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Rings with e as a radical element

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In [4] rings with identity e, having e as their radical element, were introduced. Here e is said to be a radical element of the ring R, if for every $x, y \in R$ there exists an element b in R such that $xy = bxy$.

Rings having this property are close to commutative rings, but still different. In [4], some properties of these rings are established and it is shown that for primitive rings "every left ideal is a two-sided ideal" is equivalent to "there exists an identity e and e is a radical element".

A primitive ring with e as a radical element is a division ring. In general: e is a radical element in a ring R if and only if $Rxy = Ryx$ for all $x, y \in R$. In § 1 we show that for a simple primering S the property $Sxy = Syx$ for all $x, y \in S$ implies that S has an identity and is a division ring (Theorem 3).

An easy application of the Wedderburn-Artin structure theorem gives that a nil-semisimple artinian ring R with e as a radical element is a finite direct sum of division rings (Theorem 2). We give a general structure theorem for rings R with e as a radical element and having no proper nilpotents (Theorem 4). This last theorem is analogous to a similar theorem of Reid for subcommutative rings [3]. Therefore we investigate the relationship between rings with e as a radical element and subcommutative rings in § 2. Using a fundamental result of LAWVER, we are able to give a counterexample to a conjecture in [4]. It is here that the ring $\mathbf{D}_2$ of rational quaternions with denominators prime to 2 is used. This ring has a proper Jacobson radical ($\neq 0$, $\neq \mathbf{D}_2$) which has some interest in its own and is investigated in § 3. The Jacobson radical $\mathbf{J}(\mathbf{D}_2)$ of $\mathbf{D}_2$ is a ring such that every l-ideal or r-ideal is two-sided, but it does not have an identity.

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§ 1.

Definition. Let R be a ring with identity e. Then e is called a *radical element in* R if for every $x, y \in R$ there exists a non-zero element $b \in R$ such that $xy = byx$.

Evidently any commutative ring with identity e and any division ring has e as a radical element.

Following THIERRIN [5] an ideal I of the ring R is called completely prime if $ab \in I$ implies that $a \in I$ or $b \in I$ for any two elements a and b of R, and completely semi-prime if $a^n \in I$ implies that $a \in I$ for any element a of R. Furthermore, R is called completely prime (completely semi-prime) if the zero-ideal of R is completely prime (completely semi-prime). Clearly R is completely prime if and only if R has no zero divisors i.e. if R is a domain, and completely semi-prime if and only if R has no nonzero nilpotents.

Lemma 1. *Let R be a ring with e as a radical element. Then*

$$R \text{ is prime} \Leftrightarrow R \text{ is completely prime},$$
$$R \text{ is semi-prime} \Leftrightarrow R \text{ is completely semi-prime}.$$

Proof. In [4] Proposition 4.3 it is shown that R is a prime ring implies R has no zero-divisors, hence R is completely prime. The converse is clear. Now let R be semi-prime and let $a \in R$ with $a^n = 0$. If $x \in (Ra)^n$ then x is a sum of elements of the form

$$r_1 a \cdot r_2 a \cdots a \cdot r_n a = r_1 a \cdot r_2 a \cdots a \cdot r_{n-1}[c_n(r_n a)]a = \cdots = r_1 c_2 \cdot r_2 c_3 \cdots r_{n-1} c_n \cdot r_n a^n = 0.$$

So $(Ra)^n = 0$. But R has no nonzero nilpotent l-ideals, hence $Ra = 0$. Then $a = 0$, since $a = ea \in Ra$. So R has no nonzero nilpotents and R is completely semi-prime. The converse is again clear. In the same way it may be shown that an ideal in R is a prime (semi-prime) ideal if and only if it is completely prime (completely semi-prime). This means, in particular, that the intersection of all prime ideals in R coincides with the intersection of all completely prime ideals.

In [4] it is shown that in a ring R with radical element e: $\sqrt{(0)} = \{x \in R : x^n = 0$ for some natural number $n\}$ is the intersection of all completely prime ideals not containing e i.e. the intersection of all completely prime ideals. Hence the intersection of the completely prime ideals is the set of nilpotent elements in a ring R with radical element e.

Now THIERRIN [5] has defined the so-called generalized nil-radical $\mathbf{N}_g$, which is the upper radical determined by the class of all rings without zero-divisors. $\mathbf{N}_g$ is a special radical and for any ring R one has: $\mathbf{N}_g(R) =$ intersection of all ideals I in R such that R/I has no zero-divisors i.e. I is a completely prime ideal in R. Hence for a ring R with e as a radical element, the radical $\mathbf{N}_g$ coincides with the intersection of all prime ideals which is the lower nil radical. Since the upper nil-radical $\mathbf{N} \subseteq \mathbf{N}_g$ for any ring R, one has that for a ring R with radical element e the following ideals coincide:

a) Lower nil radical β=intersection of all prime ideals,
b) Upper nil radical $\mathbf{N}$,
c) Generalized nil radical $\mathbf{N}_g$=intersection of all completely prime ideals,
d) The ideal of all nilpotent elements.

