Remark on the Jordan model for contractions of class $C_{\cdot 0}$

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In [5] SZ.-NAGY and C. FOIAŞ introduce a relation of complete injective-similarity, which is weaker than quasi-similarity [6; p. 70], and they use it to study operators of finite defect in class $C_{\cdot 0}$. In particular, they show that if Θ and Φ are quasi-equivalent $n \times m$ inner matrices over H^∞, then $S(\Theta)$ and $S(\Phi)$, the compressions of the unilateral shift of multiplicity m to the coinvariant subspaces determined by Θ and Φ, respectively, are completely injection-similar. This result partially extends, in a natural way, the theorem [1] which established, in the case $n=m<\infty$, that Θ and Φ are quasi-equivalent if and only if $S(\Theta)$ and $S(\Phi)$ are quasi-similar.

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Our object in this note is to show that in certain important cases the new relation of injection by a complete family can be replaced by the older and stronger relation of quasi-affine transform (see Theorem 2 and its Corollaries 1). One such case occurs when Φ is the normal form Θ' associated with Θ by the relation of quasi-equivalence; then on one side the result remains that there is a complete family of injections $\{X_1, X_2\}$ such that $S(\Theta)X_j = X_j S(\Theta')$, for $j=1, 2$; whereas, on the other side, our result will give the existence of a quasi-affinity X such that $XS(\Theta) = S(\Theta')X$.

Preliminaries

Let Θ and Φ be $n \times m$ matrices over the Hardy space H^∞ of bounded measurable functions on the unit circle $\boldsymbol{T}$ with vanishing Fourier coefficients of negative index. Such a matrix is called inner if $\Theta^*(e^{it})\Theta(e^{it}) = I_m$ a.e. on $\boldsymbol{T}$, where I_m is the $m \times m$ identity matrix. In this case it necessarily follows that $n \geqq m$. We will assume throughout that n is finite.

Associated with each inner Θ is a Hilbert space $\mathfrak{H}(\Theta)$ and an operator $S(\Theta)$ defined by

$$\mathfrak{H}(\Theta) = H_n^2 \ominus \Theta H_m^2 \quad \text{and} \quad S(\Theta)u = P_\theta(\chi u) \quad \text{for} \quad u \in \mathfrak{H}(\Theta),$$

where H^2_n is the Hardy space of n dimensional (column) vector valued functions on $\boldsymbol{T}$, P_Θ is the orthogonal projection of H^2_n onto $\mathfrak{H}(\Theta)$, and $\chi(z)=z$ for $z\in\boldsymbol{T}$. Operators of this type give canonical functional models for contractions in class $C_{.0}$ with finite defect. For a discussion of this operator class see [6].

A one-to-one operator X from a Hilbert space $\mathfrak{H}_1$ into a Hilbert space $\mathfrak{H}_2$ is called an *injection;* a family $\{X_\alpha\}$ of injections $X_\alpha\colon \mathfrak{H}_1\to\mathfrak{H}_2$ is called *complete* if the closed linear span of the ranges of the X_α's is $\mathfrak{H}_2$. If a complete family of injections consists of but a single operator, then the operator is called a *quasi-affinity.*

Suppose T_1 is an operator on $\mathfrak{H}_1$ and T_2 an operator on $\mathfrak{H}_2$. If there exists a complete injective family $\{X_\alpha\}$ such that $X_\alpha\colon \mathfrak{H}_2\to\mathfrak{H}_1$ and $T_1X_\alpha=X_\alpha T_2$, then T_2 is said to be *injected* into T_1 by $\{X_\alpha\}$, and we write $T_1\overset{ci}{\succ} T_2$. If $\{X_\alpha\}$ is a singleton, then T_2 is called a quasi-affine *transform* of T_1, and we write $T_1\succ T_2$. If $T_1\overset{ci}{\succ} T_2$ and $T_2\overset{ci}{\succ} T_1$ then T_1 and T_2 are said to be *completely injection-similar*, and we denote this by $T_1\overset{ci}{\sim} T_2$. This latter concept is an extension of *quasi-similarity* [6; p. 70], which can be viewed as the case when each family consists of a single quasi-affinity.

