On the strong stability by Lyapunov's direct method

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Introduction

Consider the ordinary differential equation

$$\dot{x} = X(t, x), \tag{E}$$

where $t \in I = [0, \infty)$, and x belongs to the Euclidean n-space R^n. The function $X(t, x)$: $:\Gamma \to R^n$,

$$\Gamma = \{(t, x): t \in I, \|x\| < H\} \quad (0 < H = \text{const.}),$$

is continuous together with its first partial derivatives with respect to every component of x.

The unique solution of (E) through the point (t_0, x_0) denoted by $x(t) = x(t; t_0, x_0)$ is supposed to exist in I, provided that $\|x_0\|$ is sufficiently small. In addition, assume that $X(t, 0) \equiv 0$ for $t \in I$, i.e. $x = 0$ is a solution of (E), called the zero solution.

Recall first the following classical stability concepts. The zero solution of (E) is said to be

(i) *stable:* if given any $\varepsilon > 0$ and any $t_0 \in I$ there exists a $\delta(\varepsilon, t_0) > 0$ such that $\|x_0\| < \delta(\varepsilon, t_0)$ implies $\|x(t; t_0, x_0)\| < \varepsilon$ for $t \geqq t_0$.

(ii) *uniformly stable:* if given any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $t_0 \in I$, $\|x_0\| < \delta(\varepsilon)$ imply $\|x(t; t_0, x_0)\| < \varepsilon$ for $t \geqq t_0$.

(iii) *asymptotically stable:* if it is stable and if given any $t_0 \in I$ there exists a $\sigma(t_0) > 0$ such that $\|x_0\| < \sigma(t_0)$ implies $\|x(t; t_0, x_0)\| \to 0$ as $t \to \infty$.

In this paper another type of stability, the so called strong stability will be investigated.

Definition 1. The zero solution of (E) is said to be *strongly stable* if given any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $t_0 \in I$, $\|x_0\| < \delta(\varepsilon)$ imply $\|x(t; t_0, x_0)\| < \varepsilon$ for $t \in I$.

The concept of the strong stability was introduced by G. Ascoli [1] for linear systems. Def. 1 is taken from W. A. Coppel's monograph [2]. Obviously strong stability implies uniform stability, which, in turn, implies ordinary stability. Furthermore, the strong stability and the asymptotic stability are always incompatible.

At present some criteria for this type of stability are known for linear systems and for nonlinear systems of the form $\dot{x}=A(t)x+R(t,x)$, where $R(t,x)$ is small in a sense (see [2, p. 66]).

In general, the investigation of a given stability property by Lyapunov's direct method is based on principal theorems of the following type: the existence of a function $V(t,x)\colon \Gamma\to R$ with certain properties implies the desired stability property. In Sec. 1 we establish such a principal theorem for the strong stability, and we prove also the converse of this theorem. In Sec. 2 we give a sufficient condition for the strong stability by means of differential inequalities. This condition in several important cases may be easier to apply then the previous theorem. This can be seen in Sec. 3, where it is applied to the study of perturbed nonlinear differential equations and rheonomic mechanical systems under the action of potential forces

1. Lyapunov functions and the strong stability

According to the notations of W. Hahn's monograph [3], we shall say that *a function* $a(r)\colon[0,H)\to R$ *belongs to class* K $(a(r)\in K)$ if it is continuous, strictly increasing on $[0,H)$ and $a(0)=0$.

Definition 1.1. A function $V(t,x)\colon\Gamma\to R$ having continuous first partial derivates in Γ, is said to be a *Lyapunov function* if $V(t,0)\equiv 0$ for $t\in I$, and $V(t,x)$ is positive definite i.e. there exists a function $a(r)\in K$ such that $V(t,x)\geqq a(\|x\|)$ holds for $t\in I$ and for all x belonging to a certain ball $S_\lambda=\{x\colon \|x\|<\lambda\}$ $(\lambda>0)$.

