

Strongly reductive operators

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§ 1. Reductive

An operator T on a Hilbert space \mathfrak{H} is said to be *reductive* if each subspace of \mathfrak{H} invariant under T reduces T . (In this paper operators are bounded linear transformations, Hilbert spaces are complex, separable and infinite-dimensional, and subspaces are closed linear manifolds.) Using orthogonal projections instead of subspaces the definition can be expressed algebraically: T is reductive if $PT=TP$ whenever $P^2=P=P^*$ and $(1-P)TP=0$.

All Hermitian operators are reductive, and there are many examples of non-Hermitian reductive operators. However, no non-normal operator has been shown to be reductive, which suggests the following conjecture:

Reductive operator conjecture. Every reductive operator is normal.

It is a remarkable fact that this conjecture is equivalent to the perhaps best known conjecture in operator theory:

Invariant subspace conjecture. Every operator on a Hilbert space has a non-trivial invariant subspace.

(The subspaces $\{0\}$ and \mathfrak{H} are the trivial subspaces of the Hilbert space \mathfrak{H} ; all other subspaces are non-trivial.)

Theorem 1.1. (DYER, PEDERSON and PORCELLI [3]). *The reductive operator conjecture is true if and only if the invariant subspace conjecture is true.*

The question "Which normal operators are reductive?" has been studied by several authors, beginning with WERMER in 1952. He solved the problem completely for unitary operators, and obtained certain sufficient conditions for arbitrary normal operators.

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Theorem 1.2. (WERMER [11]). *If the spectrum, $\Sigma(T)$, of the normal operator T neither divides the plane nor has interior, then T is reductive. ($\Sigma(T)$ is a compact subset of the complex plane \mathbb{C} ; we say $\Sigma(T)$ divides the plane if its complement is disconnected.)*

The proof of the theorem depends on a special case of the following well known theorem. The theorem and its proof are included because the theorem is used several times in this paper.

Theorem 1.3. *Let t be an element of a C^* -algebra. Then the following conditions are equivalent:*

- (i) *t is normal and the spectrum of t , $\Sigma(t)$, neither divides the plane nor has interior;*
- (ii) *t^* is the limit in norm of a sequence of polynomials in t .*

Proof. Suppose t satisfies either (i) or (ii). Then t is normal and \mathfrak{C} , the closed subalgebra generated by 1 , t and t^* , is commutative. By the Gelfand—Neumark theorem, the Gelfand mapping $x \rightarrow \hat{x}$ of \mathfrak{C} onto $C(\mathfrak{M})$, the algebra of continuous functions on the maximal ideal space \mathfrak{M} of \mathfrak{C} , is an isometric isomorphism. Moreover, $\mathfrak{M} = \Sigma(t)$, and $\hat{t}(z) = z$ for each z in $\Sigma(t)$. Thus t^* is the limit in norm of a sequence of polynomials in t if and only if the function $\bar{z} \in P(\Sigma(t))$, the closure in $C(\Sigma(t))$ of the set of polynomials. The Stone—Weierstrass theorem implies that $\bar{z} \in P(\Sigma(t))$ if and only if $P(\Sigma(t)) = C(\Sigma(t))$, and by Lavrentiev's theorem [5, p. 48] $P(\Sigma(t)) = C(\Sigma(t))$ if and only if $\Sigma(t)$ neither divides the plane nor has interior.

A necessary and sufficient condition for reductivity of a normal operator was obtained by Sarason:

Theorem 1.4. (SARASON [8]). *The normal operator T is reductive if and only if T^* is in the closure, with respect to the weak operator topology, of the set of polynomials in T .*

In a subsequent paper he obtains the following spectral criterion for reductivity:

Theorem 1.5. (SARASON [9]). *Let T be a normal operator and let μ be a finite positive measure in the plane which is mutually absolutely continuous with the spectral measure of T . Then T is reductive if and only if the set of polynomials is weak-star dense in $L^\infty(\mu)$.*

In [9] he solves the problem "For which finite positive measures μ are the polynomials weak-star dense in $L^\infty(\mu)$?" so the problem "Which normal operators are reductive?" is solved. However, the solution of the approximation problem for measures is not easy to write down, and is not readily applicable as a test for reductivity of an arbitrary normal operator. The interested reader is referred to the paper.