Next we show

Theorem 2. *Let R be a nil-semisimple Artinian ring with e as a radical element. Then R is a direct sum of a finite number of division rings.*

Proof. By the Wedderburn—Artin theorem $R=Re_1\oplus\ldots\oplus Re_n$, where the Re_i are minimal left ideals in T and the $e_i\in R$ satisfy $e_ie_j=e_i$ if $i=j$ and $e_ie_j=0$ if $i\neq j$ $(i,j=1,\ldots,n)$. Also $e=e_1+\ldots+e_n$ is an identity for R. We claim that the Re_i are division rings. Let $ae_i\neq 0$. Then $(Re_i)(ae_i)\neq 0$, since $(Re_i)(ae_i)=0$ would imply $(ae_i)^2=0$, hence $ae_i=0$, since R has no nonzero nilpotents. Also $(Re_i)(ae_i)\subseteqq Re_i$ and since Re_i is minimal, this implies $(Re_i)(ae_i)=Re_i$. So for any $be_i\in Re_i$, there exists $xe_i\in Re_i$ with $(xe_i)(ae_i)=be_i$. Then Re_i is a division ring.

One might expect that full matrix rings over division rings can occur as rings with e as a radical element. Our next theorem shows that this cannot happen.

Theorem 3. *Let S be a simple prime ring with $Sxy=Syx$ for all $x, y\in S$. Then S has an identity e, e is a radical element for S and S is a division ring.*

Proof. From $Sxy=Syx$ for all $x, y\in S$ and S is a prime ring, one can conclude that S has no zero-divisors in the same way as in the proof of Proposition 4.3 [4]. Now let $x\neq 0$ in S. Then Sx is a non-zero ideal in S, since $(sx)y=byx$ for $y\in S$ and some $b\in S$. Hence $Sx=S$. Thus, S has no proper left ideals and so it is a division ring. The rest of the theorem follows obviously.

Let R be a ring with radical element e. If N is the ideal of nilpotent elements of R then the ring $\bar{R}=R/N$ is a ring without nilpotent elements and with radical element $\bar{e}=e+N$, the identity of R/N. To state our theorem on such rings, we use the following:

Definition. Let D be a division ring. We call a subring S of D a *commutator subring* if, given $s_1\neq 0$, $s_2\neq 0$ in S, the element $s_1s_2s_1^{-1}s_2^{-1}\in S$.

Theorem 4. *Let R be a ring with e as a radical element. Then R has no nilpotents if and only if R is a subdirect sum of commutator subrings of division rings.*

Proof. Let R be a ring with radical element e and without nilpotent elements. Then the intersection of the prime ideals P in $R=(0)$, so that R is a subdirect sum of the rings R/P, P a prime ideal in R. The rings R/P are prime rings and have no divisors of zero. Being homomorphic images of R they have the property that $\bar{e}=e+P$ is a radical element for R/P. This last condition implies that any pair $\bar{x}$, $\bar{y}$ of non-zero elements of R/P has a non-zero common left multiple i.e. there exists an element $\bar{d}\neq\bar{o}$ in R/P such that $\bar{x}\bar{y}=(\bar{d}\bar{y})\bar{x}$. Hence by a well-known theorem of Ore there exists a division ring Δ_p containing R/P. For any pair $\bar{a}, \bar{b} \in R/P$, $\bar{a} \neq \bar{o}$, $\bar{b} \neq \bar{o}$, the equation $\bar{a}\bar{b} = \bar{c}\bar{b}\bar{a}$ has a unique solution in Δ_p, namely $\bar{a}\bar{b}\bar{a}^{-1}\bar{b}^{-1}$. The fact that $\bar{e}$ is a radical element for R/P implies that this solution must lie in R/P. Hence R/P is a commutator subring of Δ_p as required. The converse is obvious.

From the proof it follows that a prime ring having e as a radical element is a commutatoı subring of a division ring. This implies, in particular, Proposition 4.3 [4].

Remark. By Theorem 4 the rings with e as a radical element which are nil-semisimple (or β-semisimple) are characterized.