For every Lyapunov function $V(t,x)$ define the function

$$\dot{V}(t,x)=\sum_{i=1}^{n}\frac{\partial V(t,x)}{\partial x_i}X_i(t,x)+\frac{\partial V(t,x)}{\partial t}$$

which is said to be the *total derivative of* $V(t,x)$ *by virtue of equation* (E). It is easy to see that for every solution $x(t)$ of (E)

$$\frac{d}{dt}V(t,x(t))\equiv\dot{V}(t,x(t))\qquad(t\in I). \tag{1.1}$$

Theorem 1.1. *The zero solution of* (E) *is strongly stable if and only if there exists a Lyapunov function* $V(t,x)$ *having the following properties:*

(1) $V(t,x)\to 0$ *as* $x\to 0$ *uniformly on* I;

(2) $\dot{V}(t,x)\equiv 0$ $((t,x)\in\Gamma)$.

Proof. *Sufficiency.* By the assumptions there are functions $a(r)$, $b(r)\in K$ such that

$$a(\|x\|) \leqq V(t, x) \leqq b(\|x\|) \tag{1.2}$$

for $t\in I$ and $\|x\| \leqq \lambda < H$, where λ is an appropriate positive constant. Given $\varepsilon\,(0<\varepsilon<\lambda)$, let $\delta(\varepsilon)>0$ be chosen so that $a(\varepsilon)>b(\delta(\varepsilon))$. Let $x(t;\, t_0, x_0)$ be a solution of (E) with $\|x_0\|<\delta(\varepsilon)$. Then, from (1.1), (1.2) and property (2), we get $a(\|x(t;\, t_0, x_0)\|)\leqq \leqq V(t, x(t;\, t_0, x_0))=V(t_0, x_0)\leqq b(\|x_0\|)\leqq b(\delta(\varepsilon))<a(\varepsilon)$. Hence $\|x(t;\, t_0, x_0)\|<\varepsilon$ for $t\in I$, i.e. the zero solution is strongly stable.

Necessity. Suppose that the zero solution is strongly stable. Then we shall prove that $V(t, x)=\|x(0;\, t, x)\|$ is a Lyapunov function with properties (1), (2).

The continuity of the partial derivatives of $V(t, x)$ follows from the smoothness of the right hand side of (E). Furthermore, to prove that the function $V(t, x)$ is positive definite it is sufficient to show that for every $\gamma>0$

$$m_\gamma = \inf \{V(t, x) : t\in I,\ \|x\|\geqq\gamma\}>0.$$

Indeed, $m_\gamma\geqq\delta(\gamma)>0$ where $\delta(\gamma)$ corresponds to γ in the sense of Def. 1. Assuming the contrary, we have $(\bar t, \bar x)\in\Gamma\ (\|\bar x\|\geqq\gamma)$ such that $\|x(0;\, \bar t, \bar x)\|=V(\bar t, \bar x)<\delta(\gamma)$. Then, according to Def. 1, we have the estimation $\|x(t;\, \bar t, \bar x)\|<\gamma$ for $t\in I$, which contradicts the inequality $\|x(\bar t;\, \bar t, \bar x)\|=\|\bar x\|\geqq\gamma$. Consequently, $V(t, x)$ is a Lyapunov function.

Given $\varepsilon>0$ choose $\delta(\varepsilon)$ in the sense of Def. 1. If $\bar t\in I$ and $\|\bar x\|<\delta(\varepsilon)$, then $\|x(t;\, \bar t, \bar x)\|<\varepsilon$ for $t\in I$. Consequently, the inequality $\|\bar x\|<\delta(\varepsilon)$ implies $\|x(0;\, \bar t, \bar x)\|= =V(\bar t, \bar x)<\varepsilon$ for $\bar t\in I$, which proves (1).