In this paper we introduce the notion of strong reductivity for operators. We obtain some basic properties of strongly reductive operators, and study the question "Which normal operators are strongly reductive?" We show that the condition on $\Sigma(T)$ in theorem 1.2, which is sufficient for reductivity of the normal operator T , is both necessary and sufficient for strong reductivity of T .

In § 4 we consider reductivity in the Calkin algebra \mathfrak{A} (definitions will be given) and obtain a necessary and sufficient condition for reductivity in \mathfrak{A} of a normal element in \mathfrak{A} . The methods will resemble those used earlier in the paper.

§ 2. Strongly reductive

We say an operator T is strongly reductive if each subspace which is "almost invariant" under T "almost reduces" T . Precisely, the condition is expressed as follows:

Definition 2.1. The operator T is *strongly reductive* if for each $\varepsilon > 0$ there is a $\delta > 0$ such that $\|PT - TP\| < \varepsilon$ whenever $P^2 = P = P^*$ and $\|(1 - P)TP\| < \delta$.

Since $\|PT - TP\| = \max\{\|(1 - P)TP\|, \|(1 - P)T^*P\|\}$, $\|PT - TP\|$ may be replaced in the definition by $\|(1 - P)T^*P\|$, and in this alternative form the condition was mentioned by MOORE [7] as a natural strengthening of reductivity. The following theorem provides examples and summarizes some basic properties of strongly reductive operators:

Theorem 2.2. (i) *Hermitian operators are strongly reductive*, (ii) *strongly reductive operators are reductive*, and (iii) *the adjoint of a strongly reductive operator is strongly reductive*.

Proof. Parts (i) and (ii) are trivial consequences of the definitions. For any operator T and any projection P , $\|PT^* - T^*P\| = \|QT - TQ\|$ and $\|(1 - P)T^*P\| = \|(1 - Q)TQ\|$, where $Q = 1 - P$, so (iii) follows.

The following theorem provides further examples of strongly reductive operators, and, in view of theorem 1.3, it can be regarded as an extension of Theorem 1.2.

Theorem 2.3. *If T^* is the uniform limit of a sequence of polynomials in the operator T , then T is strongly reductive.*

Proof. We first show that if q is any polynomial and if $\varepsilon > 0$, then there is a $\delta > 0$ such that $\|(1 - P)q(T)P\| < \varepsilon$ whenever P is an orthogonal projection and $\|(1 - P)TP\| < \delta$. The proof is by induction on the degree of the polynomial. The statement is trivially true if q is a constant polynomial, for then $(1 - P)q(T)P = 0$. Suppose the statement is true for polynomials of degree k , and suppose q is a poly-

nomial of degree $k+1$. Let $r(z) = (q(z) - q(0))z^{-1}$. Then r is a polynomial of degree k , and

$$\begin{aligned} \|(1-P)q(T)P\| &= \|(1-P)(q(T) - q(0)I)P\| \\ &= \|(1-P)T(1-P+P)r(T)P\| \\ &\leq \|(1-P)T(1-P)r(T)P\| + \|(1-P)TPr(T)P\| \\ &\leq \|T\|\|(1-P)r(T)P\| + \|(1-P)TP\|\|r(T)\|. \end{aligned}$$

It follows that the statement is true for this polynomial q , and so by induction it is true for all polynomials.

Now choose $\varepsilon > 0$, choose a polynomial q such that $\|T^* - q(T)\| < \varepsilon/2$, and choose $\delta > 0$ such that $\|(1-P)q(T)P\| < \varepsilon/2$ whenever $\|(1-P)TP\| < \delta$. For such a P , $\|(1-P)T^*P\| < \|(1-P)q(T)P\| + \varepsilon/2 < \varepsilon$. Thus T is strongly reductive.