Corollary. *Let Δ be a division ring with identity e and let R be a subring* ($\neq 0$) *of Δ. Then R is a commutator subring of Δ if and only if e is a radical element for R.*

§ 2. Subcommutative rings

Definition. A ring R is said to be *r-subcommutative* if for every $a, b \in R$ there is an element $c \in R$ such that $ab = bc$. Similarly R is said to be *l-subcommutative* if for every $a, b \in R$ there is an element $d \in R$ such that $ab = da$.

Subcommutative rings have been introduced by Bucur [1], using the first part of the definition. This is also used by Lawver [2]. On the contrary, Reid [3] uses the second part of the definition, and calls such rings subcommutative. We shall use the terms r- and l-subcommutative respectively, according to the above definition. Now let R be a ring with identity e. It can be easily seen that R is r-subcommutative if and only if every r-ideal of R is two-sided and R is l-subcommutative if and only if every l-ideal of R is two-sided. So a ring R is both r- and l-subcommutative if and only if any one-sided ideal is two-sided. Such rings have been considered by Thierrin [6] and are called duo rings.

The following result is due to Reid [3].

Theorem 5. *Any l-stable subring of a direct product of division rings is l-subcommutative and has no proper nilpotent elements. Every l-subcommutative ring without proper nilpotent elements is a subdirect sum of l-stable subrings of division rings.*

Here an l-stable subring is defined as follows:

Let I be an index set and for each $i \in I$, Δ_i a division ring. For $a \in \pi\Delta_i$ (the ring direct product), define a' by

$$(a')_i = \begin{cases} 0 & \text{if } a_i = 0 \\ a_i^{-1} & \text{if } a_i \neq 0. \end{cases}$$

A subring A of $\pi\Delta_i$ is called an l-stable subring if $aAa' \subseteq A$ for each $a \in A$. Similarly, a subring A of $\pi\Delta_i$ is called an r-stable subring if $a'Aa \subseteq A$ for each $a \in A$, and an analogous theorem holds for r-stable subrings of $\pi\Delta_i$ and r-subcommutative rings.

Clearly, a commutator subring of a division ring Δ is an l-stable subring of Δ.

We shall give an example which shows that an ℓ- and r-stable subring of a division ring Δ need not be a commutator subring.

Let R be a ring with identity e, which is a radical element. For given $a, b \in R$ we have: $ab = e(ab) = c(ba)$ for some $c \in R$. Hence the equation $ab = xa$ always has a solution in R for given $a, b \in R$, so R is ℓ-subcommutative and every ℓ-ideal in R is two-sided. In [4] it is conjectured that the converse also holds, i.e. if R is an ℓ-subcommutative ring with identity e, then R has e as a radical element. We will now give a counterexample to this conjecture.

Let $\mathbf{Q}_2$ be the rational numbers with denominators prime to 2. Let $\mathbf{D}$ be the division algebra of rational quaternions. We will use the notation: $\mathbf{D} = \{(a, b, c, d)\colon a, b, c, d \in \mathbf{Q}\}$, where $(a, b, c, d) = a + bi + cj + dk$ and $\mathbf{Q}$ is the set of rational numbers.

In [2] Lawver characterizes r-stable subrings of $\mathbf{D}$. In the main theorem it is said, among others, that an r-stable non-commutative subring R of $\mathbf{D}$ with identity has rank 4 and has one of the following forms: $R = \mathbf{D}$, $R = \mathbf{D}_2 = \{(a, b, c, d)\colon a, b, c, d \in \mathbf{Q}_2\}$ or $R = R(m) = \{(a, b, c, d)\colon a \in \mathbf{Q}_2, b, c, d \in 2^m \mathbf{Q}_2\}$ for some positive integer m.

In [3] it is shown that $\mathbf{D}_2$ is ℓ-stable in $\mathbf{D}$, hence $\mathbf{D}_2$ is ℓ-subcommutative. Therefore $\mathbf{D}_2$ is both ℓ- and r-stable in $\mathbf{D}$, so both ℓ- and r-subcommutative, or $\mathbf{D}_2$ is a duo ring.

We want to show that the identity $(1, 0, 0, 0) \in \mathbf{D}_2$ is not a radical element for $\mathbf{D}_2$. Choose $x = (0, 2, 0, 2)$ and $y = (0, 0, 2, 2)$ in $\mathbf{D}_2$. Then $xy = (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})\, yx$, but $(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}) \notin \mathbf{D}_2$. So for $x = (0, 2, 0, 2)$, $y = (0, 0, 2, 2) \in \mathbf{D}_2$ there does not exist an element $b \in \mathbf{D}_2$ such that $(1, 0, 0, 0)\, xy = byx$ or $(1, 0, 0, 0)$ is not a radical element. Since $\mathbf{D}_2$ is ℓ-subcommutative, this provides the counterexample.