By (1.1) and the uniqueness of the solutions we have

$$\dot V(t, x) = \left[\frac{d}{d\tau} V(\tau, x(\tau;\, t, x))\right]_{\tau=t} = \left[\frac{d}{d\tau}\|x(0;\, t, x)\|\right]_{\tau=t} = 0$$

for all (t, x). Thus $V(t, x)$ has the property (2).

The theorem is proved.

This theorem is analogous to — but evidently independent of — K. P. PERSIDSKIĬ's well know theorem regarding the uniform stability (see [8]).

2. Differential inequalities and the strong stability

We begin by recalling two lemmas from the theory of differential inequalities (cf. [4]).

Lemma 2.1. *Suppose that the functions* $\omega_1(t, u):[t_0-T, t_0]\times\Omega\to R$, $\omega_2(t, v):[t_0, t_0+T]\times\Omega\to R$ *are continuous, where* Ω *is an open interval in* R*;* t_0 *and* T *are positive constants. Let* $u^*(t)$, $v^*(t)$ *be the maximal solutions of the initial value problems*

$$\begin{cases} \dot{u} = \omega_1(t, u) \\ u(t_0) = \xi \end{cases} \qquad (t_0 - T \leqq t < t_0;\ \xi\in\Omega),$$

$$\begin{cases} \dot{v} = \omega_2(t, v) \\ v(t_0) = \xi \end{cases} \qquad (t_0 < t \leqq t_0 + T)$$

in $[t_0-T, t_0]$, $[t_0, t_0+T]$ *respectively.*

If the continuously differentiable function $w(t):[t_0-T, t_0+T]\to R$ *satisfies the inequalities* $w(t_0)\leqq\xi$;

$$\dot{w}(t) \geqq \omega_1(t, w(t)) \quad (t_0 - T\leqq t \leqq t_0),$$

$$\dot{w}(t) \leqq \omega_2(t, w(t)) \quad (t_0 \leqq t \leqq t_0 + T),$$

then $w(t)\leqq u^*(t)$ *for* $t\in[t_0-T, t_0]$ *and* $w(t)\leqq v^*(t)$ *for* $t\in[t_0, t_0+T]$.

Lemma 2.2. *Suppose that the function* $\omega(t, u_1, u_2):[t_0-T, t_0+T]\times\Omega_1\times\Omega_2\to R$ *is continuous and nondecreasing in* u_1, *where* Ω_1 *and* Ω_2 *are open intervals in* R; t_0 *and* T *are positive constants. Let* $u^*(t)$ *be the maximal solution of the initial value problem*

$$\begin{cases} \ddot{u} = \omega(t, u, \dot{u}) \quad (t_0 - T \leqq t \leqq t_0 + T), \\ u(t_0) = \xi,\ \dot{u}(t_0) = \eta \quad (\xi\in\Omega_1,\ \eta\in\Omega_2) \end{cases}$$

in $[t_0-T, t_0+T]$.

If the twice continuously differentiable function $w(t):[t_0-T, t_0+T]\to R$ *satisfies the conditions* $w(t_0)\leqq\xi$, $\dot{w}(t_0)=\eta$;

$$\ddot{w}(t) \leqq \omega(t, w(t), \dot{w}(t)) \quad (t_0 - T\leqq t \leqq t_0 + T),$$

then $w(t)\leqq u^*(t)$ *for* $t\in[t_0-T, t_0+T]$.

To formulate the main theorem of this section we have need of the following stability concept:

Definition. 2.1. The zero solution of (E) is said to be *uniformly stable at the right (at the left)* if for every $\varepsilon>0$ there exists a $\delta(\varepsilon)>0$ such that $t_0\in I$ and $\|x_0\| < \delta(\varepsilon)$ imply $\|x(t; t_0, x_0)\|<\varepsilon$ for all $t\geqq t_0$ (for all $t\in[0, t_0]$).

The concept of the uniform stability at the right corresponds to the concept of the uniform stability introduced by K. P. PERSIDSKIĬ ([see [8]). Here we give a necessary and sufficient condition for the uniform stability at the left.