Finally we recall the definition of quasi-equivalence for $n\times m$ matrices over H^∞. Again let Θ and Φ be such matrices. Then Θ and Φ are said to be *quasi-equivalent* if for every inner function ω in H^∞ there exist an $n\times n$ matrix Δ and an $m\times m$ matrix Λ over H^∞ such that $\Delta\Theta=\Phi\Lambda$, and $\det\Delta\cdot\det\Lambda$ and ω are relatively prime. See [2] and [7].

A criterion for $S(\Theta)\prec S(\Phi)$

As mentioned in the introduction, Sz.-Nagy and C. Foiaş have shown [5; Theorem 1] that if Θ and Φ are quasi-equivalent $n\times m$ inner matrices over H^∞, then $S(\Theta)$ are completely injection-similar. Further, the two families of injections can always be chosen so as to consist of two operators each, and an example is given to show that a single injection may not suffice.

Before stating our main result, we note that the converse of their theorem is also true. Suppose $S(\Theta)$ and $S(\Phi)$ are completely injection-similar. If Θ' and Φ' are the quasi-equivalent normal forms[1)] [2] of Θ and Φ, respectively, then $S(\Theta')\overset{ci}{\sim} S(\Theta)\overset{ci}{\sim} S(\Phi)\overset{ci}{\sim} S(\Phi')$. Further, it was shown in [5; Theorem 3] that $S(\Theta)$ is injection-

[1)] The normal matrix corresponding to an $n\times m$ matrix Θ over H^∞ is the $n\times m$ matrix that has the j^{th} invariant factor θ_j of Θ in position jj for $1\leq j\leq m$ and zeros elsewhere. The invariant factor θ_j is the quotient δ_j/δ_{j-1} if $\delta_{j-1}\neq 0$, and 0 if $\delta_{j-1}=0$, where, $\delta_0=1$ and δ_j is the greatest common inner divisor of the j^{th} order minors of Θ.

similar to a unique Jordan operator; therefore, $S(\Theta')=S(\Phi')$, and hence $\Theta'=\Phi'$. Thus Θ and Φ are quasi-equivalent.

The following theorem gives our criterion.

Theorem 1. *Suppose θ and Φ are $n\times m$ matrices over H^∞. Necessary and sufficient conditions that $S(\theta)\prec S(\Phi)$ are that there exist square matrices Δ and Λ over H^∞ which statisfy*

(1) $\Delta\Theta = \Phi\Lambda$,

(2) $\ker[\Delta\ \Phi] \subseteqq \Theta H^2_m \oplus H^2_m$,

(3) $[\Delta\ \Phi]H^2_{n+m}$ *is dense in* H^2_n.

Remark. By $[\Delta\ \Phi]$ we mean the $n\times(n+m)$ matrix over H^∞ made up of the columns of Δ followed by those of Φ. By an abuse of notation (as in (2) above) we identify this matrix with the analytic Toeplitz operator from H^2_{n+m} to H^2_n that it induces. Similarly we will identify any matrix over H^∞ with an analytic Toeplitz operator when convenient.

Proof. Suppose there exists a quasi-affinity X from $\mathfrak{H}(\Theta)$ to $\mathfrak{H}(\Phi)$ such that

$$XS(\Theta) = S(\Phi)X.$$

By the lifting theorem (see [3] for the case of scalar $\Theta=\Phi$ and [6; p. 258] for the general case), there exists an $n\times n$ matrix Δ over H^∞ such that

$$X = P_\Phi\Delta|\mathfrak{H}(\Theta)$$

and $\Delta\Theta H^2_m\subseteqq\Phi H^2_n$. The latter condition is equivalent to the existence of a Λ satisfying (1).

Property (2) is most easily established by noting its equivalence to

(2') if $f\in H^2_n$ and $\Delta f\in H^2_m$, then $f\in\Theta H^2_m$.

To establish (2') suppose $f\in H^2_n$ and $\Delta f\in\Phi H^2_m$. Write $f=u+\Theta h$, where $u\in\mathfrak{H}(\Theta)$ and $h\in H^2_m$, and apply (1) to obtain

$$\Delta f = \Delta u+\Phi\Lambda h.$$

Since $\Delta f\in\Phi H^2_n$, an application of P_Φ yields

$$Xu = P_\Phi\Delta u = 0.$$

By the injectivity of X, $u=0$, and thus $f=\Theta h$, which establishes (2').