Corollary 2.4. *If T is normal, and if $\Sigma(T)$ neither divides the plane nor has interior, then T is strongly reductive.*

Proof. Apply theorem 1.3 and theorem 2.3.

§ 3. Spectrum

We derive certain properties of the spectrum of a strongly reductive operator. We prove that the spectrum neither divides the plane nor has interior, and, except for isolated normal eigenvalues of finite multiplicity, the spectrum equals the essential spectrum.

Let $\mathfrak{B}(\mathfrak{H})$ denote the algebra of all operators on the Hilbert space \mathfrak{H} , and let \mathfrak{A} denote the Calkin algebra, i.e. the factor algebra $\mathfrak{B}(\mathfrak{H})/\mathfrak{K}$, where \mathfrak{K} is the ideal of compact operators on \mathfrak{H} . Let π denote the canonical map from $\mathfrak{B}(\mathfrak{H})$ onto \mathfrak{A} : for each operator T , $\pi(T)$ is the coset in \mathfrak{A} containing T . For each operator T we define the following subsets of the plane:

$$\begin{aligned} \Sigma(T) &= \{\lambda: \lambda - T \text{ has no inverse in } \mathfrak{B}(\mathfrak{H})\}, \\ \Pi(T) &= \{\lambda: \lambda - T \text{ has no left inverse in } \mathfrak{B}(\mathfrak{H})\}, \\ \Sigma_{\text{ess}}(T) &= \{\lambda: \pi(\lambda - T) \text{ has no inverse in } \mathfrak{A}\}, \\ \Pi_{\text{ess}}(T) &= \{\lambda: \pi(\lambda - T) \text{ has no left inverse in } \mathfrak{A}\}. \end{aligned}$$

$\Sigma(T)$, $\Pi(T)$, $\Sigma_{\text{ess}}(T)$ and $\Pi_{\text{ess}}(T)$ are called the spectrum, the left spectrum or approximate point spectrum, the essential spectrum, and the left essential spectrum of T , respectively. Each is a non-empty compact subset of the plane, and they are related as follows:

$$(3.1) \quad \Sigma(T) = \Pi(T) \cup \Pi(T^*)^- \quad \text{and} \quad \Sigma_{\text{ess}}(T) = \Pi_{\text{ess}}(T) \cup \Pi_{\text{ess}}(T^*)^-$$

where $\bar{}$ denotes complex conjugation. The left spectra $\prod(T)$ and $\prod_{\text{ess}}(T)$ can be characterised in terms of "boundedness below":

Proposition 3.2. (i) $\lambda \in \prod(T)$ if and only if there is a sequence of unit vectors $\{\varphi_n\}$ such that $\|(T-\lambda)\varphi_n\| \rightarrow 0$ as $n \rightarrow \infty$, and (ii) $\lambda \in \prod_{\text{ess}}(T)$ if and only if there is an orthogonal sequence of unit vectors $\{\varphi_n\}$ such that $\|(T-\lambda)\varphi_n\| \rightarrow 0$ as $n \rightarrow \infty$.

Proof. See [6, p. 37] and [4].

Lemma 3.3. If T is strongly reductive, and if $\{\varphi_n\}$ is a sequence of unit vectors such that $\|(T-\lambda)\varphi_n\| \rightarrow 0$ as $n \rightarrow \infty$ for some λ , then $\|(T^*-\bar{\lambda})\varphi_n\| \rightarrow 0$ as $n \rightarrow \infty$.