Lemma 2.3. *The zero solution of (E) is uniformly stable at the left if and only if for every $\varepsilon>0$ the inequality*

$$\gamma(\varepsilon) = \inf\{\|x(t;\, t_0,\, x_0)\| : t_0\in I,\, t\geqq t_0,\, \|x_0\| = \varepsilon\} > 0 \tag{2.1}$$

holds.

Proof. *Necessity.* Suppose that the zero solution is uniformly stable at the left. We shall prove that for every $\varepsilon>0$ the inequality $\gamma(\varepsilon)\geqq\delta(\varepsilon)>0$ holds, where $\delta(\varepsilon)$ corresponds to ε in the sense of Def. 2.1. Assuming the contrary, we have $\varepsilon_0>0$ such that $\gamma(\varepsilon_0)<\delta(\varepsilon_0)$. Then, according to (2.1), there are $\bar{t}_0$, $\bar{t}$, $\bar{x}_0$ such that $\bar{t}_0\in I$, $\bar{t}\geqq\bar{t}_0$, $\|\bar{x}_0\|=\varepsilon_0$ and $\|x(\bar{t};\, \bar{t}_0,\, \bar{x}_0)\|<\delta(\varepsilon_0)$. Hence, by virtue of Def. 2.1, we obtain $\varepsilon_0= =\|\bar{x}_0\|=\|x(\bar{t}_0;\, \bar{t},\, x(\bar{t};\, \bar{t}_0,\, \bar{x}_0))\|<\varepsilon_0$ which is a contradiction.

Sufficiency. Let $\gamma(\varepsilon)>0$ for every $\varepsilon>0$. We shall prove that the zero solution is uniformly stable at the left, namely given $\varepsilon>0$ the inequality $\|x_0\|<\gamma(\varepsilon)$ implies $\|x(t;\, t_0,\, x_0)\|<\varepsilon$ for all $t_0\in I$, $t\in[0,\, t_0]$.

Suppose that this is not true. Then there exist ε_0, $\bar{t}_0$, $\bar{t}$, $\bar{x}_0$ $(\varepsilon_0>0,\ 0\leqq\bar{t}<\bar{t}_0,$ $\|\bar{x}_0\|<\gamma(\varepsilon_0))$ such that $\|x(\bar{t};\, \bar{t}_0,\, \bar{x}_0)\|=\varepsilon_0$. By (2.1) we get $\|\bar{x}_0\|=\|x(\bar{t}_0;\, \bar{t},\, x(\bar{t};\, \bar{t}_0,\, \bar{x}_0))\|\geqq \geqq\gamma(\varepsilon_0)$ in contradiction with the assumption $\|\bar{x}_0\|<\gamma(\varepsilon_0)$.

The lemma is proved.

Remark 2.1. Lemma 2.3 shows that uniform stability at the left and asymptotic stability are always incompatible.

Comparing Def. 1 with Def. 2.1, we can easily obtain the following

Lemma 2.4. *The zero solution of (E) is strongly stable if and only if it is uniformly stable at the right and at the left, simultaneously.*

Remark 2.2. Let us now consider the linear system

$$\dot{x} = A(t)x, \tag{2.2}$$

where $A(t)$ is a square matrix whose elements are continuous functions for $t\in I$. Denote by $\Phi(t)$ the fundamental matrix of (2.2) with $\Phi(0)=E$, where E is the unit matrix. It is easy to see that in this case (2.1) becomes

$$\gamma(\varepsilon) = \inf\{\varepsilon\|\Phi(t_0)\Phi^{-1}(t)\|^{-1}:\ t_0\geqq 0,\ t\geqq t_0\} > 0.$$

Consequently, the zero solution of (2.2) is uniformly stable at the left if and only if the function $\|\Phi(t)\Phi^{-1}(s)\|$ is bounded on the set $0\leqq t\leqq s<\infty$.