As for (3), the fact that X is a quasi-affinity implies $X\mathfrak{H}(\Theta)+\Phi H^2_m$ is dense in H^2_n. Since $Xf=P_\Phi\Delta f$, it follows that $X\mathfrak{H}(\Theta)+\Phi H^2_m$ is included in $[\Delta\ \Phi]H^2_{n+m}$. Therefore (3) holds.

Conversely, if there exists an $n\times n$ matrix Δ satisfying (1), (2), and (3), then define X to be $P_\Phi \Delta|\mathfrak{H}(\Theta)$. The argument that X is a quasi-affinity and satisfies $XS(\Theta)=S(\Phi)X$, is straightforward.

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The following two lemmas essentially form the key ingredients in the proof of the injectivity part of Theorem 1 in [5]. We include them here for easy reference.

Lemma 1. *A sufficient condition for* (2) *to hold is the existence of Δ and Λ having determinants which are nonzero a.e. on* $\boldsymbol{T}$, *satisfying* (1), *and such that if $g\in L^2_m$, $\Lambda g\in H^2_m$ and $\Theta g\in H^2_n$, then $g\in H^2_m$.*

Proof. The question is: does $f\in H^2_n$ and $\Delta f\in \Phi H^2_m$ imply $f\in \Theta H^2_m$? Suppose h in H^2_m is such that $\Delta f=\Phi h$. Since the determinants of Δ and Λ are nonzero a.e., both Δ^{-1} and Λ^{-1} exist a.e. on T. Consequently, the following relations hold pointwise a.e. on $\boldsymbol{T}$:

$$\Delta f = \Phi\Lambda\Lambda^{-1}h = \Delta\Theta(\Lambda^{-1}h).$$

Thus

$$f = \Theta(\Lambda^{-1}h),$$

which implies $\Lambda^{-1}h\in L^2_m$, since $f\in H^2_m$ and Θ is isometric a.e. If $g=\Lambda^{-1}h$, then g satisfies the hypothesis, and hence $g\in H^2_m$. But $f=\Theta g$, and hence the answer to our question is yes.

Lemma 2. *A sufficient condition for* (2) *to hold is the existence of Δ and Λ satisfying* (1) *such that Δ has a determinant which is nonzero a.e. and Λ has a determinant relatively prime to the greatest common divisor of the $m\times m$ minors of Θ.*

Proof. By Lemma 1 it suffices to show that if $g\in L^2_m$, $\Lambda g\in H^2_m$ and $\Theta g\in H^2_n$, then $g\in H^2_m$. If the classical adjoint of Λ is applied to Λg, then we see that $(\det \Lambda)g$ is in H^2_m. For any $m\times m$ submatrix Θ_α of Θ, we have $\Theta_\alpha g\in H^2_m$, and consequently $(\det \Theta_\alpha)g\in H^2_m$. Since $\det \Lambda$ and the collection of all $m\times m$ minors of Θ form a relatively prime set, the conclusion follows from a lemma of Sz.-Nagy [4; p. 74].

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On the basis of Lemma 2 we can obtain from Theorem 1:

Theorem 2. *Suppose Θ and Φ are quasi-equivalent $n\times m$ inner matrices over H^∞. If the rows of Φ span an m-dimensional subspace of H^2_m, then $S(\Theta)\prec S(\Phi)$.*

Proof. Select Δ_1 and Λ satisfying (1) such that each of their determinants is relatively prime to all the invariant factors of Θ and Φ.

By hypothesis, elementary row operations with complex scalars can be used to replace the last $n-m$ rows of Φ by rows of zeros, i.e. there exists an invertible $n\times n$ matrix A over $\mathbf{C}$ such that $A\Phi$ has the form $\begin{bmatrix}\Phi_1\\ 0\end{bmatrix}$ where Φ_1 is an $m\times m$ matrix over H^∞, and 0 is the $(n-m)\times m$ zero matrix. Let Δ_0 be the $(n-m)\times n$ matrix formed by the last $n-m$ rows of $A\Delta_1$. The closure $\mathfrak{M}$ of $\Delta_0 H_n^2$ is a full invariant subspace of the unilateral shift in H^2_{n-m}. (It is full since $\det\Delta_1\neq 0$ implies that at least one $(n-m)\times(n-m)$ minor of Δ_0, say δ, is nonzero. Hence $\mathfrak{M}$ includes δH^2_{n-m}.) Thus there exists an inner $(n-m)\times(n-m)$ matrix Ψ such that $\mathfrak{M}=\Psi H^2_{n-m}$. Set

