Note on integral inequalities

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In [1] we proved the integral inequality

$$\int_{-\infty}^{\infty} \sup_{x/p+y/q=t} f(x)g(y)\,dt \geqq \Big(\int_{-\infty}^{\infty} f^p(x)\,dx\Big)^{1/p}\Big(\int_{-\infty}^{\infty} g^q(x)\,dx\Big)^{1/q} \tag{1}$$

for arbitrary non-negative measurable functions $f(x)$, $g(x)$ and for fixed p and q satisfying the conditions $1 \leqq p, q \leqq \infty$ and $1/p+1/q=1$, assuming that the left-hand side of (1) has sense.

Setting $F(x,y)=f(x)g(y)$ (1) can be written in the form

$$\int_{-\infty}^{\infty} \sup_{x/p+y/q=t} F(x,y)\,dt \geqq \Big(\int_{-\infty}^{\infty}\Big\{\int_{-\infty}^{\infty} F^p(x,y)\,dx\Big\}^{q/p} dy\Big)^{1/q}. \tag{2}$$

Professor B. SZ.-NAGY raised the problem whether inequality (2) holds for an arbitrary non-negative measurable function $F(x,y)$ of two variables. The answer to this question is negative. A counter-example is yielded, say in the case $p=q=2$, by the function

$$F_1(x,y)=\begin{cases}1 & \text{if } \ 0\leqq x\leqq 3 \ \text{ and } \ -x+2\leqq y\leqq -x+3,\\ 0 & \text{otherwise.}\end{cases}$$

Indeed, straightforward computation yields, in this case, the value 1/2 for the left-hand side, and the value $\sqrt{5/2}$ for the right-hand side, of (2).

However, instead of (2) we can prove the inequality

$$\max_{\alpha} \int_{-\infty}^{\infty} \sup_{\alpha x+(1-\alpha)y=t} F(x,y)\,dt \geqq \Big(\int_{-\infty}^{\infty}\Big\{\int_{-\infty}^{\infty} F^p(x,y)\,dx\Big\}^{q/p} dy\Big)^{1/q},$$

where α runs over the two-point set $\{0, 1\}$.

More generally, we have the following

Theorem. *Let $f(x_1, x_2, \ldots, x_m)$ be a non-negative measurable function and set*

$$J=\max J_i$$

where

$$J_i = \int_{-\infty}^{\infty} S_i(x_i)\, dx_i, \quad S_i(x_i) = \sup_{x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_m} f(x_1, x_2, \dots, x_m)$$

$$(i = 1, 2, \dots, m).$$

Then we have

$$(3) \quad \left[\int_{-\infty}^{\infty} \left(\dots \left(\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f^{p_1}(x_1, \dots, x_m)\, dx_1 \right)^{p_2/p_1} dx_2 \right)^{p_3/p_2} dx_3 \dots \right)^{p_m/p_{m-1}} dx_m \right]^{1/p_m} \leqq J$$

for arbitrary numbers $p_1, p_2, \dots, p_m (\geqq 1)$ *with* $\sum_{i=1}^{m} 1/p_i = 1$.

Proof. It is clear that

$$J = \prod_{i=1}^{m} J^{1/p_i} \geqq \prod_{i=1}^{m} J_i^{1/p_i} = \left(\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} S_1(x_1)\, dx_1 \right)^{p_2/p_1} S_2(x_2)\, dx_2 \right)^{1/p_2} \prod_{i=3}^{m} J_i^{1/p_i} =$$

$$= \left(\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} S_1(x_1)\, S_2^{p_1/p_2}(x_2)\, dx_1 \right)^{p_2/p_1} dx_2 \right)^{1/p_2} \prod_{i=3}^{m} J_i^{1/p_i} =$$

$$= \left(\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} S_1(x_1)\, S_2^{p_1/p_2}(x_2)\, dx_1 \right)^{p_2/p_1} dx_2 \right)^{p_3/p_2} S_3(x)\, dx_3 \right)^{1/p_3} \prod_{i=4}^{m} J_i^{1/p_i} =$$

$$= \left(\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} S_1(x_1)\, S_2^{p_1/p_2}(x_2)\, S_3^{p_1/p_3}(x_3)\, dx_1 \right)^{p_2/p_1} dx_2 \right)^{p_3/p_2} dx_3 \right)^{1/p_3} \prod_{i=4}^{m} J_i^{1/p_i}.$$

It is easy to see that repeating this procedure we arrive at the inequality

$$J \geqq \left[\int_{-\infty}^{\infty} \left(\dots \left(\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} S_1(x_1)\, S_2^{p_1/p_2}(x_2)\, S_3^{p_1/p_3}(x_3) \cdot \dots \right.\right.\right.\right.$$

$$\left.\left.\left.\left. \dots \cdot S_m^{p_1/p_m}(x_m)\, dx_1 \right)^{p_2/p_1} dx_2 \right)^{p_3/p_2} \dots \right)^{p_m/p_{m-1}} dx_m \right]^{1/p_m}.$$

Since

$$1 + p_1/p_2 + \dots + p_1/p_m = p_1$$

and

$$S_i(x_i) \geqq f(x_1, x_2, \dots, x_m),$$

we thus get (3). This completes the proof.

Note that our theorem can be generalized from *the space R* to the space R^n. To prove this we have only to write $\mathbf{x}_i \in R^n$ instead of $x_i \in R$ throughout the proof.

Reference

[1] L. Leindler, On a certain converse of Hölder's inequality. II, *Acta Sci. Math.*, 33 (1972), 217—223.

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