Moreover, it is well known [2] that the zero solution of (2.2) is uniformly stable at the right if and only if the function $\|\Phi(t)\Phi^{-1}(s)\|$ is bounded on the set $0\leqq s\leqq t<\infty$.

Thus, the zero solution of (2.2) is strongly stable if and only if the functions $\|\Phi(t)\|$, $\|\Phi^{-1}(t)\|$ are bounded for $t\in I$ (cf. [2]), i.e. if and only if the zero solution of (2.2) is stable together with the zero solution of the adjoint system. (This latter property served originally as the definition of the strong stability for the linear system (see [1]).

Example 2.1. Let us consider the equation

$$\dot{u} = f(t)g(u) \quad (t\in I,\ u \geqq 0), \tag{2.3}$$

where the functions $f(t): I\to R$ and $g(u): I\to I$ are continuous; $g(0)=0$, $g(u)>0$ for $u>0$ and $\int_0^\eta (g(u))^{-1}\,du=\infty$ $(0<\eta=\text{const.})$. Let $G(u;\, u_0):(0,\infty)\to(0,\infty)$ be the inverse of the function $\int_{u_0}^{u} (g(s))^{-1}\,ds$ $(u_0>0)$. Then nontrivial solutions of (2.3) are given by the expression

$$u(t;\, t_0, u_0) = G\Big(\int_{t_0}^{t} f(s)\,ds;\, u_0\Big). \tag{2.4}$$

Using (2.4), by Lemmas 2.3 and 2.4 it is easy to prove the following statement: *The zero solution of* (2.3) *is strongly stable if and only if*

$$\limsup_{t\to\infty}\Big|\int_0^t f(s)\,ds\Big| < \infty.$$

Having these concepts and preliminary results, we state the following

Theorem 2.1. *Assume that there exists a Lyapunov function $V(t, x)$ satisfying the following conditions on Γ:*

(1) *$V(t, x)\to 0$ as $x\to 0$ uniformly on I;*

(2) *$\omega_1(t, V(t, x))\leqq \dot{V}(t, x)\leqq \omega_2(t, V(t, x))$, where the functions $\omega_1(t, u)$, $\omega_2(t, v)$: $I\times I\to R$ are continuous and $\omega_1(t, 0)\equiv\omega_2(t, 0)\equiv 0$ for $t\in I$;*

(3) *the zero solution $u=0$ $(v=0)$ of the equation $\dot{u}=\omega_1(t, u)$ $(\dot{v}=\omega_2(t, v))$ is uniformly stable at the left (at the right).*
Then the zero solution of (E) is strongly stable.

Proof. By the assumptions there exist functions $a(r)$, $b(r)\in K$ such that

$$a(\|x\|) \leqq V(t, x) \leqq b(\|x\|) \tag{2.5}$$

for all $t\in I$, $\|x\|<\lambda<H$, where $\lambda>0$ is an appropriate constant. Let ε $(0<\varepsilon<\lambda)$ be given. According to assumption (3), there exists a $\varkappa(\varepsilon)>0$ such that $0\leqq\eta<\varkappa(\varepsilon)$

implies

$$(2.6)\qquad \begin{aligned} &0 \leqq u^*(t;\, t_0,\, \eta) < a(\varepsilon) \quad (t_0 \in I,\ 0 \leqq t \leqq t_0),\\ &0 \leqq v^*(t;\, t_0,\, \eta) < a(\varepsilon) \quad (t_0 \in I,\ t_0 \leqq t < \infty), \end{aligned}$$

where $u^*(t;\, t_0,\, \eta)$ $(v^*(t;\, t_0,\, \eta))$ is the maximal solution of the equation $\dot{u} = \omega_1(t, u)$ $(\dot{v} = \omega_2(t, v))$, passing through the point (t_0, η).

