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Modification sets and transforms of discrete measures

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In this paper $\mathbf{T}$ is the circle group and $\mathbf{Z}$ the ring of integers. Let $M(\mathbf{T})$ denote the usual Banach convolution algebra of bounded Borel measures on $\mathbf{T}$; $M_a(\mathbf{T})$ those $\mu \in M(\mathbf{T})$ which are absolutely continuous with respect to Lebesgue measure on $\mathbf{T}$; $M_s(\mathbf{T})$ the set of $\mu \in M(\mathbf{T})$ which are concentrated on sets of Lebesgue measure zero and $M_d(\mathbf{T})$ those $\mu \in M_s(\mathbf{T})$ which are discrete.

The Fourier—Stieltjes coefficients $\hat{\mu}(n)$ of the measure $\mu \in M(\mathbf{T})$ are defined by

$$\hat{\mu}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta}\, d\mu(\theta) \qquad (n \in \mathbf{Z}).$$

A subset E of $\mathbf{Z}$ is called a *modification set* if

$$M_a(\mathbf{T})\hat{}\,|_{E^c} \subset M_s(\mathbf{T})\hat{}\,|_{E^c}. \tag{1}$$

If $S \subset \mathbf{Z}$, then let $\#(S, n)$ be the number of members of S which do not exceed n in modulus. If $\lim\limits_{n\to\infty} \frac{\#(S,n)}{2n}$ exists then we call this limit the *natural density* of S and denote it by $d(S)$.

W. RUDIN in [5] proved the existence of sets $E \subset \mathbf{Z}$ satisfying (1) with arbitrarily small natural density. In [6] RUDIN showed the existence of modification sets with natural density zero.

Using a result of Pigno and Saeki (stated below) we show the existence of arithmetically interesting sets $E \subset \mathbf{Z}$ with arbitrarily small natural density satisfying

$$M_a(\mathbf{T})\hat{}\,|_{E^c} \subset M_d(\mathbf{T})\hat{}\,|_{E^c}. \tag{2}$$

Futhermore, in contrast to Rudin's result we prove that there are no sets E of natural density zero that satisfy (2).

Let $\overline{\mathbf{Z}}$ denote the Bohr compactification of $\mathbf{Z}$ and let E^a denote the set of accumulation points of E which are in $\mathbf{Z}$ (the topology is with respect to $\overline{\mathbf{Z}}$).

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Theorem 1. (PIGNO and SAEKI [4]) *The set $A\subset \mathbf{Z}$ satisfies*

$$M_a(\mathbf{T})\hat{}|_A \subset M_d(\mathbf{T})\hat{}|_A \tag{3}$$

if and only if $A^a\cap A=\emptyset$ and there exists a $\mu\in M(\mathbf{T})$ such that $\hat{\mu}(A)=1$ and $\hat{\mu}(A^a)=0$.

Theorem 2. *Given $\varepsilon>0$ there is a set $E\subset \mathbf{Z}$ such that E has natural density less than ε and E^c satisfies* (2). *This result is best possible.*

To prove the first part of Theorem 2, we will need two lemmas.

Lemma 1. *Let $E\subset \mathbf{Z}^+$ be such that for infinitely many positive integers n, there exists a positive integer l_n, and a finite set E_n such that*

$$n-l_n \to \infty \tag{4}$$

and

$$E\subset E_n\cup \bigcup_{j=0}^{\infty} \big([jn, jn+l_n]\cap[(j+1)n-l_n, (j+1)n]\big) \text{ for each } n. \tag{5}$$

Then E has no limit point in $\mathbf{Z}$ with the relative Bohr topology.

Proof. The proof follows the lines of the proof of Proposition 5 of [3]. For $a<0$, the cited proof shows that a is not a limit point of E. For $a\geqq 0$, we can find an n with $a<n-l_n$ and the proof continues in the same way.

