varphi((a, b)-A)=0$ and

$$'\sum_n df_n/d\varphi|_x = df/d\varphi|_x \quad \textit{for} \quad x\in A.$$

The proof is so close to that of Fubini for the case where $\varphi(x)=x$ that it is omitted.

Similarly, Lebesgue's density theorem may be generalized. It is convenient to say that a sequence of open intervals (x_k, y_k) determines x if $x\in(x_k, y_k)$ for all k and $\lim_k (x_k-y_k)=0$.

Definition. Let φ increase on an open interval $I\subset\mathbf{R}$. The *μ_φ-density* of a set A at $x\in I$ is $\Delta(A, x)$ if for all sequences (x_k, y_k) which determine x

$$\lim_k \frac{\mu_\varphi(A\cap(x_k, y_k])}{\mu_\varphi((x_k, y_k])} = \Delta(A, x).$$

Proposition 15. *If $A\subset(a, b)\subset[a, b]\subset I$ is a μ_φ-measurable set then there is some $D\subset A$ such that*

$$\Delta(x, A) = 1 \quad \textit{for} \quad x\in D \quad \textit{and} \quad \mu_\varphi(A-D) = 0.$$

Proof. There is by Proposition 6(d) an open set G_n' for $n\in\mathbf{N}$ such that $A\subset G_n\subset(a, b)$ and $\mu_\varphi(G_n)<\mu_\varphi(A)+1/2^n$. Let

$$f(x) = \mu_\varphi(A\cap(a, x]), \quad \psi(x) = \varphi^\varrho(x)-\varphi^\varrho(a), \quad x\in(a, b),$$
$$f_n(x) = \mu_\varphi(G_n\cap(a, x]) \quad x\in(a, b), \quad n\in\mathbf{N}.$$

Then f, ψ, f_n increase on (a, b). Since

$$0 \leqq f_n(x)-f(x) = \mu_\varphi((G_n-A)\cap(a, x]) \leqq \mu_\varphi(G_n-A) < 1/2^n$$

and

$$f_n(y)-f(y)-(f_n(x)-f(x)) = \mu_\varphi((G_n-A)\cap(x, y]) \geqq 0 \quad \text{for} \quad x<y$$

it follows from Theorem 1 and Proposition 14 that there is some $D\subset A$ such that

$$\text{(1)} \qquad 0 \leqq \sum_n (df_n/d\psi|_x - df/d\psi|_x) < +\infty, \quad x\in D, \quad \text{and} \quad \mu_\psi(A-D)=0.$$

For $x\in D$ and any sequence (x_k, y_k) which determines x, there is some $k_{n,x}$ such that $(x_k, y_k]\subset G_n$ for $k\geqq k_{n,x}$. Then by Theorem 1, Proposition 1 and Proposition 6

$$df_n/d\psi|_x = \lim_k \frac{f_n(y_k)-f_n(x_k)}{\psi(y_k)-\psi(x_k)} = \lim_k \frac{\mu_\varphi(G_n\cap(x_k, y_k])}{\mu_\varphi((x_k, y_k])} = 1, \quad x\in D, \quad n\in \mathbf{N}.$$

By (1),

$$0 \leqq \sum_n (1-df/d\psi|_x) < +\infty \quad \text{for} \quad x\in D.$$

Hence

$$\text{(2)} \qquad df/d\psi|_x = 1 \quad \text{for} \quad x\in D.$$

Since $E(\varphi)=E(\varphi^e)=E(\psi)$ and $\varphi(x)-\psi(x)=\varphi^e(a)$ for $x\in(a, b)-E(\varphi)$, it follows from Proposition 7 that $\mu_\varrho(A-D)=\mu_\psi(A-D)=0$. Hence by (2)

$$1 = df/d\psi|_x = \lim_k \frac{f(y_k)-f(x_k)}{\psi(y_k)-\psi(x_k)} = \lim_k \frac{\mu_\varphi(A\cap(x_k, y_k])}{\mu_\varphi((x_k, y_k])} = \Delta(x, A),$$

for $x\in D$, $\mu_\varphi(A-D)=0$.

Proposition 16. *If φ increases on an open interval $I\subset\mathbf{R}$, f is μ_φ-integrable on $[a, b]\subset I$ and*

$$F(x) = \int_{(a, x]} f\, d\mu_\varphi \quad \textit{for} \quad x\in(a, b)$$

there is some $A\subset(a, b)$ such that

$$dF/d\varphi|_x = f(x) \quad \textit{for} \quad x\in A \quad \textit{and} \quad \mu_\varphi((a, b)-A)=0.$$

Proof. It is assumed, without loss of generality, that f is positive. There is a sequence of compact $C_n\subset(a, b)$ such that

$$C_n\subset C_{n+1} \quad \text{and } f \text{ is continous on } C_n, \text{ for } n\in\mathbf{N},$$

$$\lim_n \mu_\varphi((a, b)-C_n)=0, \quad \lim_n \int_{C_n} f\, d\mu_\varphi = \int_{(a, b)} f\, d\mu_\varphi < +\infty.$$