This also shows that, although $\mathbf{D}_2$ is an ℓ-and r-stable subring of $\mathbf{D}$, it is not a commutator subring, since $xyx^{-1}y^{-1} = (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$ is not in $\mathbf{D}_2$.

In fact, we have the following result:

Theorem 6. *Let R be a subring of $\mathbf{D}$ ($\neq 0, \neq \mathbf{D}$). The the following are equivalent:*

a) *R is a commutator subring of $\mathbf{D}$,*

b) *R is commutative and $e \in R$ (e = identity of $\mathbf{D}$,*

c) *e is a radical element in R.*

Proof.

a)$\Rightarrow$b). Let R be a commutator subring of $\mathbf{D}$. Then $e \in R$ by the definition of commutator subring. Also R is an ℓ-stable subring of $\mathbf{D}$. Although in [2] r-stable subrings of $\mathbf{D}$ are characterized (main theorem), it is clear that the class of ℓ-stable subrings of $\mathbf{D}$ with e coincides with the class of r-stable subrings of $\mathbf{D}$ with e. Suppose that R is non-commutative. Then either $R = \mathbf{D}_2$ or $R = R(m)$ for some positive integer m. But $\mathbf{D}_2$ is not a commutator subring of $\mathbf{D}$, as we have seen, and the same argument can be used with respect to $R(m)$ for any positive integer m. This contradiction implies that R must be commutative.

b) $\to$ c). Clear from the definition of radical element.

c) $\to$ a). See the corollary of Theorem 4. In fact, the equivalence of a) and c) is true for any division ring Δ, which is the content of the corollary of Theorem 4.

§ 3. The Jacobson radical

Our next object is to consider the Jacobson radical of the ring $\mathbf{D}_2$. Let K be the set of all elements in $\mathbf{D}_2$ which do not have inverses in $\mathbf{D}_2$. It can easily be seen that the element $(a, b, c, d) \in \mathbf{D}_2$ $(\neq 0)$ does not have an inverse in $\mathbf{D}_2$ if and only if an even number (0, 2 or 4) of the rationals a, b, c, d have the form $\frac{2p}{q}$, with $p, q \in Z$, q odd, i.e. belong to $2\mathbf{Q}_2$. A straightforward calculation shows that these elements form an ideal in $\mathbf{D}_2$. Then it is well known, that K is the Jacobson radical $\mathbf{J}(\mathbf{D}_2)$ of $\mathbf{D}_2$. As the elements not in K all have inverses in $\mathbf{D}_2$, it follows that $\mathbf{D}_2/\mathbf{J}(\mathbf{D}_2)$ is a division ring and $\mathbf{D}_2$ is a local ring with $\mathbf{J}(\mathbf{D}_2)$ as its unique maximal ideal. In fact, $\mathbf{D}_2/\mathbf{J}(\mathbf{D}_2) \cong \cong \mathbf{Z}_2$, as can easily be checked. It is easy to see that $\mathbf{J}(\mathbf{D}_2)$ can be also characterized as the set of those elements (a, b, c, d) which have a norm $N(a, b, c, d) = a^2 + b^2 + c^2 + d^2$ with even numerator: $\mathbf{J}(\mathbf{D}_2) = \{x \in \mathbf{D}_2 \colon N(x) = \frac{2p}{q}, p, q \in \mathbf{Z}, q \text{ odd}\}$. Now let $a, b \in \mathbf{J}(\mathbf{D}_2)$ with $a \neq 0$. Then $N(a^{-1}ba) = N(b) = \frac{2p}{q}$, hence $a^{-1}ba \in \mathbf{J}(\mathbf{D}_2)$. Therefore $a^{-1}\mathbf{J}(\mathbf{D}_2) a \subseteqq \mathbf{J}(\mathbf{D}_2)$ and similarly $a\mathbf{J}(\mathbf{D}_2)a^{-1} \subseteqq \mathbf{J}(\mathbf{D}_2)$, So $\mathbf{J}(\mathbf{D}_2)$ is an ℓ-and κ-stable subring of $\mathbf{D}$ and Theorem 5 implies that $\mathbf{J}(\mathbf{D}_2)$ is ℓ- and κ-subcommutative. Since both $x = (0, 2, 0, 2)$ and $y = (0, 0, 2, 2)$ are in $\mathbf{J}(\mathbf{D}_2)$, but $xyx^{-1}y^{-1} \notin \mathbf{J}(\mathbf{D}_2)$, $\mathbf{J}(\mathbf{D}_2)$ is not a commutator subring of $\mathbf{D}_2$. Also a commutator subring of a division ring must have an identity and $\mathbf{J}(\mathbf{D}_2)$ does not have an identity.

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