Proof. Let P_n be the orthogonal projection onto $\text{span}\{\varphi_n\}$. Thus $P_n\psi = (\psi, \varphi_n)\varphi_n$ for each vector ψ , and

$$\|(1-P_n)TP_n\| = \|(1-P_n)T\varphi_n\| = \|(1-P_n)(T-\lambda)\varphi_n\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Therefore, since T is strongly reductive, $\|(1-P_n)T^*P_n\| \rightarrow 0$ as $n \rightarrow \infty$. So $\|T^*\varphi_n - P_nT^*\varphi_n\| \rightarrow 0$ as $n \rightarrow \infty$. Now $P_nT^*\varphi_n = (T^*\varphi_n, \varphi_n)\varphi_n = (\varphi_n, T\varphi_n)\varphi_n$, and $(\varphi_n, T\varphi_n) \rightarrow \bar{\lambda}$ as $n \rightarrow \infty$. Thus it follows that $\|(T^*-\bar{\lambda})\varphi_n\| \rightarrow 0$ as $n \rightarrow \infty$.

Corollary 3.4. If T is strongly reductive, then $\sum(T) = \prod(T)$ and $\sum_{\text{ess}}(T) = \prod_{\text{ess}}(T)$.

Proof. From proposition 3.2 and lemma 3.3,

$$\prod(T) = \prod(T^*)^- \text{ and } \prod_{\text{ess}}(T) = \prod_{\text{ess}}(T^*)^-.$$

Now use equations 3.1.

Theorem 3.5. If T is strongly reductive, then $\sum(T)$ is the union of $\sum_{\text{ess}}(T)$ and isolated normal eigenvalues of finite multiplicity.

Proof. By Corollary 3.4 $\sum(T) = \prod(T)$ and $\sum_{\text{ess}}(T) = \prod_{\text{ess}}(T)$. Clearly $\prod_{\text{ess}}(T) \subset \prod(T)$. Suppose $\lambda \in \prod(T) \setminus \prod_{\text{ess}}(T)$. Since $\lambda \notin \prod_{\text{ess}}(T)$, $\ker(T-\lambda)$ is finite-dimensional, and $T-\lambda$ is bounded below on $\ker(T-\lambda)^\perp$. Since $\lambda \in \prod(T)$, it follows that $\ker(T-\lambda)$ is non-trivial, and thus λ is an eigenvalue of T of finite multiplicity. Since T is strongly reductive, the eigenvalue λ is a normal eigenvalue, i.e. $T\varphi = \lambda\varphi$ implies $T^*\varphi = \bar{\lambda}\varphi$.

It remains to be shown that λ is an isolated point of $\prod(T)$. Since $\ker(T-\lambda)$ is invariant under T and T is strongly reductive, $\ker(T-\lambda)$ reduces T . Let T' denote the restriction of T to $\ker(T-\lambda)^\perp$. Since $T'-\lambda$ is bounded below $\lambda \notin \prod(T')$, and since $\prod(T')$ is compact there is a $\delta > 0$ such that $\mu \notin \prod(T')$ whenever $|\mu-\lambda| < \delta$. Now $\prod(T) = \{\lambda\} \cup \prod(T')$, so λ is isolated in $\prod(T)$.

Lemma 3.6. If N is normal, T is strongly reductive and $\sum(N) \subset \sum_{\text{ess}}(T)$, then N is reductive.

Proof. We may suppose that T and N are operators on the same Hilbert space \mathfrak{H} . Let P be an orthogonal projection on \mathfrak{H} such that $(1-P)NP=0$. We shall

show that $PN - NP = 0$. Let $\mathfrak{H}^{(\infty)}$ be the orthogonal direct sum of copies of \mathfrak{H} , indexed by the non-negative integers. Define the following operators on $\mathfrak{H}^{(\infty)}$:

$$S = T \oplus N \oplus N \oplus N \oplus \dots, \quad P_1 = 0 \oplus P \oplus P \oplus P \oplus \dots,$$

$$P_2 = 0 \oplus 0 \oplus P \oplus P \oplus \dots, \quad P_3 = 0 \oplus 0 \oplus 0 \oplus P \oplus \dots, \text{ and so on.}$$