For $n\in\mathbf{N}$ let $f_n(x)=f(x)$ for $x\in C_n$, and $f_n(x)=0$, for $x\in(a, b)-C_n$, and set $A_1=\bigcup_n C_n$. Then

$$f_n\uparrow f \quad \text{on } A_1, \quad \mu_\varphi((a, b)-A_1) = \lim_n \mu_\varphi(A_1-C_n)=0.$$

Let

$$F_n(x) = \int_{(a,x]} f_n \, d\mu_\varphi \quad \text{for} \quad n\in \mathbf{N}, \quad x\in (a, b).$$

Since f_n and $f_{n+1}-f_n$ are positive on A_1, F_n and $F_{n+1}-F_n$ increase on (a, b). By the monotonic convergence theorem

$$F_1(x)+\sum_n (F_{n+1}(x)-F_n(x))$$
$$= \int_{(a,x]} f_1 \, d\mu_\varphi + \sum_n \Big(\int_{(a,x]} f_{n+1} \, d\mu_\varphi - \int_{(a,x]} f_n \, d\mu_\varphi \Big)$$
$$= \lim_n \int_{(a,x]} f_n \, d\mu_\varphi = \int_{(a,x]} \lim_n f_n \, d\mu_\varphi = F(x) < +\infty, \quad x\in (a, b).$$

Hence by Theorem 1 and the generalized Fubini theorem, Proposition 14, there is some $A_2 \subset A_1$ such that

$$0 \leqq dF_n/d\varphi|_x, \quad dF/d\varphi|_x < +\infty,$$

(1) $$\lim_n dF_n/d\varphi|_x = dF/d\varphi|_x \quad \text{for} \quad x\in A_2 \quad \text{and} \quad \mu_\varphi(A_1-A_2) = 0.$$

Consider $x\in A_2$. There is a sequence (x_k, y_k) which determines x such that $x_k, y_k \notin E(\varphi)$ for all k. Then

(2) $$dF_n/d\varphi|_x = \lim_k \frac{F_n(y_k)-F_n(x_k)}{\varphi(y_k)-\varphi(x_k)} \quad \text{for} \quad n\in \mathbf{N}.$$

Since φ is continuous at each x_k, y_k, by Proposition 6

(3) $$\mu_\varphi((x_k, y_k]) = \varphi(y_k)-\varphi(x_k) \quad \text{for all} \quad k.$$

On the compact set $C_n \cap [x_k, y_k]$, f is continuous and $f=f_n$. Hence there are $x_{n,k}$, $y_{n,k} \in C_n \cap [x_k, y_k]$, such that

(4) $$f(x_{n,k}) \leqq f(z) \leqq f(y_{n,k}) \quad \text{for} \quad z\in C_n \cap [x_k, y_k], \quad n, k\in \mathbf{N}.$$

Since $y_k - x_k \to 0$

(5) $$\lim_k f(x_{n,k}) = f(x) = \lim_k f(y_{n,k}) \quad \text{for} \quad x\in C_n.$$

By (3), (4)

$$f(x_{n,k}) \frac{\mu_\varphi(C_n \cap (x_k, y_k])}{\mu_\varphi((x_k, y_k])} \leqq (\varphi(y_k)-\varphi(x_k))^{-1} \int_{C_n \cap (x_k, y_k]} f \, d\mu_\varphi$$
$$= \frac{F_n(y_k)-F_n(x_k)}{\varphi(y_k)-\varphi(x_k)} \leqq f(y_{n,k}) \frac{\mu_\varphi(C_n \cap (x_k, y_k])}{\mu_\varphi((x_k, y_k])}, \quad n, k\in \mathbf{N}.$$

By the density theorem, Proposition 15, for each n there is some $D_n \subset C_n$ such that

(6) $$\lim_k \frac{\mu_\varphi(C_n \cap (x_k, y_k])}{\mu_\varphi((x_k, y_k])} = 1 \quad \text{for} \quad x\in D_n \quad \text{and} \quad \mu_\varphi(C_n - D_n) = 0.$$

By (2), (5), (6)

$$dF_n/d\varphi|_x = f(x) \quad \text{for} \quad x \in A_2 \cap D_n, \quad n \in \mathbf{N}. \tag{7}$$

Since $\mu_\varphi(A_1 - C_n) \to 0$, there are n_j such that $\mu_\varphi(A_1 - C_{n_j}) < 1/2^j$ for $j \in \mathbf{N}$. Let

$$D = \bigcup_k \bigcap_{j \geqq k} D_{n_j}.$$

Since $D_{n_j} \subset C_{n_j} \subset A_1$ and $\mu_\varphi(C_{nj} - D_{nj}) = 0$ for all j,

$$A_1 - D = \bigcap_k \bigcup_{j \geqq k} (A_1 - D_{n_j}) \subset \bigcup_{j \geqq k} (A_1 - C_{n_j}) \cup \bigcup_{j \geqq k} (C_{n_j} - D_{n_j}),$$

$$\mu_\varphi(A_1 - D) \leqq \sum_{j \geqq k} \mu_\varphi(A_1 - C_{n_j}) < \sum_{j \geqq k} 1/2^j = 1/2^{k-1}, \quad k \in \mathbf{N}.$$