Let now $\delta(\varepsilon) > 0$ be chosen in such a way that $b(\delta(\varepsilon)) < \varkappa(\varepsilon)$. Further let $x(t)$ be a solution of (E) satisfying $\|x(t_0)\| < \delta(\varepsilon)$ for some $t_0 \in I$. Then, in view of (2.5), $V(t_0, x(t_0)) < \varkappa(\varepsilon)$. Applying Lemma 2.1, from (2.6) and assumption (2) we have

$$V(t, x(t)) \leqq u^*\big(t;\, t_0,\, V(t_0, x(t_0))\big) < a(\varepsilon) \quad (0 \leqq t \leqq t_0),$$

$$V(t, x(t)) \leqq v^*\big(t;\, t_0,\, V(t_0, x(t_0))\big) < a(\varepsilon) \quad (t_0 \leqq t < \infty),$$

i.e. $V(t, x(t)) < a(\varepsilon)$ for $t \in I$, from which, by (2.5) it follows that $\|x(t)\| < \varepsilon$ for $t \in I$. This means that the zero solution is strongly stable, q.e.d.

Suppose now that $X(t, x)$ has a continuous derivative with respect also to t, too. Then, analogously $\dot{V}(t, x)$, we define to $V(t, x)$ the function

$$\ddot{V}(t, x) = \sum_{i=1}^{n} \frac{\partial \dot{V}(t, x)}{\partial x_i} X_i(t, x) + \frac{\partial \dot{V}(t, x)}{\partial t},$$

provided that $V(t, x)$ has continuous second partial derivatives.

Theorem 2.2. *Assume that there exists a Lyapunov function $V(t, x)$ satisfying the following conditions on Γ:*

1) *$V(t, x) \to 0$ and $\dot{V}(t, x) \to 0$ as $x \to 0$ uniformly on I;*

2) *$\ddot{V}(t, x) \leqq \omega(t, V(t, x), \dot{V}(t, x))$, where the function $\omega(t, u_1, u_2): I \times I \times R \to R$ is continuous and nondecreasing in u_1, and $\omega(t, 0, 0) \equiv 0$ for $t \in I$;*

3) *the zero solution of the equation $\ddot{u} = \omega(t, u, \dot{u})$ is strongly u-stable i.e. if given any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $t_0 \in I$, $0 \leqq u_0 < \delta(\varepsilon)$, $|\dot{u}_0| < \delta(\varepsilon)$ imply $0 \leqq u(t;\, t_0,\, u_0,\, \dot{u}_0) < \varepsilon$ for $t \in I$.*

Then the zero solution of (E) is strongly stable.

The proof of this theorem, based on Lemma 2.2, is similar to that of Theorem 2.1, and therefore it is omitted.

3. Applications

I. Let us consider the systems

$$(E)\qquad \dot{x} = X(t, x)$$

$$(3.1)\qquad \dot{y} = X(t, y) + R(t, y) \quad \big(R(t, 0) \equiv 0,\ t \in I\big).$$

Applying Theorem 2.1 we get conditions which guarantee that the strong stability of the zero solution of (E) is preserved under the perturbation $R(t, y)$.

We begin by recalling some notations and preliminary results. It is known [2] that the solution $x(t; t_0, x_0)$ of (E) is a differentiable function of t, t_0, x_0 on $I \times \Gamma$ and the $n \times n$ matrix

$$\Phi(t; t_0, x_0) = \left[\frac{\partial x_i(t; t_0, x_0)}{\partial x_{0j}}\right]_{i,j=1,2,\ldots,n}$$

is the fundamental matrix with $\Phi(t_0; t_0, x_0) = E$ of the variational systems

$$\dot{z} = X_x(t; x(t; t_0, x_0))z, \tag{3.2}$$

where the $n \times n$ matrix X_x is defined by

$$X_x(t, x) = \left[\frac{\partial X_i(t, x)}{\partial x_j}\right]_{i,j=1,2,\ldots,n}.$$

For any real square matrix A, A^* denotes the transpose of A and $\lambda(A)$ denotes the smallest eigenvalue of the symmetric matrix $1/2(A+A^*)$. We use also the notation $\alpha(t) = \inf\{\lambda(X_x(t, x)) : \|x\| < H\}$.