By proposition 3.2, Lemma 3.3 and Corollary 3.4, T is "strongly normal on $\sum_{\text{ess}}(T)$ " in the sense of Stampfli [10], i.e. for each λ in $\sum_{\text{ess}}(T)$ there is an orthonormal sequence of vectors $\{\varphi_n\}$ such that $\|(T - \lambda)\varphi_n\| \rightarrow 0$ and $\|(T^* - \bar{\lambda})\varphi_n\| \rightarrow 0$ as $n \rightarrow \infty$. Thus, by a theorem of STAMPFLI [10], there is an isometric isomorphism W , from \mathfrak{H} to $\mathfrak{H}^{(\infty)}$, and a compact operator K on $\mathfrak{H}^{(\infty)}$ such that

$$WTW^{-1} = S + K.$$

Since T is strongly reductive, so is $S + K$. Now

$$\begin{aligned} \|(1 - P_n)(S + K)P_n\| &\leq \|(1 - P_n)SP_n\| + \|(1 - P_n)KP_n\| = \\ &= \|(1 - P)NP\| + \|(1 - P_n)KP_n\| \leq \|KP_n\|. \end{aligned}$$

Now $P_n \rightarrow 0$ strongly as $n \rightarrow \infty$, and since K is compact it follows that $\|KP_n\| \rightarrow 0$ as $n \rightarrow \infty$. Thus $\|(1 - P_n)(S + K)P_n\| \rightarrow 0$ as $n \rightarrow \infty$, and since $S + K$ is strongly reductive it follows that $\|P_n(S + K) - (S + K)P_n\| \rightarrow 0$ as $n \rightarrow \infty$. Now

$$\begin{aligned} \|P_n(S + K) - (S + K)P_n\| &\geq \|P_nS - SP_n\| - \|P_nK\| - \|KP_n\| = \\ &= \|PN - NP\| - \|P_nK\| - \|KP_n\|. \end{aligned}$$

As before $\|P_nK\| \rightarrow 0$ and $\|KP_n\| \rightarrow 0$ as $n \rightarrow \infty$, so $PN - NP = 0$. Thus N is reductive.

Lemma 3.7. *If X is a compact set in the plane which either divides the plane or has interior, then there is a normal operator N which is not reductive, and whose spectrum is contained in X .*

Proof. Let \hat{X} denote the union of X and all bounded components of the complement of X . Then \hat{X} is compact and, by the hypothesis, \hat{X} has non-empty interior. Let G be a component of the interior of \hat{X} , and let λ be a point in G . Let m be the harmonic measure on \hat{X} evaluated at λ [2, p. 77]. The measure m is a probability measure, its support, $\text{supp } m$, is the boundary of G , ∂G , and it is the unique representing measure for the complex homomorphism "evaluation at λ " on the Dirichlet algebra $R(\hat{X})$ [5, chapter II.]. ($R(\hat{X})$ is the closure in $C(\hat{X})$ of the set of all rational functions with poles off \hat{X} .) That is, for any function f in $R(\hat{X})$, $f(\lambda) = \int f(z) dm(z)$.

Let N be the normal operator of multiplication by z on the Hilbert space $L^2(m)$. Then $\sum(N) = \text{supp } m = \partial G \subset \partial \hat{X} \subset \partial X \subset X$. Let $H^2(m)$ denote the closure in $L^2(m)$ of the set of polynomials. Clearly $H^2(m)$ is invariant under N . Now the constant function $1 \in H^2(m)$, and $((N - \lambda)^* 1)(z) = \bar{z} - \bar{\lambda}$. If p is any polynomial then $(p(z), \bar{z} - \bar{\lambda}) = \int p(z)(z - \lambda) dm(z) = 0$, so $\bar{z} - \bar{\lambda} \in H^2(m)^\perp$. Furthermore $\|\bar{z} - \bar{\lambda}\|^2 = \int |z - \lambda|^2 dm(z) \geq \text{dist}(\lambda, \partial G)^2 > 0$, so it follows that $H^2(m)$ does not reduce N , and thus N is not reductive.