Lemma 2. *Let $b\geqq 2$ be a fixed positive integer. Let $\{n_s\}_{s=1}^{\infty}$ be any increasing sequence of positive integers and let $\{k_s\}_{s=1}^{\infty}$ be any sequence of positive integers with $n_s>k_s$ for all s, and $n_s-k_s\to\infty$. Let E be the set of positive integers t with the property:*

If $t=d_rd_{r-1}\dots d_0$ is the representation of t in the base b and $n_{s_t}\leqq r<n_{s_t+1}$, then for each $s\leqq s_t$ at least one of the digits $d_{ns-1}, d_{ns-2}, \dots, d_{ns-ks}$ is non-zero and at least one of these digits is not $b-1$.

Then E has no limit point in $\mathbf{Z}$ with the relative Bohr topology.

Proof. Given n_s and any $t\in E$, there is some $j\geqq 0$ with $t\in[jb^{n_s}, (j+1)b^{n_s}]$. For this j we have

$$jb^{n_s}+b^{n_s-k_s} \leqq t \leqq (j+1)b^{n_s}-b^{n_s-k_s}.$$

We may now apply Lemma 1 with the b^{n_s}'s playing the role of the n's and with $b^{n_s}(1-b^{-k_s})$ playing the role of l_n.

Proof of Theorem 2. For the first part, we will, given $\varepsilon>0$, find a subset E of $\mathbf{Z}^+$ such that $d(E)>(1-\varepsilon)/2$ and E has no limit point in $\mathbf{Z}$ with the relative Bohr topology. To do this, fix any base $b\geqq 2$ and apply Lemma 2 with $\{n_s\}_{s=1}^{\infty}$, a sufficiently rapidly increasing sequence and $\{k_s\}_{s=1}^{\infty}$, also rapidly increasing with $n_s>k_s$ and

$n_s - k_s \to \infty$. For example, if we take

$$\prod_{s=1}^{\infty} (1-2/b^{k_s})/(1+2/b^{k_s}) > 1-\varepsilon/2$$

and then choose the sequence $\{n_s\}_{s=1}^{\infty}$ to be sufficiently rapidly increasing in terms of our already chosen sequence $\{k_s\}_{s=1}^{\infty}$, then E will have the desired properties. The set $E \cup -E$ satisfies the conclusion of the first part of the present theorem.

For the second part, we begin by observing that any set S for which $\overline{\lim_{n\to\infty}} \frac{\#(S,n)}{2n} = 1$ must contain arbitrarily long blocks of consecutive integers. To conclude our proof we establish the following lemma.

Lemma 3. *Any subset S of $\mathbf{Z}$ which contains arbitrarily long blocks of consecutive integers is dense in $\overline{\mathbf{Z}}$. In fact, $S^a = \overline{\mathbf{Z}}$.*

Proof. First, if U is any neighborhood of 0 in $\overline{\mathbf{Z}}$, then finitely many integer translates of U cover $\overline{\mathbf{Z}}$. If these translates are x_1+U, x_2+U, ..., x_n+U, then set $x = \max(|x_1|, |x_2|, \dots, |x_n|)$. It is now clear that any block of $2x+1$ consecutive integers contains a member of U and we are done.

Note. The sequences of Theorem 2 have a nice arithmetical structure. If one is only interested in density properties, then the derivation of Theorem 2 can be simplified as follows:

Since arithmetic progressions are open sets in $\mathbf{Z}$ with the relative Bohr topology we begin by choosing a thin arithmetic progression containing 0 and, except for 0, delete all members of this arithmetic progression from $\mathbf{Z}$. We now go to the first negative member of this new set and place it in a thin arithmetic progression and again delete all other members of this arithmetic progression from the set just constructed. We next go to the first positive member of the set we now have and continue. If all arithmetic progressions are chosen sufficiently thin, then after the n^{th} step all deleted members of the arithmetic progressions chosen will have absolute value greater than n and it is immediate that our set is well defined. In addition, the sufficient thinness of the arithmetic progressions guarantees that our constructed set has the desired density property. Finally, its lack of a limit point is clear from the construction.