Applying Ważewski's inequality [5] to the system (3.2), we obtain

$$\|\Phi(t; t_0, x_0)\| \leqq L \exp\left(\int_{t_0}^{t} \alpha(s)\, ds\right) \quad (0 \leqq t \leqq t_0;\ L = \text{const.}). \tag{3.3}$$

Theorem 3.1. *Let the solution $x=0$ of (E) be strongly stable. Then there are continuous functions $\gamma(t): I \to R$ $(\gamma(t) > 0)$ and $g(r) \in K$ such that the inequality*

$$\|R(t, y)\| \leqq \gamma(t) g(\|y\|) \quad ((t, y) \in \Gamma) \tag{3.4}$$

implies the strong stability of the solution $y=0$ of (3.1).

Proof. Let the solution $x=0$ of (E) be strongly stable. Let us consider the function $V(t, y) = \|x(0; t, y)\|^2$. In the proof of Theorem 1.1 it was verified that $[V(t, y)]^{1/2}$ is a Lyapunov function with the property $[V(t, y)]^{1/2} \to 0$ as $y \to 0$ uniformly on I. Consequently, there are functions $a(r)$, $b(r) \in K$ such that

$$a(\|y\|) \leqq V(t, y) \leqq b(\|y\|). \tag{3.5}$$

Choose continuous functions $\gamma(t) > 0$ and $g(r) \in K$ such that

$$\int_0^\infty \gamma(t) \exp\left[-\int_0^t \alpha(s)\, ds\right] dt < \infty; \quad \int_0^\eta \frac{dr}{g(a^{-1}(r))\sqrt{r}} = \infty \tag{3.6}$$

$(0 < \eta = \text{const.})$, where by $a^{-1}(r)$ the inverse of the function $a(r)$ is denoted. We have to prove that assumption (3.4) with these functions $\gamma(t)$, $g(r)$ implies the strong stability of the solution $y=0$ of (3.1).

Using (z_1, z_2) to denote the scalar product of vectors $z_1, z_2 \in R^n$, for the total derivative $V'(t, y)$ of $V(t, y)$ by virtue of (3.1) we have

$$V'(t, y) = \dot{V}(t, y) + \sum_{j=1}^{n} \frac{\partial V(t, y)}{\partial y_j} R_j(t, y) =$$

$$= 2 \sum_{j=1}^{n} \left(\frac{\partial}{\partial y_j} x(0;\, t, y),\, x(0;\, t, y) \right) R_j(t, y) = \tag{3.7}$$

$$= 2(\Phi(0;\, t, y) R(t, y),\, x(0;\, t, y)).$$

Applying the Cauchy inequality, from (3.3)—(3.5) and (3.7) we obtain the estimation

$$|V'(t, y)| \leqq 2L\gamma(t) \exp\left[-\int_0^t \alpha(s)\, ds\right] g(\|y\|)\, \|x(0;\, t, y)\| \leqq \tag{3.8}$$

$$\leqq 2L\gamma(t) \exp\left[-\int_0^t \alpha(s)\, ds\right] g(a^{-1}(V(t, y)))[V(t, y)]^{1/2}.$$

Now (3.5), (3.6), (3.8) and Example 2.1 show that we can apply Theorem 2.1 to equation (3.1). This concludes the proof.

It can be seen from (3.6) that the functions $\gamma(t)$, $g(r)$ depend, in general, on the unperturbed system (E). The following corollary shows that if (E) is linear then $\gamma(t)$, $g(r)$ are independent of (E).

Corollary 3.1. *Suppose that the functions $\gamma(t) > 0$ and $g(r) \in K$ have the properties:*

$$\int_0^\infty \gamma(t)\, dt < \infty, \quad \int_0^\eta \frac{dr}{g(r)} = \infty \quad (0 < \eta = \text{const.}), \tag{3.9}$$

and let $R(t, y)$ satisfy assumption (3.4).