$$\Delta = A^{-1}(I_m\oplus\Psi^*)A\Delta_1.$$

Then Δ is analytic since $\Psi^*\Delta_0$ is analytic, and from $(I_m\oplus\Psi^*)A\Phi=A\Phi$ we obtain

$$\Delta\Theta = A^{-1}(I_m\oplus\Psi^*)A\Delta_1\Theta = A^{-1}(I_m\oplus\Psi^*)A\Phi\Lambda = A^{-1}A\Phi\Lambda = \Phi\Lambda.$$

Thus Δ and Λ satisfy (1). From the definition of Ψ and Λ, we see that $\det\Delta$ divides $\det\Delta_1$, and thus $\det\Delta$ is relatively prime to the invariant factors of Θ and Φ.

Condition (2) now follows from Lemma 2. We shall show that $[\Delta\ \Phi]$ satisfies (3) by showing that if $\mathfrak{N}=[A\Delta\ A\Phi]H^2_{n+m}$, then $\mathfrak{N}$ is dense in H^2_n; this is equivalent because of the invertibility of A. It is convenient to regard H_n^2 as the direct sum $H^2_m\oplus H^2_{n-m}$. Note that $A\Delta H^2_n$ includes $(\det\Delta)H^2_n$, which in turn includes $(\det\Delta)H^2_m\oplus\{0\}$, and also $A\Phi H^2_m$ includes $(\det\Phi_1)H^2_m\oplus\{0\}$. Hence $\mathfrak{N}$ includes the sum of the two manifolds $(\det\Delta)H^2_m\oplus\{0\}$ and $(\det\Phi_1)H^2_m\oplus\{0\}$. But $\det\Delta$ and $\det\Phi_1$ are relatively prime, and thus Beurling's theorem implies that $\overline{\mathfrak{N}}$ includes $H^2_m\oplus\{0\}$. From the fact that $\mathfrak{N}$ includes $A\Delta H^2_n$, it now follows that $\overline{\mathfrak{N}}$ also includes $\{0\}\oplus\Psi^*\Delta_0 H^2_n$, and hence $\overline{\mathfrak{N}}\supset\{0\}\oplus\Psi^*\mathfrak{M}=\{0\}\oplus H^2_{n-m}$. Thus $\overline{\mathfrak{N}}=H^2_n$.

Corollary 1. *If Θ is $n\times m$ inner and Θ' is its normal form, then $S(\Theta)\prec S(\Theta')$.*[2]

Proof. Immediate from Theorem 2.

Finally, for any operator T on a Hilbert space $\mathfrak{H}$ the multiplicity μ_T is defined to be the minimal cardinality of a set $\mathfrak{M}$ in $\mathfrak{H}$ such that

$$\mathfrak{H} = \bigvee_{j=0}^{\infty} T^j\mathfrak{M}.$$

In [5; Proposition 3] it is shown, in particular, that if Θ' is the normal form of Θ, then

$$\mu_{S(\Theta)} \leqq 2\mu_{S(\Theta')}.$$

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[2] In the special case that Θ is also *-outer (and hence $S(\Theta)\in C_{10}$) this result is contained in [5], Corollary 2.

This follows from a general observation that if $T_1 \overset{ci}{\succ} T_2$ and if $X=\{X_\alpha\}$ is a corresponding complete system of injections, then $\mu_{T_1} \leqq (\text{card}(X)) \cdot \mu_{T_2}$.

By Corollary 1 we can add the following to Proposition 3 of [5].

Corollary 2. *If Θ is $n \times m$ inner over H^∞ and Θ' is its quasi-equivalent normal form, then*

$$\mu_{S(\Theta')} \leqq \mu_{S(\Theta)} \leqq 2\mu_{S(\Theta')}.$$

Proof. Proposition 3 of [5] and Corollary 1.

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