Hence $\mu_\varphi(A_1 - D) = 0$. Let $A = A_2 \cap D$. If $x \in A$ then, for some k and all $j \geqq k$, $x \in A_2 \cap D_{nj}$. By (1), (7)

$$dF/d\varphi|_x = \lim_j dF_{n_j}/d\varphi|_x = f(x) \quad \text{for} \quad x \in A. \tag{8}$$

Since $A = A_2 \cap D \subset A_2 \subset A_1 \subset (a, b)$

$$\begin{aligned} 0 \leqq \mu_\varphi((a, b) - A) \leqq \mu_\varphi((a, b) - A_1) + \\ + \mu_\varphi(A_1 - A_2) + \mu_\varphi(A_2 - A) \leqq \mu_\varphi(A_1 - D) = 0. \end{aligned} \tag{9}$$

By (8), (9), A satisfies the required conditions.

Theorem 2. *Let f, φ increase on an open interval $I \subset \mathbf{R}$ and let f be absolutely continuous with respect to φ, i.e., $\mu_f(A) = 0$ for all $A \subset I$ such that $\mu_\varphi(A) = 0$. Then*

$$f(b-) - f(a+) = \int_{(a,b)} df/d\varphi|_x \, d\mu_\varphi \quad \textit{for all} \quad (a, b) \subset I.$$

Proof. Consider the measures μ_f, μ_φ. By the theorem SAKS ([3], p. 33) calls the Lebesgue decomposition theorem there are, for any (a, b)$\subset I$, some $H \subset (a, b)$ such that $\mu_\varphi(H) = 0$ and a positive function g, μ_φ-integrable on (a, b), such that

$$\mu_f((a, x]) = \int_{(a,x]} g \, d\mu_\varphi + \mu_f(H \cap (a, x]) \quad \text{for all} \quad x \in (a, b).$$

Since f is absolutely continuous with respect to φ and $\mu_\varphi(H) = 0$, $\mu_f(H \cap (a, x]) = 0$ for all $x \in (a, b)$. Hence

$$\psi(x) = \mu_f((a, x]) = \int_{(a,x]} g \, d\mu_\varphi \quad \text{for} \quad x \in (a, b).$$

By Proposition 16 there is some $A_1 \subset (a, b)$ such that

$$d\psi/d\varphi|_x = g(x) \quad \text{for} \quad x \in A_1 \quad \text{and} \quad \mu_\varphi((a, b) - A_1) = 0.$$

Since f increases on I there is, by Theorem 1, some $A_2 \subset (a, b)$ such that

$$0 \leqq df/d\varphi|_x < +\infty \quad \text{for} \quad x \in A_2 \quad \text{and} \quad \mu_\varphi((a, b) - A_2) = 0.$$

Let $A = A_1 \cap A_2$. For $x \in A$ there is a sequence $(x_k, y_k) \subset (a, b)$, determining x and such that $x_k, y_k \in (a, b) - (E(\varphi) \cup E(f))$. By Proposition 6

$$f(y_k) - f(x_k) = \mu_f((x_k, y_k]) = \psi(y_k) - \psi(x_k) \quad \text{for all} \quad k.$$

By Proposition 1

$$df/d\varphi|_x = \lim_k \frac{f(y_k) - f(x_k)}{\varphi(y_k) - \varphi(x_k)} = \lim_k \frac{\psi(y_k) - \psi(x_k)}{\varphi(y_k) - \varphi(x_k)} = d\psi/d\varphi|_x = g(x), \quad x \in A,$$

and

$$0 \leqq \mu_\varphi((a, b) - A) \leqq \mu_\varphi((a, b) - A_1) + \mu_\varphi((a, b) - A_2) = 0.$$

Hence

$$\mu_f((a, x]) = \int_{(a, x]} df/d\varphi|_x \, d\mu_\varphi \quad \text{for} \quad x \in (a, b).$$

There are sequences $a_k, b_k \in (a, b) - E(f)$ such that $a_1 < b_1$ and $a_k \downarrow a$, $b_k \uparrow b$. Now

$$f(b_k) - f(a_k) = \mu_f((a_k, b_k]) = \int_{(a_k, b_k]} df/d\varphi|_x \, d\mu_\varphi \quad \text{for all} \quad k.$$

Hence

$$f(b-) - f(a+) = \lim_k \int_{(a_k, b_k]} df/d\varphi|_x \, d\mu_\varphi = \int_{(a, b)} df/d\varphi|_x \, d\mu_\varphi .$$

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