Then the strong stability of the solution $x=0$ of the system

$$\dot{x} = A(t) x \tag{3.10}$$

implies the strong stability of the solution $y=0$ of the perturbed system

$$\dot{y} = A(t) y + R(t, y). \tag{3.11}$$

Proof. In Remark 2.2 it was proved that the solution $x=0$ of (3.10) is strongly stable if and only if the fundamental matrix $\Phi(t)$ of (3.10) and its inverse are bounded.

in the matrix norm for $t\in I$. Therefore we may suppose that in (3.5) $a(r)\equiv c_1 r^2$ and $b(r)\equiv c_2 r^2$ with appropriate positive constants c_1, c_2. Furthermore, $\Phi(0; t,y)=$ $=\Phi^{-1}(t)$, consequently the estimation (3.8) has the form

$$|V'(t, y)| \leqq c_3 \gamma(t) g\left(\left[\frac{1}{c_1} V(t, y)\right]^{1/2}\right) [V(t, y)]^{1/2} \quad (c_3 = \text{const.}),$$

showing that in this case just the assumption (3.9) guarantees the applicability of Theorem 2.1.

The corollary is proved.

Corollary 3.1 contains one of W. A. Coppel's theorems [2, Theorem 7, p. 67] as a special case.

II. Let

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad (i = 1, 2, \ldots, n) \tag{3.12}$$

be the equations of a mechanical system in canonical form. Assume that $q=0$ is an equilibrium position.

Let us first suppose that (3.12) describes a conservative and scleronomic system. Then the Hamiltonian function $H(q, p)$ is the sum of the kinetic energy $T(q, p)$ and the potential energy $W(q)$; $T(q, p)$ is positive definite with respect to p, and $W(0)=0$. A well known theorem of J. L. Lagrange assures that the equilibrium position $q=0$ is stable if W has an isolated minimum there [7]. Theorem 1.1 shows that under the same condition the equilibrium position is not only stable but also strongly stable.

Consider now a rheonomic system under the action of potential forces, having Hamiltonian function of the form

$$H(t, q, p) = \sum_{i,j=1}^{n} a_{ij}(t) p_i p_j + W(q) \quad (W(0) = 0), \tag{3.13}$$

where the scalar functions $a_{ij}(t)$ are continuously differentiable and bounded for $t\in I$.

Theorem 3.2. *If the Hamiltonian function* (3.13) *is positive definite and*

$$\int_0^\infty \max\left\{\left|\frac{d}{dt} a_{ij}(t)\right| : i, j = 1, 2, \ldots, n\right\} dt < \infty, \tag{3.14}$$

then the equilibrium position is strongly stable.

Proof. Since H is positive definite, there exist a number $a>0$ and a function $b(r)\in K$ such that $H(t,q,p)\geqq a\sum_{i=1}^{n} p_i^2+b(\|q\|)$. Furthermore,

$$\dot{H}=\dot{H}(t,q,p)=\frac{\partial H(t,q,p)}{\partial t}=\sum_{i,j=1}^{n}\frac{d}{dt}a_{ij}(t)p_i p_j;$$

hence we obtain the estimation

$$|\dot{H}|\leqq\frac{1}{2}\sum_{i,j=1}^{n}\left|\frac{d}{dt}a_{ij}(t)\right|(p_i^2+p_j^2)\leqq\frac{n}{a}\max\left\{\left|\frac{d}{dt}a_{ij}(t)\right|:\ i,j=1,2,\dots,n\right\}H(t,q,p).$$

Moreover, the boundedness of the functions $a_{ij}(t)$ guarantees that $H(t,q,p)\to 0$ as $q\to 0$ and $p\to 0$ uniformly on I.

These properties of H, assumption (3.14), and Example 2.1 show that Theorem 2.1 can be applied to equations (3.12), and this concludes the proof.

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