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# On a convolution theorem

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Let $G$ be a non-discrete locally compact Abelian group and let $1 \leqq q < \infty$. DIEUDONNÉ [3] showed that $f * L^q(G) \neq L^q(G)$ for every $f \in L^1(G)$. In [2], BURNHAM and GOLDBERG proved Dieudonné's result for $q=1$ by considering Banach algebras with elements which are "generalized divisors of zero", and an extension to Banach modules yields Dieudonné's result for $1 < q < \infty$. In this note we give a simple, elementary proof of the following result.

**Theorem.** *Let $G$ be a non-discrete locally compact Abelian group and let $1 \leqq q \leqq \infty$. Then $L^1(G) * g \neq L^q(G)$ for every $g \in L^q(G)$.*

**Proof.** Suppose $L^1(G) * g = L^q(G)$ for some $g \in L^q(G)$. Then there exists $j \in L^1(G)$ such that $j * g = g$. Now if $h \in L^q(G)$, then $h = k * g$ for some $k \in L^1(G)$, and hence $j * h = j * k * g = k * j * g = k * g = h$. Thus $j * h = h$ for every $h \in L^q(G)$. Now for any $f \in L^1(G)$, choose a sequence $\{h_n\}$ in $L^1(G) \cap L^q(G)$ such that $\|h_n - f\|_1 \to 0$ as $n \to \infty$. Then $\|j * f - f\|_1 \leqq \|j * f - j * h_n\|_1 + \|j * h_n - f\|_1 \leqq \|j\|_1 \|f - h_n\|_1 + \|h_n - f\|_1 \to 0$. Thus $j$ is an identity element for $L^1(G)$. But this is impossible, since $G$ is non-discrete.

**Remark 1.** It is clear that the set $L^q(G)$ in the preceding theorem can be replaced by many other sets, and we mention some examples below.

(i) If $B$ is any dense subset of $L^1(G)$ with $L^1(G) * B \subset B$, then $L^1(G) * g \subsetneqq B$ for every $g \in B$. In particular, if $S(G)$ is a Segal algebra in $L^1(G)$, then $L^1(G) * g \subsetneqq S(G)$ for every $g \in S(G)$.

(ii) If $g \in L^{pq}(G)$, then $L^1(G) * g \subsetneqq L^{pq}(G)$. See BLOZINSKI [1, 2.9] and YAP [5, (4.2)] for the relevant facts.

**Remark 2.** KROGSTAD [4] has used the above theorem (with $q=1$) to show that the union of all proper Segal algebras on $G$ is $L^1(G)$. This answers a question of H. C. WANG.

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