Theorem 3.8. *If T is strongly reductive, then $\Sigma(T)$ neither divides the plane nor has interior.*

Proof. In view of Theorem 3.5 it is sufficient to show that $\Sigma_{\text{ess}}(T)$ neither divides the plane nor has interior; but this follows from Lemma 3.6 and Lemma 3.7.

Corollary 3.9. *Let T be a normal operator. Then the following conditions are equivalent:*

- (i) T is strongly reductive,
- (ii) $\Sigma(T)$ neither divides the plane nor has interior,
- (iii) T^* is the uniform limit of a sequence of polynomials in T .

Proof. By theorem 3.8 (i) implies (ii), and by Corollary 2.4 (ii) implies (i), so (i) and (ii) are equivalent. By Theorem 1.3 (ii) and (iii) are equivalent.

§ 4. Essentially reductive

In [7] MOORE shows how the concept of reductivity can be extended from operators to elements of an arbitrary C^* -algebra. The idea is simply to use the algebraic formulation of the definition of reductivity: say an element t in a C^* -algebra is reductive if $pt=tp$ whenever $p^2=p=p^*$ and $(1-p)tp=0$. He devotes particular attention to the Calkin algebra \mathfrak{A} , and we shall provide the solution to a problem he poses concerning reductivity in \mathfrak{A} : "Which normal elements of \mathfrak{A} are reductive elements of \mathfrak{A} ?" First he shows how the problem can be stated in terms of operators.

Definition 4.1. An operator T is *essentially reductive* if $PT-TP$ is compact whenever $P^2=P=P^*$ and $(1-P)TP$ is compact.

Definition 4.2. An operator T is *essentially normal* if T^*T-TT^* is compact.

Moore's problem in these terms is: "Which essentially normal operators are essentially reductive?" He provides the following partial answers:

Theorem 4.3. (MOORE [7]). *If T is essentially normal and if $\Sigma_{\text{ess}}(T)$ neither divides the plane nor has interior, then T is essentially reductive.*

The proof is based on theorem 1.3.

He also proves the following theorem which he uses to obtain a partial converse to theorem 4.3:

Theorem 4.4. (MOORE [7]). *If N is normal, T is essentially reductive, and $\Sigma(N) \subset \Sigma_{\text{ess}}(T)$, then N is reductive.*

The statement and proof of the theorem are analogous to the statement and proof of Lemma 3.6. Using theorem 4.2 Moore shows that if T is essentially normal and if $\Sigma_{\text{ess}}(T)$ either has interior, or (more generally) contains a closed analytic Jordan curve, then T is not essentially reductive.

Lemma 3.7 provides a full converse to theorem 4.3 when coupled with Theorem 4.4.

Theorem 4.5. *If T is essentially normal, then the following conditions are equivalent:*

- (i) T is essentially reductive,
- (ii) $\sum_{\text{ess}}(T)$ neither divides the plane nor has interior,
- (iii) T^* is the uniform limit of a sequence $\{p_n(T) + K_n\}$, where each p_n is a polynomial and each K_n is compact.

Proof. By Theorem 4.4 and Lemma 3.7 (i) implies (ii), and by Theorem 4.3 (ii) implies (i), so (i) and (ii) are equivalent. Condition (iii) is precisely the condition under which $\pi(T^*)$ is the limit in norm of a sequence of polynomials in $\pi(T)$, so by Theorem 1.3 (ii) and (iii) are equivalent.

Corollary 4.6. *If T is essentially normal and essentially reductive, then T is a compact perturbation of a strongly reductive, normal operator.*

Proof. By Theorem 4.5 $\sum_{\text{ess}}(T)$ does not divide the plane, so by a theorem of BROWN, DOUGLAS and FILLMORE [1, p. 119], T is a compact perturbation of a normal operator N such that $\sum(N) = \sum_{\text{ess}}(T)$. By corollary 2.4 N is strongly reductive.

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