We conclude with the following two results:

Theorem 3. *Let $\mathcal{P} = \{p^k\colon p$ a prime, $k \in \mathbf{Z}^+\}$ be the set of prime powers. Then $\mathcal{P}$ satisfies*

$$M_a(\mathbf{T})\hat{}\,|_{\mathcal{P}} \subset M_d(\mathbf{T})\hat{}\,|_{\mathcal{P}}.$$

Proof. We show that $\mathcal{P}^a \subset \{-1,1\}$. If $n \neq 0, \pm 1$, consider the arithmetic progression $\{2n^2k+n\colon k \in \mathbf{Z}\}$. Since $(n, 2nk+1) = 1$, it follows that $n(2nk+1) = 2n^2k+n \in \mathcal{P}$

is impossible unless $|2nk+1|=1$. But if $|2nk+1|=1$, we have $k=0$ and $2n^2k+n=n$. Thus, the arithmetic progression $\{2n^2k+n: k\in \mathbf{Z}\}$ separates n from $\mathscr{P}\setminus\{n\}$.

Finally, 0 is separated from $\mathscr{P}$ by the arithmetic progression $\{6k: k\in\mathbf{Z}\}$. This shows $\mathscr{P}^a\subset\{-1,1\}$. Thus $\mathscr{P}\cap\mathscr{P}^a=\emptyset$ and our result now follows from Theorem 1.

Let $r\in\mathbf{Z}^+$ with $r\geqq 2$. Set $\mathscr{E}=\{r^k: k\in\mathbf{Z}^+\}$ and put $\mathscr{F}=2\mathscr{E}=\{r^n+r^m: n, m\in\mathbf{Z}^+\}$. Recall that $\mathscr{E}$ is an I_0 set; see for example [1, p.85].

Theorem 4. *The set $\mathscr{F}$ satisfies*

$$M_a(\mathbf{T})\hat{}|_{\mathscr{F}} \subset M_d(\mathbf{T})\hat{}|_{\mathscr{F}}.$$

Proof. We show that $\mathscr{F}^a\subset\{0\}$. For definiteness we shall take $\mathscr{E}=\{2^k: k\in\mathbf{Z}^+\}$. Consider the one point compactification of $\mathscr{E}$ which we realize in the following manner: Put

$$\mathbf{D} = \{e^{2\pi i m/2^n}: m\in\mathbf{Z} \text{ and } n\in\mathbf{Z}^+\}$$

and consider $\mathbf{D}$ as a discrete subgroup of $\mathbf{T}$. We then identify $\mathscr{E}$ with its image in the compact group $\hat{\mathbf{D}}$ (dual to $\mathbf{D}$) in the usual way; see [2, p. 107] and [2, p. 403]. The closure of $\mathscr{E}$ in $\hat{\mathbf{D}}$ is simply $\mathscr{E}\cup\{0\}$. The set of limit points of $\mathscr{F}$ in $\hat{\mathbf{D}}$ is $\{0\}\cup\mathscr{E}$. Since $\hat{\mathbf{D}}$ is a factor group of $\bar{\mathbf{Z}}$ and $\mathbf{D}$ is dense in $\mathbf{T}$ it follows that $\mathscr{F}^a\subset\{0\}\cup\mathscr{E}$.

Fix any 2^k and look at the arithmetic progression $\{3s2^{k+1}+2^k: s\in\mathbf{Z}\}$. Suppose we have $3s2^{k+1}+2^k=2^m+2^n$ $(m\geqq n)$.

Case 1. If $m>n$, then $2^k\|3s2^{k+1}+2^k$ and $2^n\|2^m+2^n$ and so $k=n$ and $3s2^{k+1}=2^m$, which is impossible.

Case 2. If $m=n$, then $3s2^{k+1}+2^k=2^{m+1}$ and since $2^k\|3s2^{k+1}+2^k$ we see that $k=m+1$. Thus, $s=0$ whence $3s2^{k+1}+2^k=2^m+2^n=2^k$.

It now follows from cases 1 and 2 that $\{3s2^{k+1}+2^k: s\in\mathbf{Z}\}$ separates 2^k from $\mathscr{F}\setminus\{2^k\}$. Thus $\mathscr{F}^a\cap\mathscr{F}=\emptyset$ and our result again follows from Theorem 1.

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